Numerical simulation of the incompressible Navier-Stokes equations in a moving domain: application to hemodynamics

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Outline

Incompressible Navier-Stokes equations in a moving domain

Model for a viscous fluid with constant density in a domain that changes shape in time

Examples:

- Static domain: flow in a channel
- Fluid-fluid: bubbles of air in water
- Fluid-structure: hemodynamics, bridges, sail design

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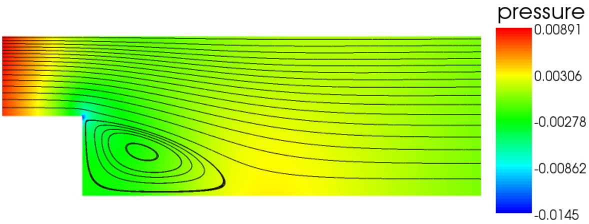

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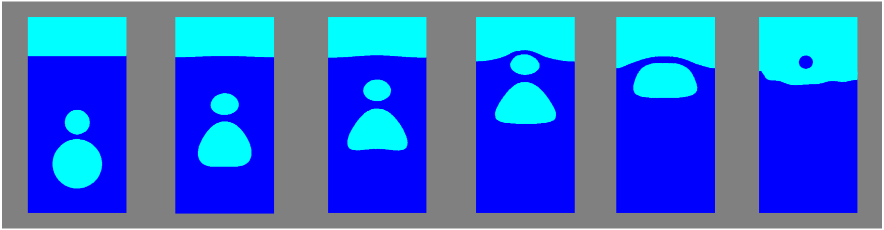

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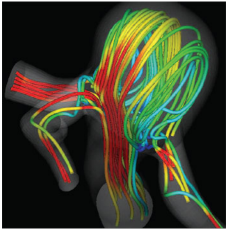
  

Navier-Stokes equations (Eulerian coordinate system)

$$\rho\frac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta} \mathbf{u} + \rho(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega_t \times (0,T) \tag{1}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0, \quad \text{in } \Omega_t \times (0,T) \tag{2}$$

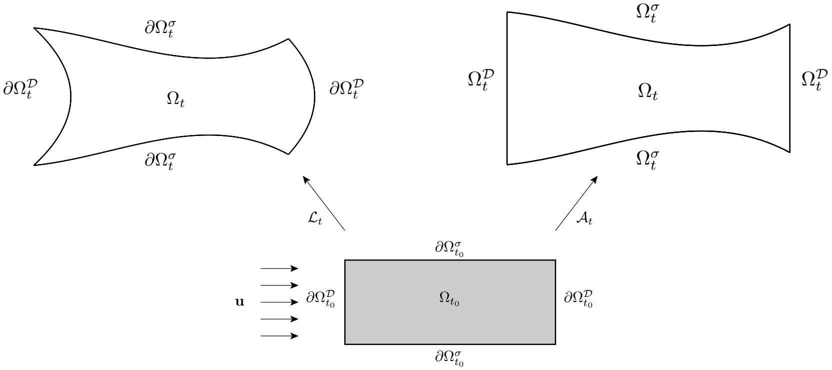
$$\mathcal{B}(\mathbf{u},p) = \mathbf{g}, \quad \text{on } \partial\Omega_t \times [0,T] \tag{3}$$

Model for viscous fluid

- $\mathbf{u}$ is the velocity field, p is the pressure field
- ν is the viscosity constant, ρ is the density
- Momentum conservation: (1)
- Mass conservation: (2)
- $\mathcal{B}(\mathbf{u},p)$ is an operator with boundary conditions

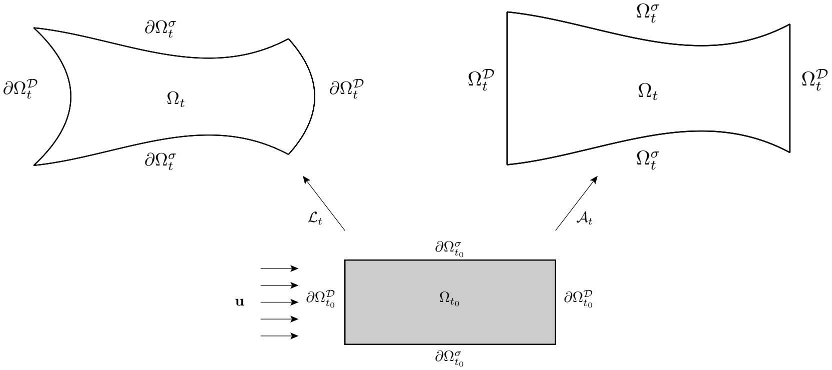
Coordinate systems:

