ASSESSING THE IMPACT OF HEAD STARTS IN THE PERFORMANCE OF ONE-SIDED MARKOV-TYPE CONTROL SCHEMES

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Key Words: statistical process control; run length; phase-type distributions; stochastic ordering.

ABSTRACT

Head start values — that is, non-zero initial values for the summary statistics of Markov-type control schemes such as the well known *CUSUM* schemes — are thoroughly recommended in the literature and used by quality control practitioners. The rationale of it is as follows: if the process is operating in-control, the summary statistic of the control scheme is soon brought to zero, so that the expected effect of the head start is minimal; otherwise, the operator is alerted to the out-of-control situation much sooner, which may prevent start-up problems.

This paper emphasizes on the assessment of the impact of the adoption of a head start on the run length, the indisputedly popular performance measure of control schemes. It also brings the issue of stochastically monotone matrices into focus in the statistical process control setting and investigates their influence on the alarm rate function, the ageing character and the equilibrium rate function of the run length of one-sided Markov-type control schemes.

1. INTRODUCTION

Manufacturing processes are typically monitored using data: a random sample of n items is usually taken on a regular basis and the observed values of a summary statistic are se-

quentially plotted together with appropriate control limits. The resulting graphical device is grandly termed as quality control scheme. Underlying any control scheme analysis, there is an indisputedly popular performance measure, the number of samples taken before a signal is triggered by the control scheme, also known as run length (RL).

The average run length, $ARL = E(RL)$, has been throughly used to describe the likely performance of a control scheme, although it is a unidimensional and possibly misleading snapshot of the scheme performance (Luceño and Puig-Pey (1)). According to Lai (2), it was Aroian and Levene (3) who proposed the average efficiency number, now more commonly known as the average run length, in the assessment of the performance of quality control schemes.

The survival function of RL , $\bar{F}_{RL}(m) = P(RL > m)$, $m \in \mathbb{N} = \{1, 2, \dots\}$, represents the probability of triggering a signal after collecting at least m samples and provides what can be thought as a bidimensional snapshot of the performance of the control scheme. Some authors have argued that $p \times 100\%$ upper percentage points of the run length, $F_{RL}^{-1}(p) = \inf\{m \in \mathbb{N} : \bar{F}_{RL}(m) \leq p\}$, provide a more appropriate performance measure when *Shewhart* control schemes are not used (see, e.g., Yashchin (4)).

The alarm (or hazard) rate of RL , as proposed by Margavio *et al.* (5), equals

$$\lambda_{RL}(m) = \frac{P(RL = m)}{P(RL \geq m)}, \quad m \in \mathbb{N}, \quad (1)$$

and represents the conditional probability of a signal at sample m , given that the previous $m - 1$ samples were not responsible for triggering a signal. Thus, it provides a conditional snapshot and a rather more insightful portrait of the scheme ability to signal than the ones based on ARL or the survival function of RL .

Similarly, the reversed hazard rate function (see Kijima (6, p.113) or Shanthikumar *et al.* (7)) of RL is given by

$$\bar{\lambda}_{RL}(m) = \frac{P(RL = m)}{P(RL \leq m)}, \quad m \in \mathbb{N}. \quad (2)$$

It represents the probability of a signal at sample m , given that the control scheme triggered at least one signal up to sample m .

The equilibrium rate function (see Shaked and Shanthikumar (8, p. 436)) of RL is another useful but rather technical characteristic. It is defined by $r_{RL}(1) = 0$ and

$$r_{RL}(m) = \frac{P(RL = m - 1)}{P(RL = m)}, \quad m = 2, 3, \dots \quad (3)$$

It is worth adding that if the relative decrease in the probability that the m^{th} sample triggers a signal relative to the probability of the signal being given by the previous sample is denoted by $d_{RL}(m)$ then we have

$$d_{RL}(m) = \frac{P(RL = m - 1) - P(RL = m)}{P(RL = m - 1)} = 1 - \frac{1}{r_{RL}(m)}, \quad m = 2, 3, \dots \quad (4)$$

One way of increasing the sensitivity to permanent shifts involves the accumulation of information across successive observations such as in cumulative sum (*CUSUM*) schemes (Page (9)) like the one described in the next example.

Example 1 — Consider that the detection of upward shifts in the expected number of defectives per random sample of size n , np , is done by using the summary statistic

$$S_N = \begin{cases} u, & N = 0 \\ \max\{0, S_{N-1} + (X_N - k)\}, & N \in \mathbb{N}, \end{cases} \quad (5)$$

where: u represents the initial value given to the summary statistic and if u is larger than 0 a head start has been given to the control scheme as proposed by Lucas and Crosier (10); X_N denotes the defectives count for the N^{th} sample and we assume that $X_N \stackrel{i.i.d.}{\sim} \text{Binomial}(n, p)$; and k , usually called reference value, is a positive (integer) constant, smaller than n for this scheme. The process is deemed out-of-control at time N if the observation of the summary statistic exceeds the (integer) upper control limit x . This is what we call an upper one-sided *CUSUM* scheme for binomial output (Hawkins and Olwell (11, pp.122-3)) and the suitable choice of x and k depends upon a trade-off between a large in-control RL and a swift detection of a specific change in np . ◦

This and other similar control schemes make use of summary statistics with a recursive character which can be regarded as forming (or being approximated by) an absorbing Markov

chains in discrete time and discrete state space e.g. $\{0, 1, \dots, x, x+1\}$ governed by the transition matrix $\mathbf{P}$, as originally proposed by Brook and Evans (12), thus, the designation of *Markov-type scheme*.

