Behavioral algebraization of logics

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Outline

1. Motivation
2. Algebraization
   - Algebraization
   - AAL
   - Limitations
3. Generalizing to many-sorted logics
   - Many-sorted signatures
   - Behavioral algebraization
   - Behavioral AAL
4. Conclusions and further work

Motivation

- Theory of algebraization of logics

  [Lindenbaum & Tarski, Blok & Pigozzi, Czelakowski, Nemeti et al]

  - Representation of logics in algebraic setting (theory restricted to propositional based logics)
  - Study the process by which a class of algebras is associated with a logic

- The scope of applicability is limited

- Generalization of the notion of algebraizable logic

  - Behavioral equivalence

Algebraization

$\mathcal{L}$ Logic over a signature Σ	Strong representation $\Longleftrightarrow$	**Equational logic** over a class K of Σ-algebras

Logic

Definition

A *structural propositional logic* is a pair $\mathcal{L} = \langle \Sigma, \vdash \rangle$, where Σ is a propositional signature and $\vdash \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$ is a *consequence relation* satisfying the following conditions, for every $T_1 \cup T_2 \cup \{\varphi\} \subseteq L_\Sigma(X)$:

Reflexivity: if $\varphi \in T_1$ then $T_1 \vdash \varphi$

Cut: if $T_1 \vdash \varphi$ for all $\varphi \in T_2$, and $T_2 \vdash \psi$ then $T_1 \vdash \psi$

Weakening: if $T_1 \vdash \varphi$ and $T_1 \subseteq T_2$ then $T_2 \vdash \varphi$

Structurality: if $T_1 \vdash \varphi$ then $\sigma[T_1] \vdash \sigma(\varphi)$

Equational logic

Let K be a class of Σ-algebras.

Definition (Equational logic Eqn_K^Σ)

$$\{t_i \approx t'_i : i \in I\} \vDash_{Eqn_K^\Sigma} t \approx t'$$

iff

for every $A \in K$ *and homomorphism* $h : T_\Sigma(X) \to A$,
$h(t) = h(t')$ *whenever* $h(t_i) = h(t'_i)$ *for every* $i \in I$

Algebraizable logic

$\mathcal{L}$ is algebraizable if there are translations θ and Δ, and a class of algebras K s.t.

$$\mathcal{L} \underset{\Delta(x_1,x_2)}{\overset{\Theta(x)}{\rightleftarrows}} Eqn_K^\Sigma$$

$T \vdash \varphi$ iff $\Theta[T] \vDash_{Eqn_K^\Sigma} \Theta(\varphi)$

$\{\Delta(\delta_i, \epsilon_i) : i \in I\} \vdash \Delta(\varphi_1, \varphi_2)$ iff $\{\delta_i \approx \epsilon_i : i \in I\} \vDash_{Eqn_K^\Sigma} \varphi_1 \approx \varphi_2$

$\varphi \dashv\vdash \Delta[\Theta(\varphi)]$ $\qquad$ $\varphi_1 \approx \varphi_2 =\!\mid\vDash_{Eqn_K^\Sigma} \Theta[\Delta(\varphi_1, \varphi_2)]$

Examples

Example (Classical logic)

CPL $\leftrightarrows$ Boolean algebras

Example (Intuitionistic logic)

IPL $\leftrightarrows$ Heyting algebras

In both cases
$\theta = \{x \approx \top\}$ and $\Delta = \{p \leftrightarrow q\}$

Bridge theorems

$\mathcal{L}$ algebraizable and K its equivalent algebraic semantics.

$\mathcal{L}$ **has**	**K has**
the local deduction theorem	congruence extension property
Craig's interpolation theorem	amalgamation property
the deduction theorem	EDPC

Leibniz operator - The main tool

Definition (Leibniz operator)

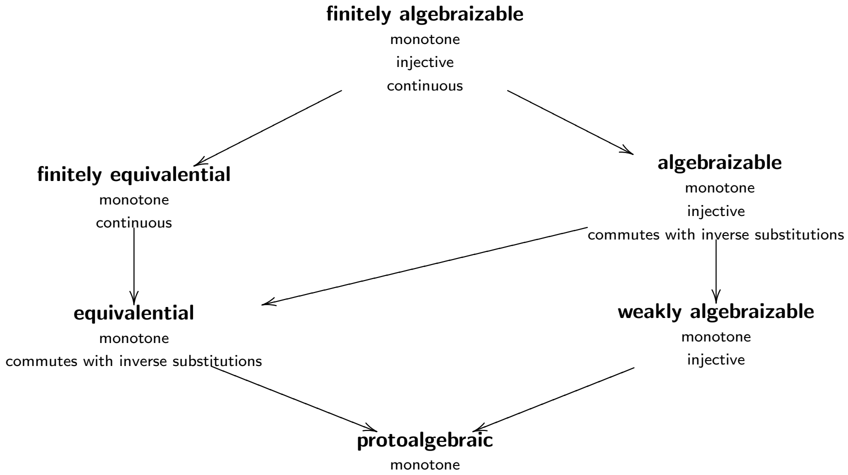
Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a structural propositional logic. Then the *Leibniz operator* on the formula algebra can be given by:

$$\Omega : Th_{\mathcal{L}} \longrightarrow \text{Congr}_{\mathbf{L}_{\mathbf{\Sigma}}}$$

$$T \mapsto \text{ largest congruence compatible with } T.$$

θ is compatible with T if $\langle \varphi, \psi \rangle \in \theta$ implies $(\varphi \in T)$ iff $(\psi \in T)$

