Dynamical Systems : Chaos and Equilibrium

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Motivations:

- Modeling physical phenomena
- Describe future asymptotic behavior

Given a map f and x consider the *orbit of x* as the set

$$\mathcal{O}^+(x) = \{x, f(x), f^2(x), f^3(x), \dots\}$$

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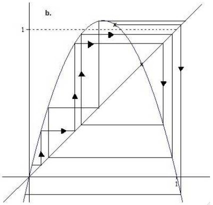
Example 1:

$f(x) = ax(1-x)$ describes population size in a time instant



Example 2:

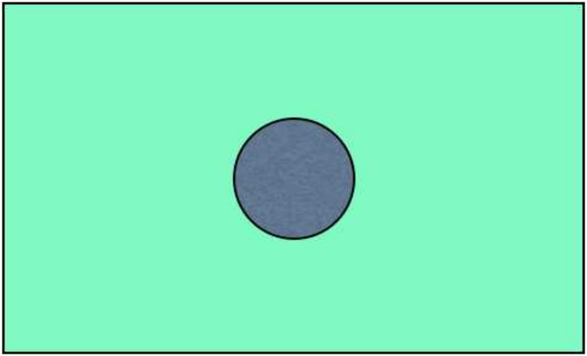
The position of a person in a giant wheel at constant velocity can be modelled by

$$\begin{array}{rccc} R_\alpha : & S^1 & \longrightarrow & S^1 \\ & e^{2\pi i\theta} & \longmapsto & e^{2\pi i(\theta+\alpha)} \end{array}$$

where $S^1 = \{z \in \mathbb{C} : z = e^{2\pi i\theta} \text{ with } \theta \in \mathbb{R}\}$.
Position of a person at time n seconds $= R_\alpha^n(x) = e^{2\pi i(\theta+\alpha n)}$

Order! Not chaos!

Example 3: Description of the position of a ball in a billiard table without holes



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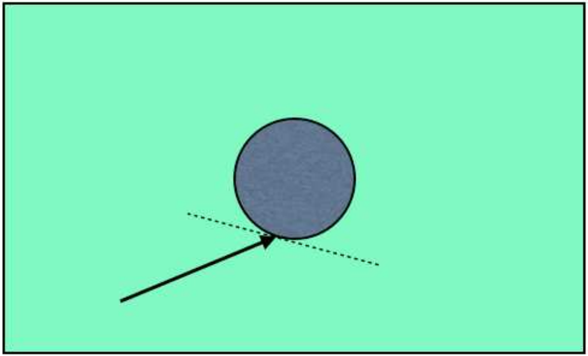

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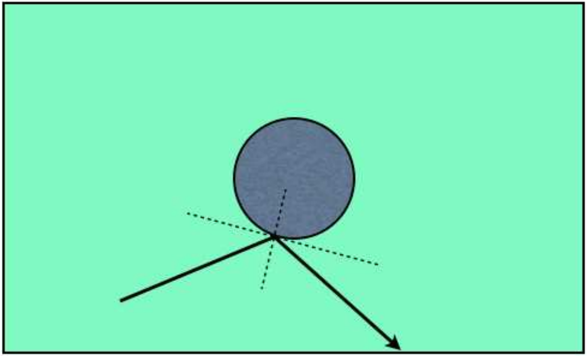

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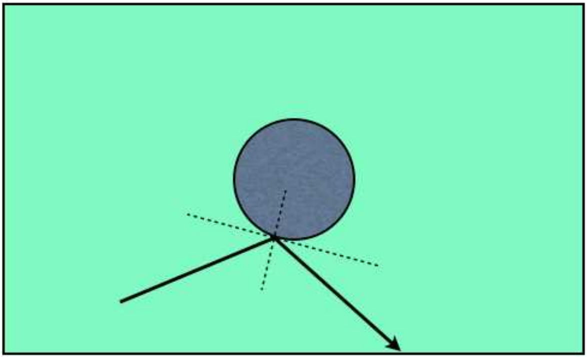

What happens if one consider different trajectories?