In addition, the *RL* of a general Markov-type scheme for discrete (continuous) output is (or can be approximated by) a first passage time for the absorbing state $x+1$, with a phase-type distribution characterized by: an initial state u in $\{0, 1, \dots, x\}$ that corresponds to the initial value of the summary statistic; and a $(x+1) \times (x+1)$ sub-stochastic matrix obtained from $\mathbf{P}$ by deleting its final row and column, $\mathbf{Q}$, that governs the transitions between the transient states $0, 1, 2, \dots, x$. (See Neuts (13, Chap.2) for a discussion on phase-type distributions in discrete and continuous time.)

Let

$$RL^u = \min\{N : S_N > x \mid S_0 = u\}, \quad u = 0, 1, \dots, x, \quad (6)$$

be the run length (or its approximation) of a Markov-type control scheme whose summary statistic has initial value u and whose upper control limit equals x . Then the average run length, the survival function, the alarm rate function, the reversed hazard rate function and the equilibrium rate function of RL^u (or their approximations) follow from Brook and Evans (12) and are respectively given by:

$$ARL^u = \mathbf{e}_u^\top (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1} \quad (7)$$

$$\bar{F}_{RL^u}(m) = \mathbf{e}_u^\top \mathbf{Q}^m \mathbf{1}, \quad m \in \mathbb{N} \quad (8)$$

$$\lambda_{RL^u}(m) = 1 - \frac{\mathbf{e}_u^\top \mathbf{Q}^m \mathbf{1}}{\mathbf{e}_u^\top \mathbf{Q}^{m-1} \mathbf{1}}, \quad m \in \mathbb{N} \quad (9)$$

$$\bar{\lambda}_{RL^u}(m) = \frac{\mathbf{e}_u^\top \mathbf{Q}^{m-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1}}{1 - \mathbf{e}_u^\top \mathbf{Q}^m \mathbf{1}}, \quad m \in \mathbb{N} \quad (10)$$

$$r_{RL^u}(m) = \begin{cases} 0, & m = 1 \\ \frac{\mathbf{e}_u^\top \mathbf{Q}^{m-2} (\mathbf{I} - \mathbf{Q}) \mathbf{1}}{\mathbf{e}_u^\top \mathbf{Q}^{m-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1}}, & m = 2, 3, \dots, \end{cases} \quad (11)$$

where $\mathbf{e}_u$ denotes the $(u+1)^{th}$ unit-vector of dimension $x+1$ and $\mathbf{1}$ is a column vector of $(x+1)$ ones.

The intuition on the impact of the adoption of a head start is so strong that devoting a paper to such matter could be thought as unnecessary or excessive. However, Lucas and Crosier (10) only assessed the influence of the adoption of a head start numerically, and Morais and Pacheco (14) were only able to analitically prove that there is a reduction of $\overline{F}_{RL^u}(m)$ when a head start is given to an upper one-sided Markov-type control scheme. Moreover, a similar result cannot be extended to standard or two-sided Markov-type control schemes such as the ones described by Page (9) and Lucas and Crosier (10), respectively.

This paper investigates the impact of the adoption of a head start not only in $\overline{F}_{RL^u}(m)$, but also in $\lambda_{RL^u}(m)$ and $r_{RL^u}(m)$, requiring some mathematical work on some special features of the transition matrix underlying the absorbing Markov chain. The results achieved will be stated as monotonicity properties, which are among the most important qualitative properties of stochastic models, as noted by Stoyan (15, p. 39).

Two finals remarks. The proofs included in this paper can all be found in the Appendix so that any technical details will not obfuscate the essence of their interpretation and the illustrations provided. All the numerical results were produced by programs for the package *Mathematica* (Wolfram (16)); these programs will be made available to those who are interested and request them from the authors.

2. STOCHASTICALLY MONOTONE MATRICES

Before we proceed we have to recall some important ageing notions and a few stochastic order relations adapted from Shaked and Shanthikumar (8, pp. 4, 13, 25 and 27) and Kijima (6, Section 3.2), respectively.

Definition 2 — Let X and Y be two positive integer random variables. Then X is said to be stochastically smaller than Y in the:

- usual sense — $X \leq_{st} Y$ — iff $\overline{F}_X(m) \leq \overline{F}_Y(m), m \in \mathbb{N}$
- hazard rate sense — $X \leq_{hr} Y$ — iff $\lambda_X(m) \geq \lambda_Y(m), m \in \mathbb{N}$
- reversed hazard rate sense — $X \leq_{rh} Y$ — iff $\overline{\lambda}_X(m) \leq \overline{\lambda}_Y(m), m \in \mathbb{N}$

- likelihood ratio sense — $X \leq_{lr} Y$ — iff $P(X = m)/P(Y = m) \downarrow_{m \in \mathbb{N}}$, or equivalently, iff $r_X(m) \geq r_Y(m)$, $m \in \mathbb{N}$. ◦

Clearly (see Theorems 1.C.1 and 1.B.1 of Shaked and Shanthikumar (8, pp. 28, 14)),

$$X \leq_{lr} Y \Rightarrow \begin{cases} X \leq_{hr} Y \\ X \leq_{rh} Y \end{cases} \Rightarrow X \leq_{st} Y. \quad (12)$$

Definition 3 — Consider a positive integer random variable X . Then X is said to have:

- increasing hazard rate — $X \in IHR$ — iff $\lambda_X(m) \uparrow_{m \in \mathbb{N}}$
- decreasing reversed hazard rate — $X \in DRHR$ — iff $\bar{\lambda}_X(m) \downarrow_{m \in \mathbb{N}}$
- decreasing likelihood ratio — $X \in DLR$ — iff $P(X = m)/P(X = m - 1) = 1/r_X(m) \downarrow_{m \in \{2, 3, \dots\}}$. ◦