Characterization theorems



Limitations

- Theory restricted to propositional based logics
- Logics over many-sorted languages
  - In practice, propositional based logics are not enough for reasoning about complex systems
- Logics with non-truth-functional connectives

Many-sorted signatures

Definition (Many-sorted signature)

A *many-sorted signature* is a pair $\Sigma = \langle S, F\rangle$ where S is a set (of *sorts*) and $F = \{F_{ws}\}_{w\in S^*, s\in S}$ is an indexed family of sets (of operations).

Example (stacks of natural numbers)

$\Sigma_{Stack} = \langle S, F\rangle$ such that

- $S = \{nat, stack\}$
- $F_{nat} = \{0\}$
- $F_{nat\ nat} = \{s\}$
- $F_{stack} = \{Empty\}$
- $F_{stack\ nat} = \{Top\}$
- $F_{stack\ stack} = \{Pop\}$
- $F_{stack\ nat\ stack} = \{Push\}$

Many-sorted signatures

Example (FOL)

$\Sigma_{FOL} = \langle S, F \rangle$ such that

- $S = \{\phi, t\}$
- $F_{\phi\phi} = \{\forall_x : x \in X\} \cup \{\neg\}$
- $F_{\phi\phi\phi} = \{\Rightarrow, \wedge, \vee\}$
- $F_{t^n\phi} = \{P : \ P \ \ n\text{-ary predicate symbol}\}$
- $F_{t^n t} = \{f : \ f \ \ n\text{-ary function symbol}\}$

Many-sorted behavioral logic

The motivation for the term behavioral:
algebraic approach to the specification and verification of object oriented systems.

Its distinctive feature is that sorts are split between visible and hidden, the visible sorts being for the outputs, while the hidden sorts are for objects.

Hidden data can only be indirectly compared through the visible data using experiments.

Two values are behaviorally equivalent if they cannot be distinguished by the set of available experiments.

The set of experiments may not coincide with the set of all contexts.

Behavioral equivalence: motivation

Example (Stacks revisited)

In the signature Σ_{Stack} the only visible sort is nat.

Given a Σ_{Stack}-algebra A, two hidden elements, $a, b \in A_{stack}$ are behaviorally equivalent if for every $n \in \mathbb{N}$

$$top(pop^n(a)) = top(pop^n(b))$$

An operation is congruent if it is compatible with behavioral equivalence.
Example: if $a \equiv b$ then $push(a, s(0)) \equiv push(b, s(0))$

In this example all operations are congruent

Restricted behavioral equivalence

Sometimes we need to restrict the admissible set of experiments.

Example (Non-deterministic stacks)

In a non-deterministic stack natural numbers are pushed non-deterministically $\Sigma_{NDStack} = \langle S, F \rangle$ such that

- $S = \{nat, stack\}$
- $F_{nat} = \{0\}$
- $F_{nat\ nat} = \{s\}$
- $F_{stack} = \{Empty\}$
- $F_{stack\ nat} = \{Top\}$
- $F_{stack\ stack} = \{Pop\}$
- $F_{stack\ stack} = \{Push\}$

In this case we do not want to consider experiments that contain the operation $Push$. This operation is considered a non-congruent operation.

How to generalize?

$\mathcal{L}$	Strong representation $\Longleftrightarrow$	**Equational logic**
unsorted		unsorted
	Strong representation $\Longleftrightarrow$	

How to generalize?

$\mathcal{L}$ unsorted	Strong representation $\Longleftrightarrow$	**Equational logic** unsorted
$\mathcal{L}$ many-sorted	Strong representation $\Longleftrightarrow$	

How to generalize?

$\mathcal{L}$ unsorted	Strong representation $\Longleftrightarrow$	**Equational logic** unsorted
$\mathcal{L}$ many-sorted	Strong representation $\Longleftrightarrow$	**Behavioral equational logic** many-sorted

Γ-Behavioral equivalence

Consider given a subsignature Γ of Σ.

Given a Σ-algebra A, two elements a_1, a_2 of a hidden sort are Γ-behavioral equivalent, $a_1 \equiv_\Gamma a_2$, if they can not be distinguished by a Γ-experiment.