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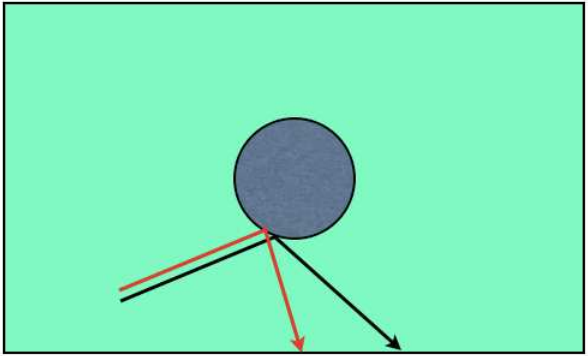

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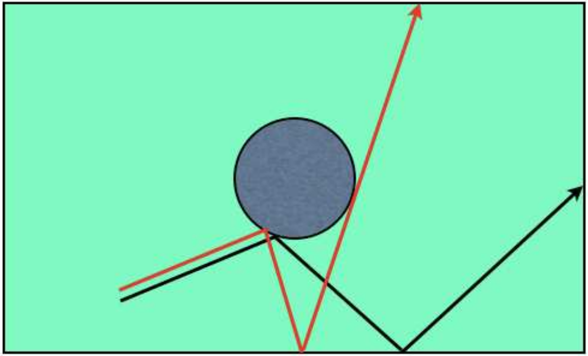

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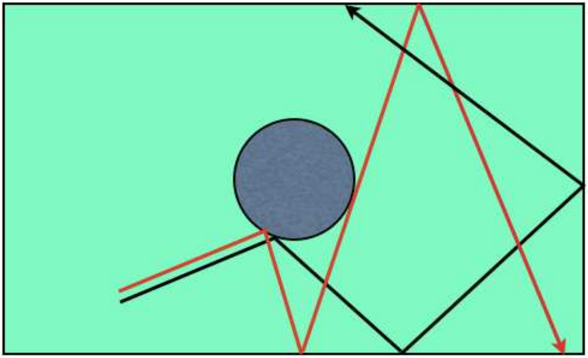


Sensitivity to initial conditions!

Example 4: Lorenz equations for weather forecast
$a = 10, r = 28, b = 8/3$

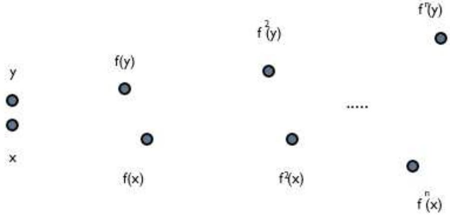
$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

Sensitivity to initial conditions!

Chaotic dynamical systems

Sensitivity to initial conditions (topological sense)

$\exists \varepsilon > 0$ so that if $x \neq y$ there exists $n \in \mathbb{N}$ s.t. $d(f^n(x), f^n(y)) > \varepsilon$



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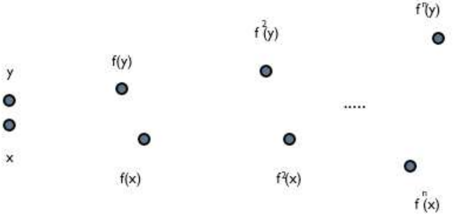
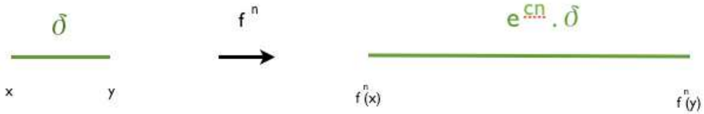


Positive Lyapunov exponents (measure theoretical sense)

$$\lambda(f,x) := \lim_{n\to+\infty} \frac{1}{n} \log |(f^n)'(x)| \quad \text{(Lyapunov exponent at } x \text{ if exists)}$$



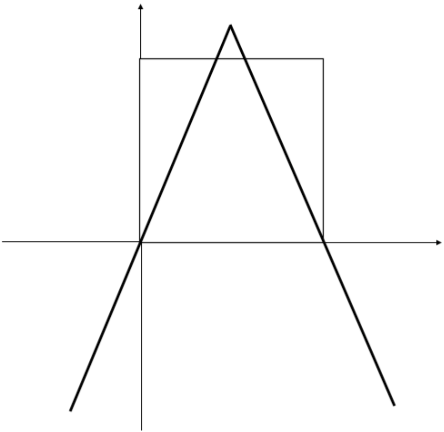
Order in chaotic dynamical systems

Going back to the original questions:

Question 1: Do chaotic dynamical systems have some "good" topological/geometrical structure?