Kijima (6, pp. 120-121) provides some illustrative examples of random variables with decreasing likelihood ratio, namely the binomial and *Poisson* distributions, and adds, in page 118, a well known fact:

$$Y \in DLR \Rightarrow \begin{cases} Y \in IHR \\ Y \in DRHR. \end{cases} \quad (13)$$

Definition 4 — The Markov chain $\{S_N, N = 0, 1, \dots\}$ is said to be stochastically monotone in the $*$ -sense ($*$ = st, hr, rh, lr) — $\mathbf{P} \in \mathcal{M}_*$ — iff

- $(S_{N+1}|S_N = i) \leq_* (S_{N+1}|S_N = m)$, $i \leq m$, for $N = 0, 1, \dots$ ◦

This special feature of $\mathbf{P}$ essentially mean that, if we associate probability functions of discrete random variables to each row of $\mathbf{P}$, the associated random variables increase stochastically in the $*$ -sense, as we progress in the rows of this stochastic matrix. In addition, following Equation (12), we have

$$\mathbf{P} \in \mathcal{M}_{lr} \Rightarrow \begin{cases} \mathbf{P} \in \mathcal{M}_{hr} \\ \mathbf{P} \in \mathcal{M}_{rh} \end{cases} \Rightarrow \mathbf{P} \in \mathcal{M}_{st}. \quad (14)$$

Since the classes of stochastic matrices $\mathcal{M}_*, * = st, hr, rh, lr$, include many of the common Markov models of applied probability theory (see, e.g., Keilson and Kester (17) and Bäuerle and Rolsky (18)) it is natural to inquire whether there are Markov-type control schemes associated to these four classes of matrices. For that matter we present Proposition 5, that shows that the ageing behaviour of the quality characteristic plays a vital role in the stochastically monotone character of the transition matrix $\mathbf{P}$ that governs the absorbing Markov chain.

Proposition 5 — Consider an upper one-sided *CUSUM* control scheme with summary statistic defined by Equation (5) where: $X_N \stackrel{i.i.d.}{\sim} X$, with X being a nonnegative integer quality characteristic; and k is a positive integer reference value. Then the following results hold:

$$\mathbf{P} \in \mathcal{M}_{st} \tag{15}$$

$$X \in IHR \Rightarrow \mathbf{P} \in \mathcal{M}_{hr} \tag{16}$$

$$X \in DRHR \Rightarrow \mathbf{P} \in \mathcal{M}_{rh} \tag{17}$$

$$X \in DLR \Rightarrow \mathbf{P} \in \mathcal{M}_{lr}. \tag{18}$$

This proposition was inspired by Example 3.11 from Kijima (6, p. 131) that only provides proofs of results analogous to (15) and (16); a sketch of the proof of results (17) and (18) follows in the Appendix.

The influence of the initial value u upon the characteristics of RL^u vitally depends on the stochastic monotone character of the transition matrix $\mathbf{P}$, as we shall shortly see.

3. ASSESSING THE IMPACT OF A HEAD START IN THE RUN LENGTH

By capitalizing on two results from the first passage time literature due to Durham *et al.* (19) and Assaf *et al.* (20), we can assess the ageing behaviour of the run length of some upper one-sided CUSUM schemes without head start, RL^0 :

$$\mathbf{P} \in \mathcal{M}_{rh} \Rightarrow RL^0 \in IHR \tag{19}$$

$$\mathbf{P} \in \mathcal{M}_{lr} \Rightarrow RL^0 \in DLR. \tag{20}$$

For instance, if the quality characteristic has decreasing reversed hazard rate — like the binomial or *Poisson* distribution — then $\mathbf{P} \in \mathcal{M}_{rh}$ and, thus, RL^0 has increasing alarm rate.

Adopting a head start can, however, alter the monotone behaviour of $\lambda_{RL^u}(m)$ and $r_{RL^u}(m)$, as illustrated by the next example.

Example 6 — Consider the upper one-sided *CUSUM* scheme for the detection of upward shifts in the expected number of defectives per random sample of size n that makes use of the summary statistic defined by Equation (5) with: reference value $k = 3$; upper control limit $x = 6$; initial values $u = 0$ (scheme with no head start) and $u = 3$ (scheme with a 50% head start); and $X_N \stackrel{i.i.d.}{\sim} \text{Binomial}(n = 100, p_0 = 0.02)$ in the absence of assignable causes. In this case, the in-control run length, $RL^u = RL^u(0)$, has a discrete phase-type distribution with in-control sub-stochastic matrix equal to

$$\mathbf{Q} = \mathbf{Q}(0) = \begin{bmatrix} 0.8590 & 0.0902 & 0.0353 & 0.0114 & 0.0031 & 0.0007 & 0.0002 \\ 0.6767 & 0.1823 & 0.0902 & 0.0353 & 0.0114 & 0.0031 & 0.0007 \\ 0.4033 & 0.2734 & 0.1823 & 0.0902 & 0.0353 & 0.0114 & 0.0031 \\ 0.1326 & 0.2707 & 0.2734 & 0.1823 & 0.0902 & 0.0353 & 0.0114 \\ 0 & 0.1326 & 0.2707 & 0.2734 & 0.1823 & 0.0902 & 0.0353 \\ 0 & 0 & 0.1326 & 0.2707 & 0.2734 & 0.1823 & 0.0902 \\ 0 & 0 & 0 & 0.1326 & 0.2707 & 0.2734 & 0.1823 \end{bmatrix}. \quad (21)$$

The parameters yield to a scheme with *ARLs* at the in-control value $np_0 = 2$ equal to $ARL^u(0) = 1015.71, 995.07$, for $u = 0, 3$ (respectively), as shown by Table 1, where we can also find out-of-control *ARL* values for an upward shift with magnitude $\theta = p - p_0 = 0.02$, and some values of the alarm rate and the equilibrium rate functions of $RL^u(\theta'), u = 0, 3$ and $\theta' = 0, 0.02$.

