Algebraic structures parametrised by manifolds

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Axioms for a monoid

A *monoid* is a group without inverses.
It has only a unit and an associative multiplication.

In detail, a monoid is given by

- a set A
- a unit $e \in A$
- a multiplication map $\begin{aligned} A \times A &\longrightarrow A \\ (a, b) &\longmapsto a \cdot b \end{aligned}$

which verify

- associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- the unit axiom: $a \cdot e = a = e \cdot a$

Examples of monoids

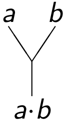
Examples of monoids:

- any group is also a monoid
- $\mathbb{N}$ equipped with addition (unit is 0)
- any ring (e.g. $\mathbb{Z}$, $\mathbb{Q}$) with its multiplication (unit is 1)

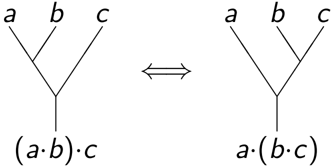
Visual representation of a monoid

Unit:

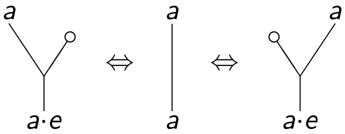
Multiplication:



Associativity:

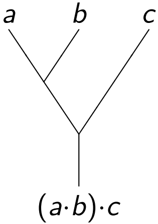


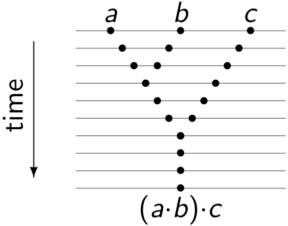
Unit axiom:



Particles in $\mathbb{R}$ (merging)

The pictures represent movies of particles moving on a line:

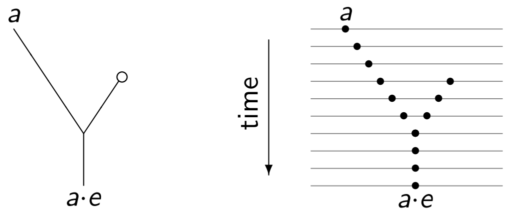




The particles are *solid* and *sticky*.
If they bump each other then they stick together.

Particles in $\mathbb{R}$ (creation)

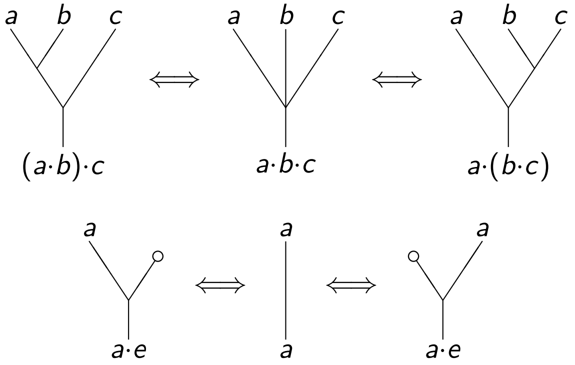
Particles can also appear out of nothing.



However, particles cannot disappear once created.

Axioms and deformation

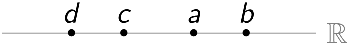
The unit and associativity axioms for a monoid correspond to deformation of movies:



Configurations of particles

An instantaneous time slice of a movie — suitably labeled by elements of a monoid A — is a *labeled configuration* of particles.

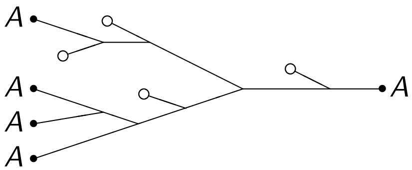
E.g.



The space of all such configurations is called $C^A(\mathbb{R})$. Movies of particles are *directed* paths in that space. What does $C^A(\mathbb{R})$ look like?

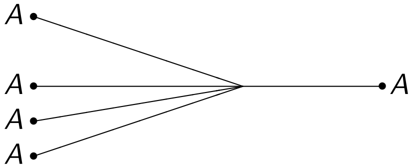
Convergence

Given any configuration in $\mathbb{R}$, we can merge all the particles:



This movie flows from several copies of A to one single copy of A.

Furthermore, the movie can always be deformed to look like



Convergence (conclusion)

This means that the configurations labeled in A, when flowing along all possible movies, *converge* to a single copy of A.

In other words, the space of labeled configurations can be deformed into A:

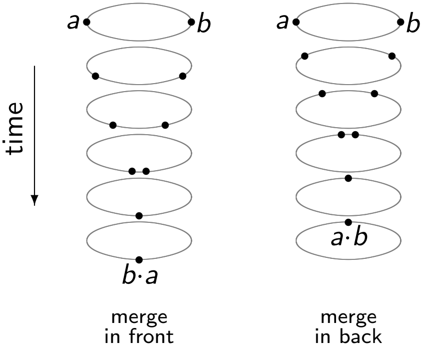
$$C^A(\mathbb{R}) \simeq A$$

That is rather boring.
However, $\mathbb{R}$ is homotopically boring to begin with...

Other manifolds: circle (S^1)

The circle, S^1, looks locally like $\mathbb{R}$.
Therefore, we can also take configurations in S^1 (labeled in A).

Movies on S^1:



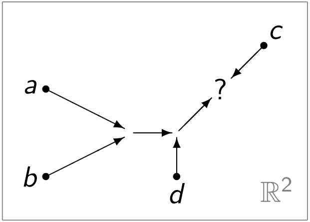
S^1 and Hochschild homology

The space of labeled configurations on S^1, $C^A(S^1)$, is very interesting.
It is named the topological *Hochschild homology* of A.

Hochschild homology is closely related to algebraic K-theory, so it carries a lot of useful information about rings (e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{F}_p$).
In a nutshell, K-theory tells us about the homotopical properties of GL_n for large n.

Other manifolds: higher dimensions

Manifolds of dimension n locally look like $\mathbb{R}^n$.
Do these parametrise any algebraic structure?



Yes! But these are not monoids like before.
On $\mathbb{R}$ there is only one way to multiply (one direction).
On $\mathbb{R}^n$ there are n ways to multiply (n directions).
So we'll call them *n-monoids*.

Higher Hochschild homology

Let A be a n-monoid. Like before, we can see that the space of configurations on $\mathbb{R}^n$ is not very interesting:

$$C^A(\mathbb{R}^n) \simeq A$$

But we can consider other manifolds, M, of dimension n. $C^A(M)$ is a higher dimensional generalization of Hochschild homology. It relates to:

- algebraic K-theory
- quantum field theories
- embedding spaces of manifolds
- non-abelian Poincaré duality