Behavioral logic

Definition (Many-sorted behavioral equational logic $BEqn^{\Sigma}_{K,\Gamma}$)

$$\{t_i \approx t'_i : i \in I\} \vDash_{BEqn^{\Sigma}_{K,\Gamma}} t \approx t'$$

iff

for every $A \in K$ *and* h *homomorphism over* A

$h(t) \equiv_{\Gamma} h(t')$ *whenever* $h(t_i) \equiv_{\Gamma} h(t'_i)$ *for every* $i \in I$

Behavioral algebraization

$$\boldsymbol{\mathcal{L}} \underset{\Delta(x_1,x_2)}{\overset{\Theta(x)}{\rightleftarrows}} {\color{blue} BEqn^{\Sigma}_{K,\Gamma}}$$

$$T \vdash \varphi \quad \text{iff} \quad \Theta[T] \vDash_{\color{blue} BEqn^{\Sigma}_{K,\Gamma}} \Theta(\varphi)$$

$$\{\Delta(\delta_i, \epsilon_i) : i \in I\} \vdash \Delta(\varphi_1, \varphi_2) \quad \text{iff} \quad \{\delta_i \approx \epsilon_i : i \in I\} \vDash_{\color{blue} BEqn^{\Sigma}_{K,\Gamma}} \varphi_1 \approx \varphi_2$$

$$\varphi \dashv\vdash \Delta[\Theta(\varphi)] \qquad \varphi_1 \approx \varphi_2 \,=\!\mid\!\vDash_{\color{blue} BEqn^{\Sigma}_{K,\Gamma}} \Theta[\Delta(\varphi_1 \approx \varphi_2)]$$

Behavioral Leibniz operator

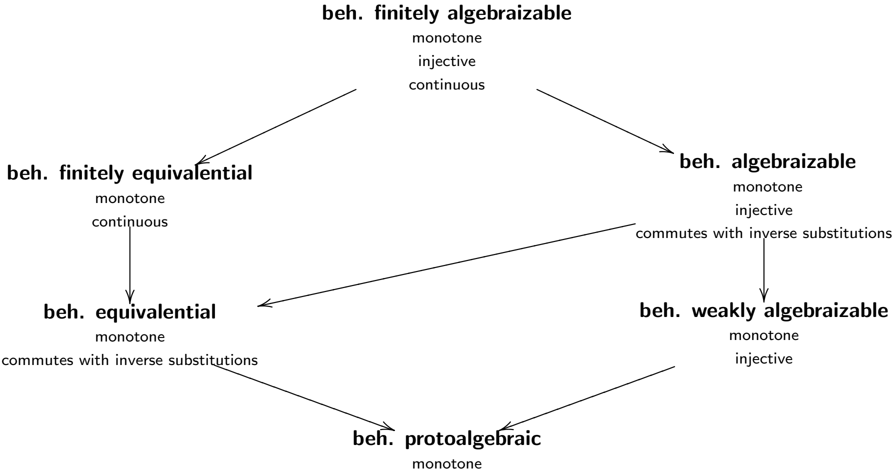
Definition (Behavioral Leibniz operator)

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a structural many-sorted logic and* Γ *a subsignature of* Σ*. The* Γ*-behavioral Leibniz operator on the term algebra,*

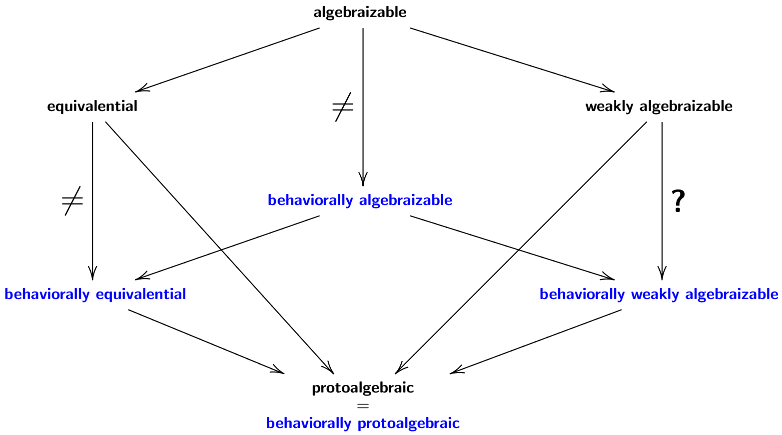
$$\Omega^{bhv}_{\Gamma} : Th_{\mathcal{L}} \to Cong^{\Sigma}_{\Gamma}(\mathbf{T_{\Sigma}}(\mathbf{X}))$$

$$T \mapsto \textit{largest } \Gamma\textit{-congruence compatible with } T.$$

Leibniz hierarchy (Behavioral)



Current vs. Behavioral



Examples

- da Costa's paraconsistent logic $\mathcal{C}_1$
  - New algebraic semantics
  - Connection with existing semantics
- First-order classical logic
  - shed light on the essential distinction between terms and formulas
- Constructive logic with strong negation
  - Give an extra insight on the role of Heyting algebras in the algebraic counterpart of N

Conclusions

- Generalization of the notion of algebraizable logic
- Characterization results using the behavioral Leibniz operator
- Covering many-sorted logics as well as some non-algebraizable logics (according to the old notion)