The use of a 50% head start yields a mild relative reduction (of 2.0% in this case) in the in-control *ARL* because the right tail behaviour of the in-control *RL* distribution is practically independent of the head start (see Lucas and Crosier (10)). However, the adoption of the same head start is responsible for a larger relative reduction (of 31.6% in this example) in

the out-control ARL .

Other more relevant facts are apparent from Figure 1 and Table 1.

Table 1: ARL s, alarm and equilibrium rates of $RL^u(\theta')$, for $u = 0, 3$, and $\theta' = 0, 0.02$.

m	$\lambda_{RL}(m)$				$r_{RL}(m)$			
	$RL^0(0)$	$RL^3(0)$	$RL^0(0.02)$	$RL^3(0.02)$	$RL^0(0)$	$RL^3(0)$	$RL^0(0.02)$	$RL^3(0.02)$
1	0.000034	0.004062	0.006844	0.106392	0	0	0	0
2	0.000234	0.005879	0.055859	0.213480	0.147055	0.693733	0.123375	0.557708
3	0.000481	0.004654	0.113719	0.235822	0.487032	1.270664	0.520264	1.150968
4	0.000671	0.003349	0.154287	0.237546	0.716424	1.396338	0.831632	1.299096
5	0.000797	0.002431	0.179720	0.234804	0.843264	1.381891	1.015104	1.326869
10	0.000974	0.001092	0.218448	0.225339	0.991105	1.068362	1.259663	1.297173
20	0.000987	0.000988	0.223523	0.223641	1.000940	1.001357	1.287543	1.288175
30	0.000987	0.000987	0.223609	0.223611	1.000988	1.000990	1.288005	1.288016
100	0.000987	0.000987	0.223611	0.223611	1.000988	1.000988	1.288014	1.288014
ARL	1015.710	995.070	7.194	4.920	—	—	—	—

The alarm rate values also show how unlikely (likely) is the emission of a false alarm (of a correct signal) at sample m , given that no previous signal has been triggered.

With no head start the alarm rate of the control scheme increases as we proceed with the sampling procedure (in opposition with the constant alarm rate of the geometric run length of *Shewhart* schemes), as shown by Columns 2 and 4 of Table 1 and by Figure 1. Thus, signalling in the absence of assignable causes, given that no observation has previously exceeded the upper control limit, becomes more likely, as we proceed with the sampling procedure.

However, if we adopt a head start the alarm rate of the scheme is no longer increasing as we proceed with the collection of samples, enabling us to add that e.g. $RL^3(0), RL^3(0.02) \notin IHR, DLR$.

Figure 1: Alarm rates of $RL^u(\theta')$, $\theta' = 0, 0.02$, for $u = 0$ (on the left) and $u = 3$ (on the right).

We should also add that the limiting form of the probability function of the RL is geometric-like with parameter $1 - \xi(\theta)$, where $\xi(\theta)$ is the maximum real eigenvalue of $\mathbf{Q}(\theta)$ (see Brook and Evans (12)), regardless of the initial value u of the summary statistic. Thus, it comes as no surprise that the values of the alarm rate functions of $RL^0(0)$ and $RL^3(0)$, and $RL^0(0.02)$ and $RL^3(0.02)$ rapidly converge to

$$\lim_{m \rightarrow +\infty} \lambda_{RL^u(0)}(m) = 1 - \xi(0) = 0.000987 \quad (22)$$

$$\lim_{m \rightarrow +\infty} \lambda_{RL^u(0.02)}(m) = 1 - \xi(0.02) = 0.223611, \quad (23)$$

respectively, as seen in Table 1 and Figure 1. A similar convergence hold for the equilibrium rate function:

$$\lim_{m \rightarrow +\infty} r_{RL^u(\theta')}(m) = 1/\xi(\theta'), \theta' = 0, 0.02. \quad (24)$$

Finally, if we compare Columns $2i$ and $2i + 1$, $i = 1, 2, 3, 4$, of Table 1 we can see that

the alarm rate and the equilibrium functions increase with the initial value of the summary statistic u suggesting the following theorem. ◦

Theorem 7 — We can add another stochastic flavour to the impact of the adoption of a head start u in the distribution of the run length of an upper one-sided Markov-type scheme:

$$\mathbf{P} \in \mathcal{M}_{st} \Rightarrow RL^u \downarrow_{st} \text{ with } u \quad (25)$$

$$\mathbf{P} \in \mathcal{M}_{rh} \Rightarrow RL^u \downarrow_{hr} \text{ with } u \quad (26)$$

$$\mathbf{P} \in \mathcal{M}_{lr} \Rightarrow RL^u \downarrow_{lr} \text{ with } u. \quad (27)$$

◦

The stochastic monotonicity results (25)–(27) certainly deserve a few remarks.