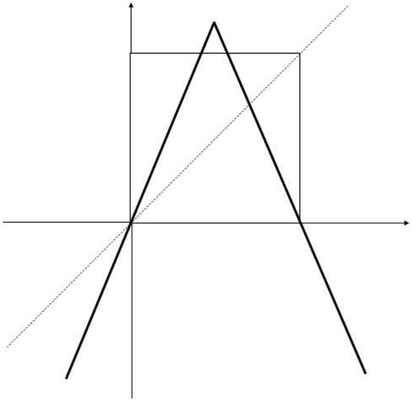
Question 2: Do chaotic dynamical systems have invariant probability distributions? Among those do equilibrium states exist?

Very simple model



Since $|f'(x)| > 1$ **at "every" point then** f **is chaotic in both senses** (exercise!)

Very simple model



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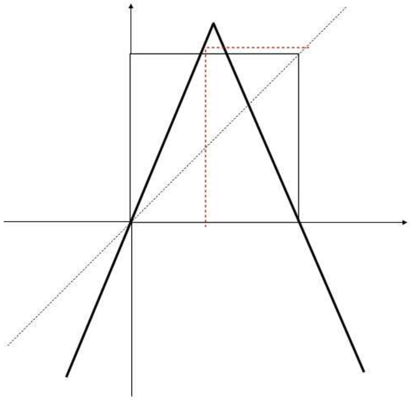

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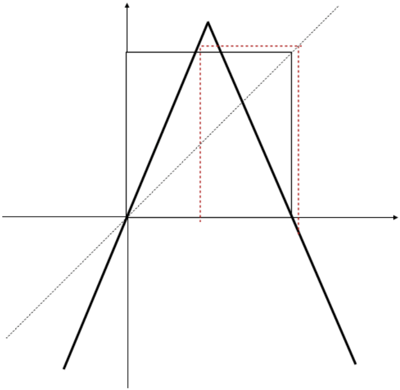

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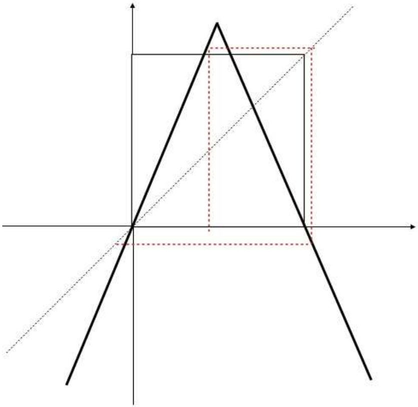

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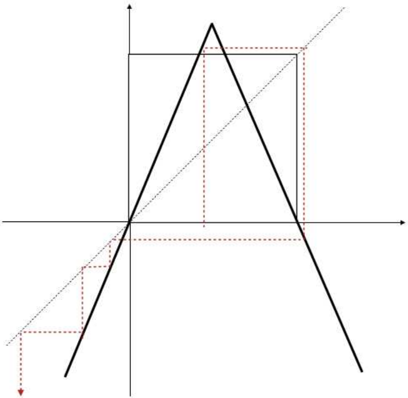

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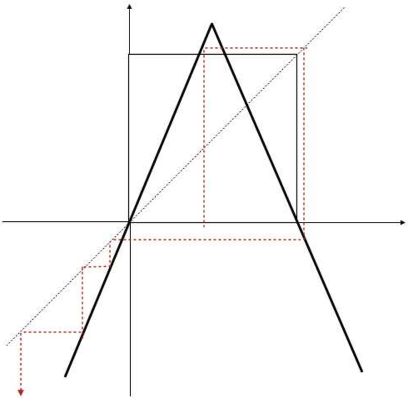
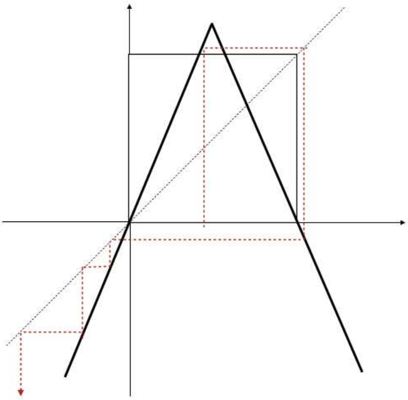


$x \in [\frac{1}{3}, \frac{2}{3}] \Rightarrow f^n(x) \to -\infty$ as n tends to infinity.

