Compactness of commutators arising in the Fredholm theory of singular integral operators with shifts

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Commutators $aS - SaI$ and $W_\alpha S - SW_\alpha$ of the Cauchy singular integral operator S with the operator of multiplication aI by a function $a \in L^\infty$ and with the shift operator W_α defined by $W_\alpha f = f \circ \alpha$ play a very important role in the Fredholm theory of singular integral operators with shifts. The compactness criterion for the first commutator on $L^2(\mathbf{T})$ over the unit circle $\mathbf{T}$ is well known for a long time, while the compactness criterion for the second commutator on $L^2(\mathbf{T})$ was obtained recently (1995) by P. Muhly and J. Xia when α is an orientation preserving homeomorphism of $\mathbf{T}$ onto itself.

Passing from $L^2(\mathbf{T})$ to $L^p(\mathbf{T})$ with the help of the Krasnoselskii interpolation theorem for compact operators, and "cutting" the unit circle $\mathbf{T}$, we prove compactness criteria for these commutators on Lebesgue spaces $L^p(\mathbf{J})$ where $\mathbf{J} := (0, 1)$.

Theorem 1. *Suppose $1 < p < \infty$ and $a \in L^\infty(\mathbf{J})$. The operator $aS - SaI$ is compact on the Lebesgue space $L^p(\mathbf{J})$ if and only if a has vanishing mean oscillation on $\mathbf{J}$.*

Theorem 2. *Suppose $1 < p < \infty$ and α is an orientation preserving homeomorphism of $[0, 1]$ onto itself such that $\alpha(0) = 0$ and $\alpha(1) = 1$. Suppose $\log \alpha' \in L^\infty(\mathbf{J})$. The operator $W_\alpha S - SW_\alpha$ is compact on the Lebesgue space $L^p(\mathbf{J})$ if and only if α' has vanishing mean oscillation on $\mathbf{J}$.*

Passage from $\mathbf{T}$ to $\mathbf{J}$ ("cutting") involves an operator R with two fixed singularities at the endpoints of half-circles. The main difficulty here consists of the proof of compactness for the commutators of R with the operators of multiplication and with the shift operator.

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