Early results concerning the monotonicity behaviour of first passage times in terms of the initial value can be found in two papers, and immediately derived from a third one:

- The stochastic result (25) follows from Corollary 1 (ii) of Theorem 2 by Kamae, Krengel and O’Brien (21). Particular cases of result (25) are given in Morais and Pacheco (14); this reference provides an induction-based proof of a result that implies that the run length of some upper one-sided *EWMA* and *CUSUM* control schemes is stochastically decreasing in the head start value.
- In order to prove the *IHR* ageing character of the continuous first passage times, $T^u = \min\{t \in \mathbb{R}^+ : S_t > x | S_0 = u\}$, $u = 0, 1, \dots, x$, Lee and Lynch (22) state and prove that this random variable decreases with u , in the hazard rate sense, in case the underlying discrete-state non-homogeneous continuous-time Markov chain, $\{S_t, t \in \mathbb{R}_0^+\}$, is stochastically monotone in the reversed hazard rate sense.
- Karlin (23) is an account of the connections between totally positive of order r transition probability matrices/kernels and the TP_r and RR_r characters of several first passage times of one-dimensional Markov chains in discrete/continuous time with discrete/continuous state space. In the light of result (i) of Theorem 2.1 of this paper we successfully manage to prove a stochastic decreasing behaviour of T^u in the likelihood ratio sense.

4. EXTENDING THE STOCHASTIC RESULTS

The Markov approach was originally introduced by Brook and Evans (12) as an alternative to the integral equation method proposed by Van Dobben de Bruyn (24) (see also Crowder (25)) for approximating the *RL* related measures of *CUSUM* schemes designed to control any continuous quality characteristic and can also be used when *EWMA* schemes are at use. This approach begins by approximating the problem — through the discretization of the summary statistic continuous state space —, and then by obtaining the exact solution to the approximate problem.

Since the resulting approximating run length converges in law to the exact run length and the stochastic orderings $\leq_{st,hr,rh,lr}$ have what Stoyan (15, p. 2) calls the weak convergence property (i.e., are closed under the limit operation), Theorem 7 still holds for continuous output.

Adding *Shewhart* limits is probably the simplest possible modification of a *CUSUM* scheme resulting in what is usually called a combined *CUSUM–Shewhart* scheme. Also considerable advantage is to be gained by using the combination of these two types of schemes since it takes advantage of two well known facts: the *Shewhart* schemes are favored when a large shift has occurred, and the *CUSUM* schemes allows a fast detection of small and moderate shifts.

The simultaneous use of standard upper one-sided control schemes of types *CUSUM* and *Shewhart* (also called an combined upper one-sided *CUSUM–Shewhart* scheme), with summary statistics S_N and X_N triggers a signal at time N if

$$S_N > x \text{ or } X_N > y + k, \quad (28)$$

where y ($0 \leq y \leq x$) turns out to be the critical value for the increment in the summary statistic S_N that should also be taken as an indication of a shift in the process parameter, even if the summary statistic S_N does not exceed the upper control limit x . Numerical results concerning combined upper one-sided *CUSUM–Shewhart* schemes for *Poisson* data can be found in Yashchin (4) and Abel (26).

The run length of this combined scheme is given by

$$\begin{aligned} RL_y^u &= \min\{N : S_N > x \text{ or } S_N - S_{N-1} > y \mid S_0 = u\} \\ &= \min\{N : S_N > x \text{ or } X_N > y + k \mid S_0 = u\}. \end{aligned} \quad (29)$$

Since any transition corresponding to an increment $S_N - S_{N-1}$ larger than y is not allowed, there are a few null entries above the main diagonal of the transition matrix $\bar{\mathbf{P}}$ that governs the associated absorbing Markov chain. As a result, these combined schemes are not governed by a stochastically monotone matrix in the hazard rate or likelihood ratio senses. In fact $\bar{\mathbf{P}}$ has the following properties (whose proofs are omitted and can be found in Morais (27)):

$$\bar{\mathbf{P}} \in \mathcal{M}_{st} \quad (30)$$

$$X \in DLR \Rightarrow \bar{\mathbf{P}} \in \mathcal{M}_{rh} \quad (31)$$

$$\bar{\mathbf{P}} \notin \mathcal{M}_{hr,lr}, \quad (32)$$

for $0 < y < x$. As a consequence:

$$RL_y^u \downarrow_{st} \text{ with } u \quad (33)$$

$$X \in DLR \Rightarrow \begin{cases} RL_y^u \downarrow_{hr} \text{ with } u \\ RL_y^0 \in IHR \end{cases} \quad (34)$$

$$\begin{cases} RL_y^u \not\downarrow_{lr} \text{ with } u \\ RL_y^0 \notin DLR. \end{cases} \quad (35)$$

Thus, the adoption of a head start does not yield a stochastic decrease in the RL in the likelihood sense although it implies an increase in the alarm rate function of the RL .

5. CONCLUDING REMARKS

In this paper run lengths of Markov-type schemes are viewed as random variables with discrete phase-type distributions with related stochastic matrices with special features, adding up to a few properties mentioned by Brook and Evans (12).

We proved that the stochastic matrices $\mathbf{P}$ that usually arise from upper one-sided control schemes (and, analogously, in lower one-sided schemes, as observed by Morais and Pacheco

(14)) are stochastically monotone in the usual sense. If, in addition, the (discrete) quality characteristic is $IHR/DRHR/DLR$ then $\mathbf{P}$ is stochastically monotone in the hazard rate/reversed hazard rate/likelihood ratio senses, respectively. By trivial capitalization on results from the literature on first passage times we were able to characterize some run lengths in terms of ageing properties.

We finally gave some stochastic flavour to the influence of the adoption of a head start in the RL distribution: this random variable stochastically decreases as we increase the initial value u of the summary statistic of the control scheme. The implications of giving a scheme a head start are surely guaranteed by the stochastic monotone character of the associated Markov chain, devised in Theorem 7.