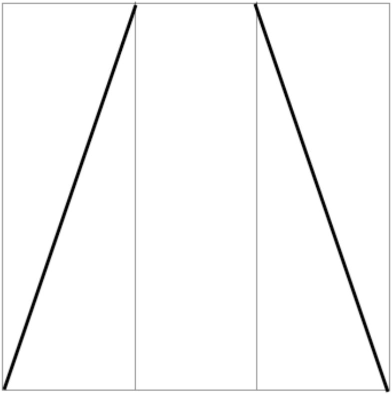
Very simple model



In fact: $x \notin [0, \frac{1}{3}] \cup [\frac{1}{3}, \frac{2}{3}] \Rightarrow f^n(x) \to -\infty$ as n tends to infinity

Very simple model

Study of $f \mid ([0, \frac{1}{3}] \cup [\frac{2}{3}, 1])$



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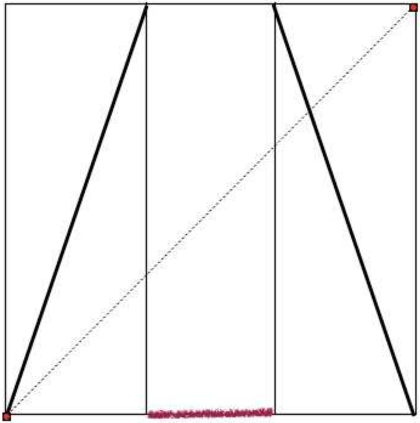
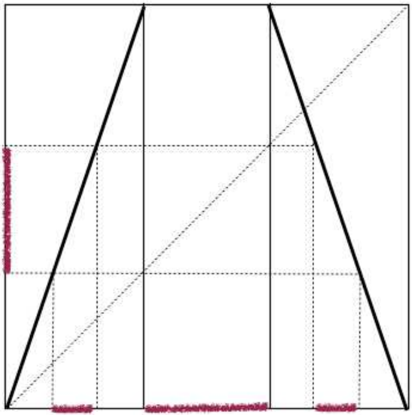


"Hole"

Very simple model



Very simple model

So. the points that do not fall in the “hole” form ...

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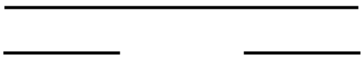

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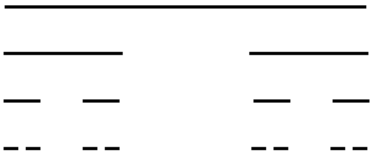

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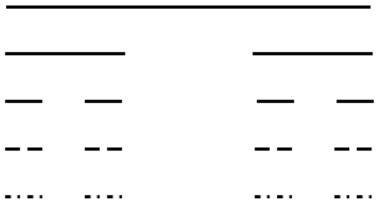

.... a Cantor set.

Simple Model

Answer 1: YES, this chaotic dynamical system has a good geometrical invariant set, a *fractal set* as relevant part of the dynamics! (self-similar Cantor set)

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Answer 2: YES, there exists a unique *equilibrium state* for smooth potentials! (selection of invariant probability measures)

$\downarrow$

Inspired by statistical mechanics

Crash course in measure theory:

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- Probability measure

$$\begin{array}{rcl} \mu : \mathcal{A} \subset \mathcal{P}(X) & \rightarrow & [0,1] \\ A & \mapsto & \mu(A) \end{array}$$

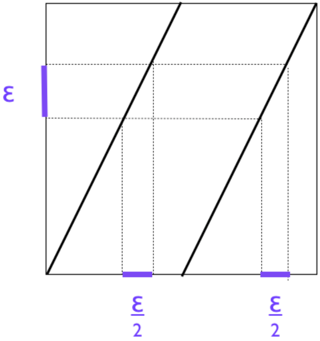
Example:

