Compactified Jacobians of Singular Curves

Margarida Melo
CMUC, Departamento de Matemática da Universidade de Coimbra

July 15, 2010

Algebraic Geometry

Algebraic geometry is concerned with the study of algebraic varieties: geometric incarnations of solutions of systems of polynomial equations.

Algebraic Geometry

Algebraic geometry is concerned with the study of algebraic varieties: geometric incarnations of solutions of systems of polynomial equations.
It is a wide area of mathematics that combines tools from many different disciplines as Abstract Commutative Algebra, Number Theory, Complex Analysis, Differential and Complex Geometry, Algebraic Topology, Category Theory, Homological Algebra, Algebraic Combinatorics and Representation Theory to study problems arising from Geometry.

Algebraic Geometry

Algebraic geometry is concerned with the study of algebraic varieties: geometric incarnations of solutions of systems of polynomial equations.
It is a wide area of mathematics that combines tools from many different disciplines as Abstract Commutative Algebra, Number Theory, Complex Analysis, Differential and Complex Geometry, Algebraic Topology, Category Theory, Homological Algebra, Algebraic Combinatorics and Representation Theory to study problems arising from Geometry.
Even if it is a very classical area of mathematics it is one of the most actives as well, with many surprising interactions with other areas of Mathematics and with Physics.

Projective varieties

Projective varieties

Projective varieties are algebraic varieties that can be embedded in projective space.
The n-dimensional projective space is a **compactification** of the affine space $\mathbb{C}^n$ by "adding points at the infinity":

$$\mathbb{P}^n := \mathbb{C}^{n+1}/\mathbb{C}^*$$

where $\lambda(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$.

Projective varieties

Projective varieties are algebraic varieties that can be embedded in projective space.
The n-dimensional projective space is a **compactification** of the affine space $\mathbb{C}^n$ by "adding points at the infinity":

$$\mathbb{P}^n := \mathbb{C}^{n+1}/\mathbb{C}^*$$

where $\lambda(x_0, \ldots, x_n) = (\lambda x_0, \ldots, \lambda x_n)$.
To define algebraic varieties inside the projective space one must consider homogeneous polynomials: if $f \in \mathbb{C}[x_0, \ldots, x_n]$ is homogeneous of degree d,
$f(\lambda x_0, \ldots, \lambda x_n) = 0 \Leftrightarrow \lambda^d f(x_0, \ldots, x_n) = 0$.

Projective varieties

Projective varieties are algebraic varieties that can be embedded in projective space.
The n-dimensional projective space is a **compactification** of the affine space $\mathbb{C}^n$ by "adding points at the infinity":

$$\mathbb{P}^n := \mathbb{C}^{n+1}/\mathbb{C}^*$$

where $\lambda(x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n)$.
To define algebraic varieties inside the projective space one must consider homogeneous polynomials: if $f \in \mathbb{C}[x_0, \dots, x_n]$ is homogeneous of degree d,
$f(\lambda x_0, \dots, \lambda x_n) = 0 \Leftrightarrow \lambda^d f(x_0, \dots, x_n) = 0$.
The geometry of projective space is very rich and has many interesting features:
e.g. two distinct curves in $\mathbb{P}^2$ of degrees d and e, respectively, meet in exactly $d.e$ points (counted with multiplicity!).

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

- $\{\text{Hyperplanes of } \mathbb{P}^n\} \cong \mathbb{P}^n$;

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

- $\{\text{Hyperplanes of } \mathbb{P}^n\} \cong \mathbb{P}^n$;
  $a_0x_0 + \cdots + a_nx_n$ and $b_0x_0 + \cdots + b_nx_n$ define the same hyperplane of $\mathbb{P}^n$ if and only if $(a_0, \ldots, a_n) = \lambda(b_0, \ldots, b_n)$ for some $\lambda \in \mathbb{C}^*$.

