On the Essential Norm of the Cauchy Singular Integral Operator in Rearrangement-Invariant Spaces

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Abstract. This talk is devoted to a lower estimate for the essential norm of the Cauchy singular integral operator S in reflexive weighted rearrangement-invariant spaces $X(\Gamma, w)$ over Carleson (or Ahlfors-David regular) curves Γ. These spaces are a wide generalization of classic Lebesgue, Orlicz, and Lorentz spaces. Using results on Fredholmness of singular integral operators with piecewise continuous coefficients in the space $X(\Gamma, w)$, we prove that

$$|S| := \inf \|S + \mathit{compact}\|_{\mathcal{L}(X(\Gamma,w))} \geq \cot\big(\pi\lambda/2\big) \tag{1}$$

where

$$\lambda = \inf_{t\in\Gamma} \min\{\alpha(Q_t w), 1 - \beta(Q_t w)\},$$

and $0 < \alpha(Q_t w) \leq \beta(Q_t w) < 1$ are the indices of a submultiplicative function $(Q_t w)(x) : (0,\infty) \to (0,\infty)$, which is associated with local properties of the space, of the curve, and of the weight at the point $t \in \Gamma$. In some cases we give formulas for computation of $\alpha(Q_t w)$ and $\beta(Q_t w)$. In particular, if we consider the non-weighted case ($w = 1$), then $\alpha(Q_t 1)$ and $\beta(Q_t 1)$ coincide with Zippin (fundamental) indices [3] of the rearrangement-invariant space $X(\Gamma)$. For Lebesgue spaces $L^p(\Gamma), 1 \leq p \leq \infty$, Zippin indices coincide and equal $1/p$. So, the estimate (1) correlates with the well-known result by Pichorides [2] and Gohberg-Krupnik (see [1]) for the Lebesgue space $L^p(\mathbf{T}), 1 < p < \infty$, over the unit circle $\mathbf{T}$:

$$|S| = \|S\|_{\mathcal{L}(L^p(\mathbf{T}))} = \cot\big(\pi/2 \cdot \min\{1/p, 1 - 1/p\}\big).$$

1. I. Gohberg and N. Krupnik, *One-Dimensional Linear Singular Integral Equations*, Vols. 1, 2, Birkhäuser Verlag, Basel, Boston, Berlin, 1992. Russian original: Shtiintsa, Kishinev, 1973.
2. S. K. Pichorides, *On the best values of the constants in the theorems M. Riesz, Zygmund and Kolmogorov*, Studia Math., **19**, 2 (1972), 165–179.
3. M. Zippin, *Interpolation of operators of weak type between rearrangement invariant spaces*, J. Functional Analysis **7** (1971), 267–284.

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