APPENDIX

In the first part of the appendix we present a few preparatory notions and results required to prove Proposition 5 and Theorem 7.

First, we consider the square matrix $\mathbf{U}$, introduced by Keilson and Kester (17), with ones on and below the diagonal and zeroes elsewhere.

Then, we recall the notion of total positivity of order 2 (abbreviated TP_2) matrices and of sign-regular of order 2 (RR_2) matrices taken from Karlin (28, Chap.1). The nonnegative matrix $\mathbf{A} = [a_{ij}]$ is said to be totally positive of order 2, denoted here by $\mathbf{A} \in TP_2$, iff all the 2×2 minors of $\mathbf{A}$ are nonnegative; i.e.,

$$a_{ij} \times a_{i'j'} \geq a_{i'j} \times a_{ij'}, \quad i \leq i', \quad j \leq j'. \quad (36)$$

If the inequality (36) is reversed, $\mathbf{A}$ is called a sign-regular of order 2 matrix, i.e. $\mathbf{A} \in RR_2$.

TP_2 matrices verify what is called the *dual composition law* taken from Kijima (6, p. 110). Let $\mathbf{A} = [a_{ij}]_{i,j \in N}$ and $\mathbf{B} = [b_{ij}]_{i,j \in N}$ be nonnegative matrices. If $\mathbf{AU}^\top \in TP_2$, $\mathbf{BU}^\top \in TP_2$ and $(\mathbf{U}^\top)^{-1}\mathbf{BU}^\top \geq \mathbf{O}$, then $\mathbf{ABU}^\top \in TP_2$.

The stochastic orderings in Definition 2 can be more succinctly expressed using matrix $\mathbf{U}$, according to Kijima (6, p.122). Let X and Y be discrete random variables defined on N ,

with probability vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, respectively. Then, for instance:

$$X \leq_{rh} Y \Leftrightarrow (\text{or } \underline{\mathbf{a}} \leq_{rh} \underline{\mathbf{b}}) \Leftrightarrow \begin{bmatrix} \underline{\mathbf{a}}^\top \\ \underline{\mathbf{b}}^\top \end{bmatrix} \mathbf{U}^\top \in TP_2. \quad (37)$$

Let us now remind the reader the following equivalent characterizations of properties *IHR*, *DRHR* and *DLR* hold for the positive integer random variable X — with probability function $P(m)$ and distribution function $F(m)$ —, as reported by Kijima (6, p.113-115):

$$X \in DRHR \Leftrightarrow \begin{bmatrix} F(1) & F(2) & F(3) & \dots \\ F(0) & F(1) & F(2) & \dots \end{bmatrix} \in TP_2 \quad (38)$$

$$X \in DLR \Leftrightarrow \begin{bmatrix} P(0) & P(1) & P(2) & \dots \\ 0 & P(0) & P(1) & \dots \end{bmatrix} \in TP_2. \quad (39)$$

In addition and as pointed out in Definition 3.11 from Kijima (6, p.129), the following results are valid:

$$\mathbf{P} \in \mathcal{M}_{st} \Leftrightarrow \mathbf{U}^{-1}\mathbf{P}\mathbf{U} \geq \mathbf{O} \Leftrightarrow (\mathbf{U}^\top)^{-1}\mathbf{P}\mathbf{U}^\top \geq \mathbf{O} \quad (40)$$

$$\mathbf{P} \in \mathcal{M}_{hr} \Leftrightarrow \mathbf{P}\mathbf{U} \in TP_2 \quad (41)$$

$$\mathbf{P} \in \mathcal{M}_{rh} \Leftrightarrow \mathbf{P}\mathbf{U}^\top \in TP_2 \quad (42)$$

$$\mathbf{P} \in \mathcal{M}_{lr} \Leftrightarrow \mathbf{P} \in TP_2. \quad (43)$$

where $\mathbf{O}$ denotes a matrix with all entries equal to zero.

Finally, consider that the stochastic matrix $\mathbf{P}$ can be represented in the partitioned form

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & (\mathbf{I} - \mathbf{Q}) \times \underline{\mathbf{1}} \\ \underline{\mathbf{0}}^\top & 1 \end{bmatrix}, \quad (44)$$

where $\underline{\mathbf{1}}$ ($\underline{\mathbf{0}}$) denotes a vector of ones (zeroes). Then:

$$(\mathbf{U}^\top)^{-1}\mathbf{Q}\mathbf{U}^\top \geq \mathbf{O} \Leftrightarrow (\mathbf{U}^\top)^{-1}\mathbf{P}\mathbf{U}^\top \geq \mathbf{O} \Leftrightarrow \mathbf{U}^{-1}\mathbf{P}\mathbf{U} \geq \mathbf{O} \quad (45)$$

$$\mathbf{P} \in TP_2 \Rightarrow \mathbf{Q} \in TP_2 \quad (46)$$

$$\mathbf{P}\mathbf{U}^\top \in TP_2 \Rightarrow \mathbf{Q}\mathbf{U}^\top \in TP_2 \quad (47)$$

In one hand Equation (45) follows from the second equivalence in (40) and the fact that

$$(\mathbf{U}^\top)^{-1}\mathbf{P}\mathbf{U}^\top = \begin{bmatrix} (\mathbf{U}^\top)^{-1}\mathbf{Q}\mathbf{U}^\top & \underline{\mathbf{0}} \\ \underline{\mathbf{0}}^\top & 1 \end{bmatrix}. \quad (48)$$

On the other hand, results (46) and (47) are immediate consequences of the definition of total positivity of order 2 and the form of $\mathbf{P}$ and $\mathbf{P}\mathbf{U}^\top$, respectively, since

$$\mathbf{P}\mathbf{U}^\top = \begin{bmatrix} \mathbf{Q}\mathbf{U}^\top & \underline{\mathbf{1}} \\ \underline{\mathbf{0}}^\top & 1 \end{bmatrix}. \quad (49)$$

The second part of the appendix includes the proofs of results (17) and (18) from Proposition 5, and results (26) and (27) from Theorem 7.