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

- $\{\text{Hyperplanes of } \mathbb{P}^n\} \cong \mathbb{P}^n$;
- $M_g := \{ \text{ Smooth projective curves of given genus } g \ \}_{/\cong}$

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

- $\{\text{Hyperplanes of } \mathbb{P}^n\} \cong \mathbb{P}^n$;
- $M_g := \{ \text{ Smooth projective curves of given genus } g \ \}_{/\cong}$

$\{$Smooth projective curves of genus g over $\mathbb{C}\}$

$$\updownarrow$$

$\{$Compact Riemann surfaces with g holes$\}$

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

- $\{\text{Hyperplanes of } \mathbb{P}^n\} \cong \mathbb{P}^n$;
- $M_g := \{ \text{ Smooth projective curves of given genus } g \ \}_{/\cong}$
    - $g = 0$: (rational curves) $M_0 = \{[\mathbb{P}^1]\}$;

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

- $\{\text{Hyperplanes of } \mathbb{P}^n\} \cong \mathbb{P}^n$;
- $M_g := \{ \text{ Smooth projective curves of given genus } g \ \}_{/\cong}$
  - $g = 0$: (rational curves) $M_0 = \{[\mathbb{P}^1]\}$;
  - $g = 1$: (elliptic curves) $M_1 \cong \mathbb{A}^1$;

Moduli problems in algebraic geometry

Many problems in algebraic geometry are concerned with **classifying** certain types of varieties: **moduli problems**.
Often the set of parameters for objects of certain geometric type (**moduli space**) is **again** itself an **algebraic** variety (scheme, stack).

- $\{$Hyperplanes of $\mathbb{P}^n\} \cong \mathbb{P}^n$;
- $M_g := \{$ Smooth projective curves of given genus g $\}_{/\cong}$
    - $g = 0$: (rational curves) $M_0 = \{[\mathbb{P}^1]\}$;
    - $g = 1$: (elliptic curves) $M_1 \cong \mathbb{A}^1$;
    - $g \geq 2$:

Theorem (Riemann'1857)

The space of non-isomorphic complex structures definable over a compact, connected, topological surface of genus $g \geq 2$ has complex dimension $3g - 3$.

Compactifying the moduli space of curves

M_g is **not compact**!
Families of curves over non complete bases may degenerate to non-smooth curves.

Compactifying the moduli space of curves

M_g is **not compact**!
Families of curves over non complete bases may degenerate to non-smooth curves.
In 1969 Deligne and Mumford solved this problem by building a remarkable compactification of M_g, denoted by $\overline{M}_g$.

Compactifying the moduli space of curves

M_g is **not compact**!
Families of curves over non complete bases may degenerate to non-smooth curves.
In 1969 Deligne and Mumford solved this problem by building a remarkable compactification of M_g, denoted by $\overline{M}_g$.
It has the important property of being **modular**, i.e., its points parametrize again geometric objects of certain type.

$\{\overline{M}_g\} \leftrightarrow \{$ Stable curves of genus $g\}$.

Compactifying the moduli space of curves

M_g is **not compact**!
Families of curves over non complete bases may degenerate to non-smooth curves.
In 1969 Deligne and Mumford solved this problem by building a remarkable compactification of M_g, denoted by $\overline{M}_g$.
It has the important property of being **modular**, i.e., its points parametrize again geometric objects of certain type.

$\{\overline{M}_g\} \leftrightarrow \{$ Stable curves of genus $g\}$.

Definition

A stable curve X is a projective connected nodal curve such that $\forall E \subseteq X$ such that $E \cong \mathbb{P}^1$, $\sharp\{E \cap \overline{X \setminus E}\} \geq 3$.

Moduli space of curves with marked points

It is often useful to work with moduli spaces parametrizing pairs of curves together with a set of marked points on it.

$M_{g,n} = \{(X; p_1, \ldots, p_n), p_i \in X \text{ distinct points}\}$

Moduli space of curves with marked points

It is often useful to work with moduli spaces parametrizing pairs of curves together with a set of marked points on it.

