Compactness of some commutators on rearrangement-invariant spaces

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Commutators $aS_\Gamma - S_\Gamma aI$ and $W_\alpha S_\Gamma - S_\Gamma W_\alpha$ of the Cauchy singular integral operator S_Γ with the operator of multiplication aI by a function $a \in L^\infty(\Gamma)$ and with the shift operator W_α defined by $W_\alpha f = f \circ \alpha$ play an important role in the Fredholm theory of singular integral operators with shifts. Compactness criteria for the first commutator on $L^2(\mathbf{T})$ over the unit circle $\mathbf{T}$ are well known for a long time, while compactness criteria for the second commutator on $L^2(\mathbf{T})$ were obtained recently (1995) by P. Muhly and J. Xia when α is an orientation-preserving bi-Lipschitz homeomorphism of $\mathbf{T}$ onto itself.

We obtain analogous results for the case of the unit segment $\mathbf{J} := [0,1]$. Using interpolation of compact operators, we extend these results on general rearrangement-invariant spaces. Note that these spaces include classical Lebesgue, Orlicz, and Lorentz spaces.

Theorem 1. *Suppose Γ is either the unit circle or the unit segment and $a \in L^\infty(\Gamma)$. The operator $aS_\Gamma - S_\Gamma aI$ is compact on a rearrangement-invariant space $X(\Gamma)$ with nontrivial Boyd indices if and only if a has vanishing mean oscillation on Γ.*

Theorem 2. *Let Γ be either the unit circle or the unit segment. Suppose α is an orientation-preserving bi-Lipschitz homeomorphism of Γ onto itself. If $\Gamma = [0,1]$ then suppose in addition that $\alpha(0) = 0$ and $\alpha(1) = 1$. The operator $W_\alpha S_\Gamma - S_\Gamma W_\alpha$ is compact on a rearrangement-invariant space $X(\Gamma)$ with nontrivial Boyd indices if and only if α' has vanishing mean oscillation on Γ.*

Passage from $\mathbf{T}$ to $\mathbf{J}$ ("cutting") involves an operator N with two fixed singularities at the endpoints of half-circles. The main difficulty here consists of the proof of compactness for the commutator of R with the multiplication operators. Note that the results of Theorems 1 and 2 for $\mathbf{J}$ are new even in the case of Lebesgue spaces $L^p(\mathbf{J}), 1 < p < \infty$.

Theorems 1 and 2 are obtained in collaboration with Yuri Karlovich.

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