- Eulerian coordinate system



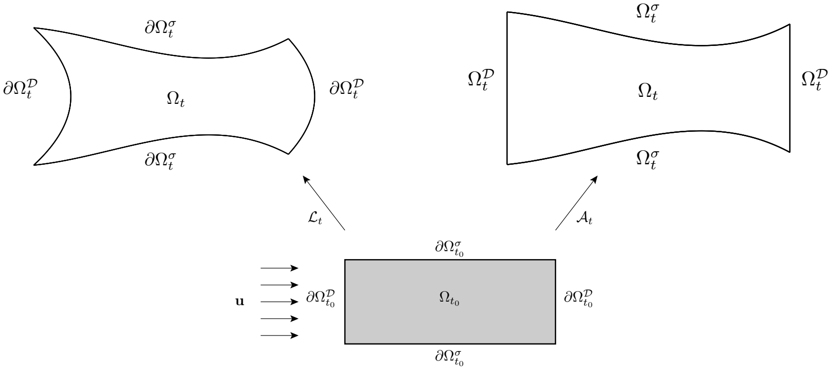
Coordinate systems:

- ~~Eulerian coordinate system~~
- Lagrangian coordinate system



Coordinate systems:

- ~~Eulerian coordinate system~~
- ~~Lagrangian coordinate system~~
- Arbitrary lagrangian-eulerian (ALE) coordinate system

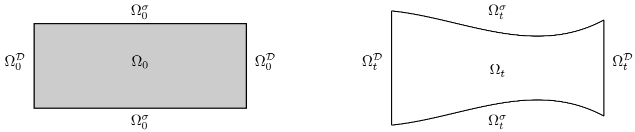


ALE approach

Navier-Stokes equations in ALE coordinates

$$\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\mathbf{Y}} + [(\mathbf{u}-\mathbf{w})\cdot\nabla]\mathbf{u} + \nabla p - \nu\Delta\mathbf{u} = \mathbf{f}, \text{ in } \Omega_t$$

$$\nabla\cdot\mathbf{u} = 0, \text{ in } \Omega_t$$



- ALE map: homeomorfism $\mathcal{A}_t : \Omega_{t_0} \longrightarrow \Omega_t$
- Domain's deformation velocity $\mathbf{w} = \frac{\partial \mathcal{A}_t}{\partial t} \circ \mathcal{A}_t^{-1}$
- ALE time derivative: $\left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\mathbf{Y}}$

Weak formulation

For $\mathbf{u}, \mathbf{v}, \boldsymbol{\beta} \in \mathbf{V}(\Omega_t)$ and $p, q \in Q(\Omega_t)$, let

$$(\mathbf{u},\mathbf{v})_{\Omega_t} = \int_{\Omega_t} \mathbf{u}\cdot\mathbf{v}\,dx \qquad a(\mathbf{u},\mathbf{v})_{\Omega_t} = \nu\int_{\Omega_t} \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{u} : \boldsymbol{\nabla}_{\mathbf{x}}\mathbf{v}\,dx$$

$$b(\mathbf{v},p)_{\Omega_t} = \int_{\Omega_t} \mathrm{div}_{\mathbf{x}}(\mathbf{u})\,p\,dx \qquad c(\mathbf{u},\mathbf{v};\boldsymbol{\beta})_{\Omega_t} = \rho\int_{\Omega_t} [\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_{\mathbf{x}}]\,\mathbf{u}\cdot\mathbf{v}\,dx.$$

Problem

For $t \in I$, find $\mathbf{u}(t) \in \mathbf{V}(\Omega_t)$, with $\mathbf{u}(t_0) = \mathbf{u}_0$ in Ω_{t_0} and $p(t) \in Q(\Omega_t)$, such that

$$\begin{aligned} \rho\left(\left.\frac{\partial\mathbf{u}}{\partial t}\right|_{\mathbf{Y}}, \mathbf{v}\right)_{\Omega_t} + c(\mathbf{u},\mathbf{v};\mathbf{u}-\mathbf{w})_{\Omega_t} + \\ a(\mathbf{u},\mathbf{v})_{\Omega_t} + b(\mathbf{v},p)_{\Omega_t} &= (\mathbf{f},\mathbf{v})_{\Omega_t}, \quad \forall\mathbf{v}\in\mathbf{V}(\Omega_t) \\ b(\mathbf{u},q)_{\Omega_t} &= 0, \qquad \forall q\in Q(\Omega_t) \end{aligned} \tag{4}$$