$M_{g,n} = \{(X; p_1, \ldots, p_n), p_i \in X \text{ distinct points}\}$

The existence of this kind of moduli space has important consequences for instance in **enumerative geometry**.
It also allowed the development of **Gromov-Witten** theory leading to surprising connections between algebraic geometry and mathematical physics (string theory.)

Moduli space of curves with marked points: compactified

Again, $M_{g,n}$ is **not compact**.

Moduli space of curves with marked points: compactified

Again, $M_{g,n}$ is **not compact**.
A (modular) compactification of $M_{g,n}$ was constructed by Knudsen in the 80's:
$\overline{M}_{g,n} = \{n$-**pointed stable curves** of genus $g.\}$

Moduli space of curves with marked points: compactified

Again, $M_{g,n}$ is **not compact**.
A (modular) compactification of $M_{g,n}$ was constructed by Knudsen in the 80's:
$\overline{M}_{g,n} = \{n$-**pointed stable curves** of genus $g.\}$

Definition

An n-pointed stable curve is a projective connected nodal curve of genus g, X, together with n-disctinct smooth points $p_1, \ldots, p_n$ of X such that
$\forall E \subseteq X$ with $E \cong \mathbb{P}^1$,

$$\sharp\{\text{marked points on } E\} + \sharp\{E \cap \overline{X \setminus E}\} \geq 3.$$

Picard varieties of smooth curves

Picard varieties of smooth curves

C smooth curve

$\mathrm{Pic}(C) := \{\text{line bundles on } C\}_{/\cong}$

Picard varieties of smooth curves

C smooth curve

$\mathrm{Pic}(C) := \{\text{line bundles on } C\}_{/\cong}$

Since C is a curve we can write
$\mathrm{Pic}^d(C) = \{\sum a_i p_i, p_i \in C, a_i \in \mathbb{Z}$ and $\sum a_i = d\}_{/\sim}$.

Picard varieties of smooth curves

C smooth curve

$\mathrm{Pic}(C) := \{\text{line bundles on } C\}_{/\cong}$

Since C is a curve we can write
$\mathrm{Pic}^d(C) = \{\sum a_i p_i, p_i \in C, a_i \in \mathbb{Z} \text{ and } \sum a_i = d\}_{/\sim}$.
$\mathrm{Pic}(C) = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d(C)$.

- $\mathrm{Pic}^d(C)$ is a projective variety;
- For $d = 0$, $\mathrm{Pic}^0(C)$ is endowed with a natural group structure given by tensor products of line bundles.

Picard varieties of smooth curves

C smooth curve

$\mathrm{Pic}(C) := \{\text{line bundles on } C\}_{/\cong}$

Since C is a curve we can write
$\mathrm{Pic}^d(C) = \{\sum a_i p_i, p_i \in C, a_i \in \mathbb{Z} \text{ and } \sum a_i = d\}_{/\sim}$.
$\mathrm{Pic}(C) = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d(C)$.

- $\mathrm{Pic}^d(C)$ is a projective variety;
- For $d = 0$, $\mathrm{Pic}^0(C)$ is endowed with a natural group structure given by tensor products of line bundles.

$\mathrm{Pic}^0(C)$ is an abelian variety of dimension g.

Picard varieties of singular curves

Let X be a nodal, possibly reducible, curve. Then, in general, the Picard variety of X, $\operatorname{Pic} X$, is not compact.

Picard varieties of singular curves

Let X be a nodal, possibly reducible, curve. Then, in general, the Picard variety of X, $\mathrm{Pic}\, X$, is not compact.
The problem of compactifying the Picard variety of singular curves has been widely studied in the last decades. It goes back to the work of Igusa and Mayer and Mumford on the 50's and since then several solutions were found.