Proof ((17)) — If $X \in DRHR$ then we get from (38)

$$F(i)F(j) \geq F(i-1) \times F(j+1), \quad 1 \leq i \leq j. \quad (50)$$

This result in turn implies that the transition matrix governing the absorbing Markov chain associated to an upper one-sided *CUSUM* scheme for discrete output is given by

$$\mathbf{P} = \begin{bmatrix} F(k) & P(k+1) & P(k+2) & \cdots & P(k+x) & \overline{F}(k+x) \\ F(k-1) & P(k) & P(k+1) & \cdots & P(k+x-1) & \overline{F}(k+x-1) \\ F(k-2) & P(k-1) & P(k) & \cdots & P(k+x-2) & \overline{F}(k+x-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F(k-x) & P(k-x+1) & P(k-x+2) & \cdots & P(k) & \overline{F}(k) \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad (51)$$

thus, verifying

$$\mathbf{P}\mathbf{U}^\top = \begin{bmatrix} F(k) & F(k+1) & F(k+2) & \cdots & F(k+x) & 1 \\ F(k-1) & F(k) & F(k+1) & \cdots & F(k+x-1) & 1 \\ F(k-2) & F(k-1) & F(k) & \cdots & F(k+x-2) & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F(k-x) & F(k-x+1) & F(k-x+2) & \cdots & F(k) & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \in TP_2, \quad (52)$$

i.e., $\mathbf{P} \in \mathcal{M}_{rh}$, in view of (42). ◦

Proof ((18)) — In case $X \in DLR$, it follows from (39) that

$$P(i) P(j) \geq P(i-1) P(j+1), \quad 1 \leq i \leq j. \quad (53)$$

Moreover, due to (36), we conclude that $\mathbf{P} = [p_{ij}]_{i,j=0}^{x+1} \in \mathcal{M}_{lr}$ if

$$\frac{p_{i0}}{p_{i+10}} \geq \frac{p_{i1}}{p_{i+11}} \geq \frac{p_{i2}}{p_{i+12}} \geq \dots \geq \frac{p_{ix}}{p_{i+1x}} \geq \frac{p_{ix+1}}{p_{i+1x+1}} \quad (54)$$

holds for $0 \leq i \leq x-1$. We shall now prove (54) using (53). The first inequality of (54) is valid since

$$p_{i0} p_{i+11} - p_{i+10} p_{i1} = F(k-i) P(k-i) - F(k-i-1) P(k-i+1) \quad (55)$$

equals the nonnegative sum

$$\begin{aligned} P(0) P(k-i) &+ [P(1) P(k-i) - P(0) P(k-i+1)] \\ &+ [P(2) P(k-i) - P(1) P(k-i+1)] \\ &+ \dots \\ &+ [P(k-i) P(k-i) - P(k-i-1) P(k-i+1)]. \end{aligned} \quad (56)$$

As for the last inequality, we can state that

$$\begin{aligned} p_{ix} p_{i+1x+1} - p_{i+1x} p_{ix+1} &= P(k+x-i) [1 - F(k+x-i-1)] \\ &\quad - P(k+x-i-1) [1 - F(k+x-i)] \end{aligned} \quad (57)$$

equals

$$\sum_{j=0}^{+\infty} [P(k+x-i) P(k+x-i+j) - P(k+x-i-1) P(k+x-i+j+1)] \quad (58)$$

which is nonnegative as well, by (53).

The remaining inequalities in (54) are particular cases of (53). ◦

Proof ((26)) — Let $\{S_N^*, N \in \mathbb{N}_0\}$ and $\{S_N', N \in \mathbb{N}_0\}$ be two absorbing Markov chains, with initial values $S_0^* = u^*$ and $S_0' = u'$ such that $0 \leq u^* \leq u' \leq x$, both governed by

the stochastically monotone matrix $\mathbf{P} \in \mathcal{M}_{rh}$. In addition, consider: $\underline{\mathbf{u}}_0^* = \underline{\mathbf{e}}_{u^*}$, $\underline{\mathbf{u}}_0' = \underline{\mathbf{e}}_{u'}$; $\underline{\mathbf{u}}_m^* = \underline{\mathbf{u}}_{m-1}^* \mathbf{Q}$ and $\underline{\mathbf{u}}_m' = \underline{\mathbf{u}}_{m-1}' \mathbf{Q}$ for $m \in \mathbb{N}$, where $\mathbf{Q}$ is the substochastic matrix in the block representation (44) of $\mathbf{P}$.

Note that, for $m \in \mathbb{N}$, $\underline{\mathbf{u}}_{m-1}^* \mathbf{1} = P(RL^{u^*} \geq m)$, $\underline{\mathbf{u}}_{m-1}' \mathbf{1} = P(RL^{u'} \geq m)$, $\underline{\mathbf{u}}_{m-1}^* (\mathbf{I} - \mathbf{Q}) \mathbf{1} = P(RL^{u^*} = m)$ and $\underline{\mathbf{u}}_{m-1}' (\mathbf{I} - \mathbf{Q}) \mathbf{1} = P(RL^{u'} = m)$.