$$\mathbf{V}(\Omega_t) = \left\{\mathbf{v} : \Omega_t \times I \longrightarrow \mathbb{R}^d,\ \ \mathbf{v} = \hat{\mathbf{v}}\circ\mathcal{A}_t^{-1},\ \ \hat{\mathbf{v}} \in \mathbf{H}^1_{\Gamma^D}(\Omega_{t_0})\right\}$$

Family of numerical methods

- Get a weak formulation of the problem
- Discretization in space
  - Finite/spectral element method
  - Construction of the ALE map
- Discretization in time
  - Finite differences/Runge-Kutta
  - Linearization of the convective term/Newton
  - Discretization of the domain's deformation velocity
- Strategy to solve algebraic system of equations
  - Direct method
  - GMRES with preconditioner
  - Algebraic factorization methods

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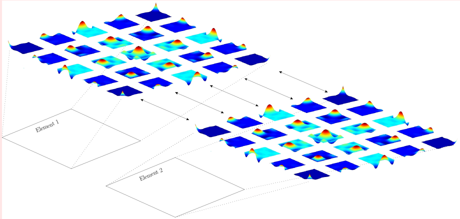
Build basis for the space

$$\mathcal{F}_N(\mathcal{T}_h) = \left\{ v \in C^0(\overline{\Omega}) : \ v_{|\Omega_e} \in \mathbb{P}_N(\Omega_e), \ \forall \Omega_e \in \mathcal{T}_h \right\}$$

Plan: reference element approach

1. Construct a basis in $\mathbb{P}_N(\hat{\Omega})$
   - Lagrange polynomials
   - Fekete/Gauss-Lobatto-Legendre points



2. Use geometrical transformation + glue similar functions (continuity)

How to build the discrete spaces?

Define spaces in the reference configuration and use the ALE map:

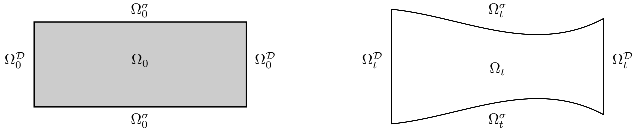
$$\mathbf{V}_\delta(\Omega_{t,\delta}) = \left\{ \mathbf{v} : \Omega_{t,\delta} \times I \longrightarrow \mathbb{R}^d, \;\; \mathbf{v} = \hat{\mathbf{v}} \circ \mathcal{A}_{t,\delta}^{-1}, \;\; \hat{\mathbf{v}} \in \mathbf{H}^1_{\Gamma^D}(\Omega_{t_0}) \cap (\mathcal{F}_N(\mathcal{T}_{t_0,\delta}))^d \right\}$$

$$Q_\delta(\Omega_{t,\delta}) = \left\{ q : \Omega_{t,\delta} \times I \longrightarrow \mathbb{R}, \;\; q = \hat{q} \circ \mathcal{A}_{t,\delta}^{-1}, \;\; \hat{q} \in \mathcal{F}_M(\mathcal{T}_{t_0,\delta}) \right\}$$

How to build the ALE map?

- Usual approach: use finite elements and harmonic extension
- Less usual approach: use spectral elements (Stokes or Laplace)
  - Representation of the boundary with curved elements
  - Keep inner elements with straight edges

Steps in the construction of $\mathcal{A}_{\cdot,\delta}$



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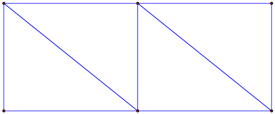
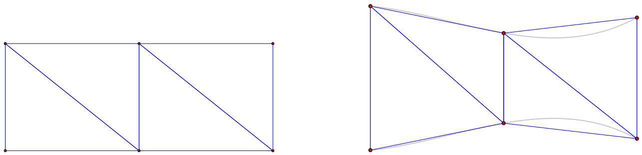


1. Generate a straight edge mesh

Steps in the construction of $\mathcal{A}_{\cdot,\delta}$



1. Generate a straight edge mesh
2. Given the description of the boundary $\partial\Omega_t$, calculate the discrete harmonic extension, $\mathcal{A}_h^1$