- For a singular curve: Igusa '56, D'Souza '79 (irreducible), Oda-Seshadri '79 (reducible);
- for families of curves: Altman-Kleimann '80, Simpson '79, Esteves '97;
- over M_g: Caporaso '93, Pandharipande '94, Schmitt '04.

- For a singular curve: Igusa '56, D'Souza '79 (irreducible), Oda-Seshadri '79 (reducible);
- for families of curves: Altman-Kleimann '80, Simpson '79, Esteves '97;
- over M_g: Caporaso '93, Pandharipande '94, Schmitt '04.

These constructions differ from one another in aspects like the modular description of the boundary points or the functorial properties.

- For a singular curve: Igusa '56, D'Souza '79 (irreducible), Oda-Seshadri '79 (reducible);
- for families of curves: Altman-Kleimann '80, Simpson '79, Esteves '97;
- over M_g: Caporaso '93, Pandharipande '94, Schmitt '04.

These constructions differ from one another in aspects like the modular description of the boundary points or the functorial properties. Given a nodal curve X, complete $\operatorname{Pic} X$ by either:

allowing more general sheaves than line bundles

allowing "some" semistable curves and line bundles over these.

- For a singular curve: Igusa '56, D'Souza '79 (irreducible), Oda-Seshadri '79 (reducible);
- for families of curves: Altman-Kleimann '80, Simpson '79, Esteves '97;
- over M_g: Caporaso '93, Pandharipande '94, Schmitt '04.

These constructions differ from one another in aspects like the modular description of the boundary points or the functorial properties. Given a nodal curve X, complete $\operatorname{Pic} X$ by either:

allowing more general sheaves than line bundles

allowing "some" semistable curves and line bundles over these.

Study and compare the different approaches!

The object we want to compactify

The object we want to compactify

$\mathcal{P}ic_{d,g,n}$: **universal Picard stack** over $\mathcal{M}_{g,n}$.

The object we want to compactify

$\mathcal{P}ic_{d,g,n}$: **universal Picard stack** over $\mathcal{M}_{g,n}$.

parametrizing triples $(C; p_1, \ldots, p_n; L)$
where

The object we want to compactify

$\mathcal{P}ic_{d,g,n}$: **universal Picard stack** over $\mathcal{M}_{g,n}$.

parametrizing triples $(C; p_1, \ldots, p_n; L)$
where

- C is a smooth curve of genus g;

The object we want to compactify

$\mathcal{P}ic_{d,g,n}$: **universal Picard stack** over $\mathcal{M}_{g,n}$.

parametrizing triples $(C; p_1, \ldots, p_n; L)$
where

- C is a smooth curve of genus g;
- $p_1, \ldots, p_n$ are n distinct points of C;

The object we want to compactify

$\mathcal{P}ic_{d,g,n}$: **universal Picard stack** over $\mathcal{M}_{g,n}$.

parametrizing triples $(C; p_1, \ldots, p_n; L)$
where

- C is a smooth curve of genus g;
- $p_1, \ldots, p_n$ are n distinct points of C;
- L is a line bundle of degree d over C.

The object we want to compactify

$\mathcal{P}ic_{d,g,n}$: **universal Picard stack** over $\mathcal{M}_{g,n}$.

parametrizing triples $(C; p_1, \ldots, p_n; L)$
where

- C is a smooth curve of genus g;
- $p_1, \ldots, p_n$ are n distinct points of C;
- L is a line bundle of degree d over C.

$\mathcal{P}ic_{d,g,n}$ has a natural map onto $\mathcal{M}_{g,n}$ and it is not complete.

Our Problem

Compactify $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$!

Our Problem

Compactify $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$!

Construct an algebraic stack $\overline{\mathcal{P}}_{d,g,n}$ and a morphism $\Psi_{d,g,n} : \overline{\mathcal{P}}_{d,g,n} \to \overline{\mathcal{M}}_{g,n}$ such that:

Our Problem

Compactify $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$!