Now, remember that since $\mathbf{P} \in \mathcal{M}_{rh}$ we have $\mathbf{P}\mathbf{U}^\top \in TP_2$, which implies $\mathbf{P} \in \mathcal{M}_{st}$, i.e., $(\mathbf{U}^\top)^{-1} \mathbf{P} \mathbf{U}^\top \geq \mathbf{O}$. As a consequence we get $\mathbf{Q}\mathbf{U}^\top \in TP_2$ and $(\mathbf{U}^\top)^{-1} \mathbf{Q} \mathbf{U}^\top \geq \mathbf{O}$. Moreover, by virtue of the fact that $0 \leq u^* \leq u' \leq x$, we have $\underline{\mathbf{u}}_0^* \leq_{rh} \underline{\mathbf{u}}_0'$, i.e.,

$$\begin{bmatrix} \underline{\mathbf{u}}_0^{*\top} \\ \underline{\mathbf{u}}_0'^\top \end{bmatrix} \mathbf{U}^\top \in TP_2, \quad (59)$$

according to result (37). By the dual of the composition law and using induction, we conclude that

$$\begin{bmatrix} \underline{\mathbf{u}}_{m-1}^{*\top} \\ \underline{\mathbf{u}}_{m-1}'^\top \end{bmatrix} \mathbf{U}^\top \in TP_2. \quad (60)$$

for all $m \in \mathbb{N}$. Applying the fact that the entries of $(\mathbf{I} - \mathbf{Q}) \mathbf{1}$ increase with i (i.e. $(\mathbf{U}^\top)^{-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1} \leq \underline{\mathbf{O}}$) to result (60) we get, for any $m \in \mathbb{N}$,

$$\begin{aligned} \begin{bmatrix} \underline{\mathbf{u}}_{m-1}^{*\top} \\ \underline{\mathbf{u}}_{m-1}'^\top \end{bmatrix} \mathbf{U}^\top \in TP_2 &\Rightarrow \frac{\underline{\mathbf{u}}_{m-1}^{*\top}}{\underline{\mathbf{u}}_{m-1}^{*\top} \mathbf{1}} \leq_{rh} \frac{\underline{\mathbf{u}}_{m-1}'^\top}{\underline{\mathbf{u}}_{m-1}'^\top \mathbf{1}} \\ &\Rightarrow \frac{\underline{\mathbf{u}}_{m-1}^{*\top}}{\underline{\mathbf{u}}_{m-1}^{*\top} \mathbf{1}} \leq_{st} \frac{\underline{\mathbf{u}}_{m-1}'^\top}{\underline{\mathbf{u}}_{m-1}'^\top \mathbf{1}} \\ &\Rightarrow \frac{\underline{\mathbf{u}}_{m-1}^{*\top} \mathbf{U}^\top}{\underline{\mathbf{u}}_{m-1}^{*\top} \mathbf{1}} \geq \frac{\underline{\mathbf{u}}_{m-1}'^\top \mathbf{U}^\top}{\underline{\mathbf{u}}_{m-1}'^\top \mathbf{1}} \\ &\Rightarrow \frac{\underline{\mathbf{u}}_{m-1}^{*top} \mathbf{U}^\top (\mathbf{U}^\top)^{-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1}}{\underline{\mathbf{u}}_{m-1}^{*\top} \mathbf{1}} \\ &\leq \frac{\underline{\mathbf{u}}_{m-1}'^\top \mathbf{U}^\top (\mathbf{U}^\top)^{-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1}}{\underline{\mathbf{u}}_{m-1}'^\top \mathbf{1}} \\ &\Leftrightarrow \frac{P(RL^{u^*} = m)}{P(RL^{u^*} \geq m)} \leq \frac{P(RL^{u'} = m)}{P(RL^{u'} \geq m)} \\ &\Leftrightarrow \lambda_{RL^{u^*}}(m) \leq \lambda_{RL^{u'}}(m). \end{aligned} \quad (61)$$

That is, $RL^{u^*} \geq_{hr} RL^{u'}$, $0 \leq u^* \leq u' \leq x$. ◦

Proof ((26)) — A simplified version of result (i) of Theorem 2.1 of Karlin (23) reads as follows: if $\mathbf{P} \in TP_2$ (equivalently, $\mathbf{P} \in \mathcal{M}_{lr}$) then $P(RL^u = m)$ is RR_2 in u and m , i.e.,

$$P(RL^{u^*} = m - 1) \times P(RL^{u'} = m) \leq P(RL^{u^*} = m) \times P(RL^{u'} = m - 1) \quad (62)$$

for $0 \leq u^* \leq u' \leq x$ and $m \in \mathbb{N}$. This inequality is equivalent to

$$\begin{aligned} \frac{P(RL^{u^*} = m - 1)}{P(RL^{u^*} = m)} &\leq \frac{P(RL^{u'} = m - 1)}{P(RL^{u'} = m)}, \quad m \in \mathbb{N} \Leftrightarrow \\ r_{RL^{u^*}}(m) &\leq r_{RL^{u'}}(m), \quad m \in \mathbb{N} \Leftrightarrow \\ RL^{u^*} &\geq_{lr} RL^{u'}, \end{aligned} \quad (63)$$

for $0 \leq u^* \leq u' \leq x$. That is, the run length decreases with the head start value in the likelihood ratio sense. $\circ$

ACKNOWLEDGEMENTS

This work was supported in part by *Programa Operacional “Ciência, Tecnologia, Inovação”* (POCTI) of the *Fundação para a Ciência e a Tecnologia* (FCT), cofinanced by the European Community fund FEDER, and the project POSI/42069/CPS/2001.

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