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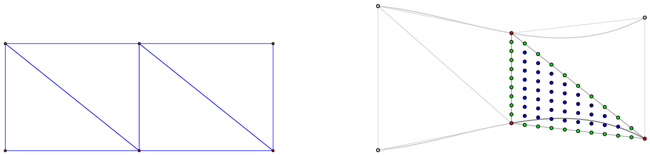
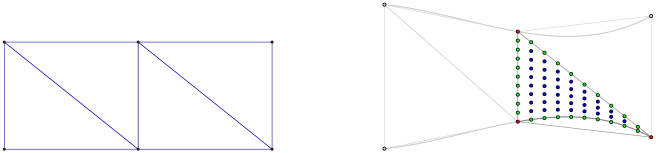


1. Generate a straight edge mesh
2. Given the description of the boundary $\partial\Omega_t$, calculate the discrete harmonic extension, $\mathcal{A}_h^1$
3. Project $\mathcal{A}_h^1$ in the space $\mathbb{P}_N$, $\mathcal{A}_h^N$

Steps in the construction of $\mathcal{A}_{\cdot,\delta}$



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2. Given the description of the boundary $\partial\Omega_t$, calculate the discrete harmonic extension, $\mathcal{A}_h^1$
3. Project $\mathcal{A}_h^1$ in the space $\mathbb{P}_N$, $\mathcal{A}_h^N$
4. Change the values of the degrees of freedom of $\mathcal{A}_h^N$ to fit the boundary, $\mathcal{A}_{\cdot,\delta}$

Weak formulation (discrete problem)

For each $n \geqslant q-1$, find $(\mathbf{u}_\delta^{n+1}, p_\delta^{n+1}) \in \mathbf{V}_\delta(\Omega_{t_{n+1},\delta}) \times Q_\delta(\Omega_{t_{n+1},\delta})$, with $\mathbf{u}_\delta^0 = \mathbf{u}_{0,\delta}$ in $\Omega_{t_0,\delta}$, such that

$$\begin{aligned}
\rho\frac{\beta_{-1}}{\Delta t}\left(\mathbf{u}_\delta^{n+1},\mathbf{v}\right)_{\Omega_{t_{n+1},\delta}} + \\
a\left(\mathbf{u}_\delta^{n+1},\mathbf{v}\right)_{\Omega_{t_{n+1},\delta}} + b\left(\mathbf{v},p_\delta^{n+1}\right)_{\Omega_{t_{n+1},\delta}} + \\
c\left(\mathbf{u}_\delta^{n+1},\mathbf{v};\mathbf{u}_\delta^{*}-\mathbf{w}_\delta^{n+1}\right)_{\Omega_{t_{n+1},\delta}} &= \left(\tilde{\mathbf{f}}_\delta^{n+1},\mathbf{v}\right)_{\Omega_{t_{n+1},\delta}}, \quad \forall \mathbf{v} \in \mathbf{V}_\delta(\Omega_{t_{n+1},\delta}) \\
b\left(\mathbf{u}_\delta^{n+1},q\right)_{\Omega_{t_{n+1},\delta}} &= 0, \quad \forall q \in Q_\delta(\Omega_{t_{n+1},\delta})
\end{aligned}$$

onde

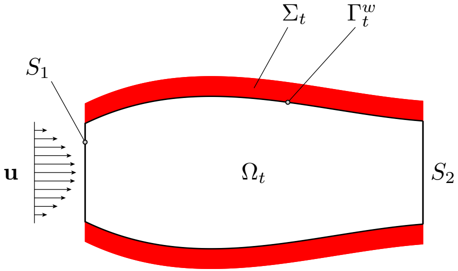
$$\tilde{\mathbf{f}}_\delta^{n+1} = \mathbf{f}^{n+1} + \rho\sum_{j=0}^{q-1}\frac{\beta_j}{\Delta t}\mathbf{u}_\delta^{n-j}$$

Assembling the matrices, we obtain

$$\begin{bmatrix} F_N & G_N \\ D_N & 0 \end{bmatrix}\begin{bmatrix} \mathbf{U}_N^{n+1} \\ \mathbf{P}_N^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_N^{n+1} \\ 0 \end{bmatrix}.$$

Fluid-structure interaction (FSI) problem in hemodynamics

An interaction exists between the blood flow and the arterial wall



- Ω_t represents the domain occupied by the blood
- Σ_t represents the arterial wall
- Γ_t^w is the interface between Ω_t and Σ_t
- S_1 are S_2 ficticious boundaries of the blood vessel