Construct an algebraic stack $\overline{\mathcal{P}}_{d,g,n}$ and a morphism $\Psi_{d,g,n} : \overline{\mathcal{P}}_{d,g,n} \to \overline{\mathcal{M}}_{g,n}$ such that:

1. $\overline{\mathcal{P}}_{d,g,n}$ and $\Psi_{d,g,n}$ fit in the diagram;

$$\begin{tikzcd} \mathcal{P}ic_{d,g,n} \arrow[r, hook] \arrow[d] & \overline{\mathcal{P}}_{d,g,n} \arrow[d, "\Psi_{d,g,n}"] \\ \mathcal{M}_{g,n} \arrow[r, hook] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{1}$$

Our Problem

Compactify $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$!

Construct an algebraic stack $\overline{\mathcal{P}}_{d,g,n}$ and a morphism $\Psi_{d,g,n} : \overline{\mathcal{P}}_{d,g,n} \to \overline{\mathcal{M}}_{g,n}$ such that:

1. $\overline{\mathcal{P}}_{d,g,n}$ and $\Psi_{d,g,n}$ fit in the diagram;

$$\begin{tikzcd} \mathcal{P}ic_{d,g,n} \arrow[r, hook] \arrow[d] & \overline{\mathcal{P}}_{d,g,n} \arrow[d, "\Psi_{d,g,n}"] \\ \mathcal{M}_{g,n} \arrow[r, hook] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{1}$$

2. fibers of $\Psi_{d,g,n}$ are compact;

Our Problem

Compactify $\mathcal{P}ic_{d,g,n}$ over $\overline{\mathcal{M}}_{g,n}$!

Construct an algebraic stack $\overline{\mathcal{P}}_{d,g,n}$ and a morphism $\Psi_{d,g,n} : \overline{\mathcal{P}}_{d,g,n} \to \overline{\mathcal{M}}_{g,n}$ such that:

1. $\overline{\mathcal{P}}_{d,g,n}$ and $\Psi_{d,g,n}$ fit in the diagram;

$$\begin{tikzcd} \mathcal{P}ic_{d,g,n} \arrow[r, hook] \arrow[d] & \overline{\mathcal{P}}_{d,g,n} \arrow[d, "\Psi_{d,g,n}"] \\ \mathcal{M}_{g,n} \arrow[r, hook] & \overline{\mathcal{M}}_{g,n} \end{tikzcd} \tag{1}$$

2. fibers of $\Psi_{d,g,n}$ are compact;
3. $\overline{\mathcal{P}}_{d,g,n}$ has a geometrically meaningful modular description.

Our strategy

Our strategy

Use **induction** in the number of marked **points** n.

Our strategy

Use **induction** in the number of marked **points** n.

- $n = 0$: give a **stack theoretical modular interpretation** of Caporaso's compactification;

Our strategy

Use **induction** in the number of marked **points** n.

- $n = 0$: give a **stack theoretical modular interpretation** of Caporaso's compactification;
- $n > 0$: Proceed as Knudsen did in the construction of $\overline{\mathcal{M}}_{g,n}$: give a **modular description of the universal family** $\mathcal{Z}_{d,g,n}$ over $\overline{\mathcal{P}}_{d,g,n}$ and construct $\Psi_{d,g,n}$.

$$\begin{tikzcd}
\mathcal{Z}_{d,g,n} \arrow[r, "c"] \arrow[d] & \overline{\mathcal{P}}_{d,g,n+1} \arrow[d, "\Psi_{d,g,n+1}"] \\
\overline{\mathcal{P}}_{d,g,n} \arrow[d, "\Psi_{d,g,n}"'] & \overline{\mathcal{M}}_{g,n+1} \arrow[dl] \\
\overline{\mathcal{M}}_{g,n} &
\end{tikzcd}$$

Main Theorem

Main Theorem

There exists a smooth and irreducible algebraic (Artin) stack $\overline{\mathcal{P}}_{d,g,n}$ of dimension $4g-3+n$ endowed with a universally closed map $\Psi_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$.