Crash course in measure theory:

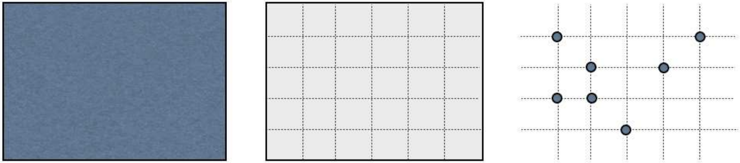
- Probability measure
- Invariant probability measure

$$\mu(f^{-1}(A)) = \mu(A) \quad \text{for all } A \in \mathcal{A}$$

Example: $f(x) = 2x (\mod 1)$, $\mu =$ length



Motivation from statistical mechanics:



External action of a potential ϕ

Discretization $\rightsquigarrow$ simplification

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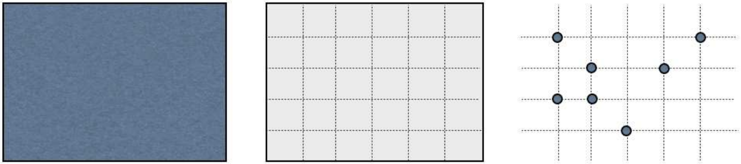


External action of a potential ϕ

Discretization $\rightsquigarrow$ simplification

1 dim lattice

(0,0,1,1,1,0,0,0,1,1,0,1,1,1,....)

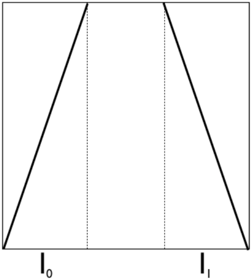
Configurations = elements in $\{0,1\}^{\mathbb{N}}$

Question: Is the gas "distributed" according to some probability?

Reduction to symbolic space

K Cantor set

$x \in K \rightsquigarrow \iota(x) = (a_0, a_1, a_2, a_3, a_4, \dots) \in \{0,1\}^{\mathbb{N}}$ itinerary



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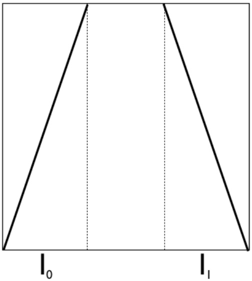

$$a_i = 0 \leftrightarrow f^j(x) \in I_0$$

Moreover,

$$\iota(f(x)) = (a_1, a_2, a_3, a_4, \dots)$$

Shift: $\sigma(a_0, a_1, a_2, a_3, a_4, \dots) = (a_1, a_2, a_3, a_4, \dots)$

Equilibrium probability distribution for shifts

Probability measures on $\{0, 1\}$: $(p, 1 - p)$

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These are σ-invariant probability measures (easy exercise!)

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Classical result: Given a regular potential ϕ is there a unique probability ν_p maximizing the pressure

$$P(\phi) = \sup_{\substack{(p_0,p_1) \\ p_0+p_1=1}} \Bigg\{ \underbrace{\sum_i -p_i \log p_i}_{\text{entropy}} + \underbrace{\frac{1}{\beta} \sum_i p_i \phi_i}_{\text{potential energy}} \Bigg\}$$

Reformulation of Question 2

$f : K \to K$ dynamical system on the Cantor set K
$\phi : M \to R$ (regular) potential
Set the topological pressure of f with respect to ϕ as

$$P(f, \phi) = \sup_{\mu} \left\{ h_\mu(f) + \int \phi \, d\mu \right\}$$

Question 2: Does there exist (unique) invariant probabilities that realize the topological complexity of the dynamics?

Some recent results

- (Pinheiro, V. 2010) The description of invariant measures in the chaotic region of a dynamical system is equivalent to the one for a shift $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ with S countable; moreover equilibrium states do exist

- (V., Viana 2009) There are many examples where the chaotic region is more relevant for computing the topological pressure

Further questions and open problems

A LOT!!

Many thanks to Fundação Calouste Gulbenkian for the excellent program!

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Hope to attend:

New and "not so new" Talents in Mathematics 2020