FSI problem

$$
\begin{aligned}
\rho \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\mathbf{Y}} + \rho((\mathbf{u}-\mathbf{w})\cdot \boldsymbol{\nabla}_{\mathbf{x}})\mathbf{u} - 2\nu \mathbf{D}(\mathbf{u}) + \nabla p &= \mathbf{f}, \quad \text{in } \Omega_t, t \in I \\
\mathrm{div}_{\mathbf{x}}(\mathbf{u}) &= 0, \quad \text{in } \Omega_t, t \in I \\
\rho_w h \frac{\partial^2 \eta}{\partial t^2} - kGh \frac{\partial^2 \eta}{\partial x^2} + \frac{Eh}{1-\nu^2}\frac{\eta}{R_0^2} - \gamma_v \frac{\partial^3 \eta}{\partial x^2 \partial t} &= \Phi_r \quad \text{in } (0,L), t \in I \\
\mathbf{u} &= \left(\dot{\eta} \circ \boldsymbol{\varphi}_\eta^{-1}\right) \mathbf{e}_2, \quad \text{in } \Gamma_t^w \\
\Phi_r &= -(\mathbf{Tn} \cdot \mathbf{e}_2) \circ \boldsymbol{\varphi}_\eta
\end{aligned}
$$

Numerical method for the FSI problem

- Modular approach: structure algorithm + fluid algorithm + interface operators
- Non-modular approach

FSI problem

$$\begin{aligned}
\rho \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\mathbf{Y}} + \rho((\mathbf{u}-\mathbf{w})\cdot\boldsymbol{\nabla}_{\mathbf{x}})\mathbf{u} - 2\nu\mathbf{D}(\mathbf{u}) + \nabla p &= \mathbf{f}, \quad \text{in } \Omega_t, t \in I \\
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\Phi_r &= -(\mathbf{Tn}\cdot\mathbf{e}_2)\circ\boldsymbol{\varphi}_\eta
\end{aligned}$$

Numerical method for the FSI problem

- Modular approach: structure algorithm + fluid algorithm + interface operators
- Non-modular approach

A carregar...

Figura: Pressure wave propagating through the blood vessel. The displacement is magnified five times.

Go

Thank you

$$t = 0ms$$

Figura: Pressure wave propagating through the blood vessel. The displacement is magnified five times.

Back

$$t = 2ms$$

Figura: Pressure wave propagating through the blood vessel. The displacement is magnified five times.

Back

$$t = 4ms$$

Figura: Pressure wave propagating through the blood vessel. The displacement is magnified five times.

Back

$$t = 6ms$$

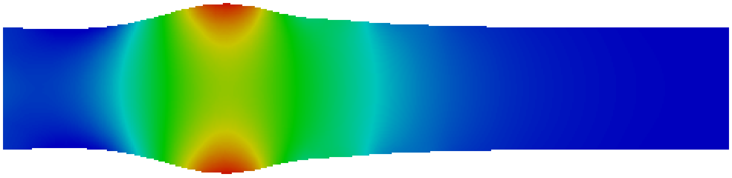

Figura: Pressure wave propagating through the blood vessel. The displacement is magnified five times.

Back

$$t = 8ms$$

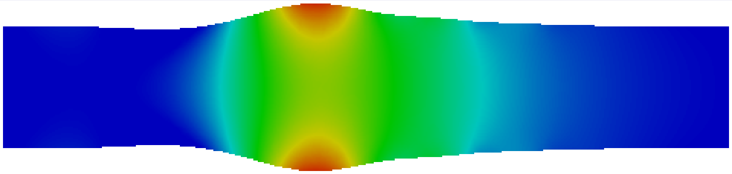

Figura: Pressure wave propagating through the blood vessel. The displacement is magnified five times.

Back

$$t = 10ms$$

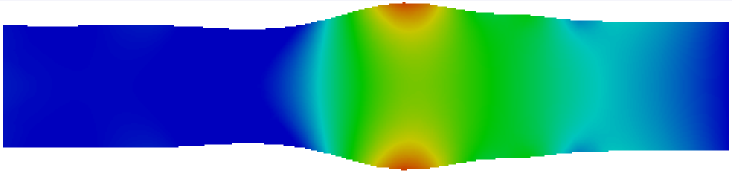

Figura: Pressure wave propagating through the blood vessel. The displacement is magnified five times.

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