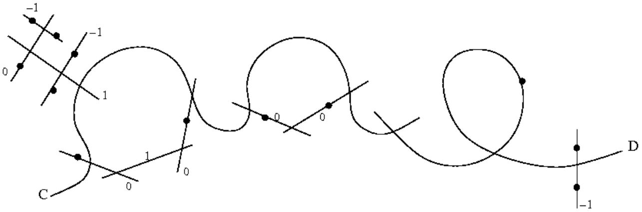
Main Theorem

There exists a smooth and irreducible algebraic (Artin) stack $\overline{\mathcal{P}}_{d,g,n}$ of dimension $4g-3+n$ endowed with a universally closed map $\Psi_{d,g,n}$ onto $\overline{\mathcal{M}}_{g,n}$.
$\overline{\mathcal{P}}_{d,g,n}$ parametrizes triples $\{(X; p_1, \ldots, p_n; L)\}$ where:

- $(X; p_1 \ldots, p_n)$ is an "**n-pointed quasistable**" curve of genus g;
- L "**balanced**" line bundle of degree d.

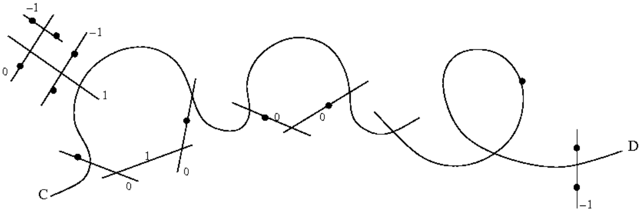
Example

Example of a 12-pointed quasistable curve X with assigned balanced multidegree in rational tails and rational bridges.



Example

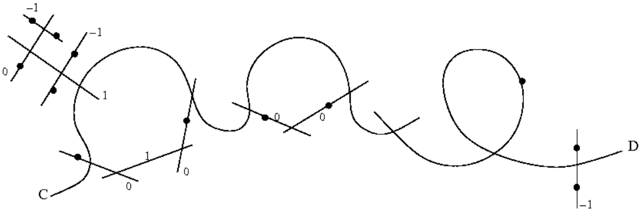
Example of a 12-pointed quasistable curve X with assigned balanced multidegree in rational tails and rational bridges.



Let L be a balanced line bundle on X.

Example

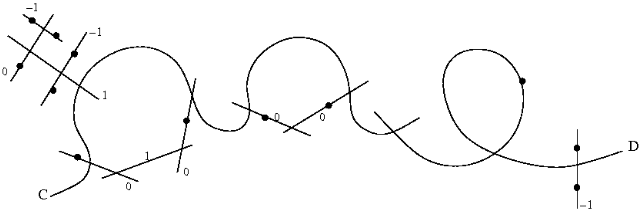
Example of a 12-pointed quasistable curve X with assigned balanced multidegree in rational tails and rational bridges.



Let L be a balanced line bundle on X.
If $\deg(L) = 0$, then $(\deg_C L, \deg_D L) = (0, 1)$;

Example

Example of a 12-pointed quasistable curve X with assigned balanced multidegree in rational tails and rational bridges.



Let L be a balanced line bundle on X.
If $\deg(L) = 0$, then $(\deg_C L, \deg_D L) = (0, 1)$;
If $\deg(L) = g - 1$, then
$(\deg_C L, \deg_D L) = (g_C + 2, g_D)$ or $(g_C + 1, g_D + 1)$.

Plans for future work

- Generalize the construction for higher rank bundles and ‘allowing the polarizations to vary”;

Plans for future work

- Generalize the construction for higher rank bundles and ‘allowing the polarizations to vary”;
- Study intersection theory on $\overline{\mathcal{P}}_{d,g,n}$ and its applications to enumerative geometry (e.g. Hurwitz numbers);

Thank you!