Confidence intervals for a threshold parameter

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1 Introduction

In this paper we consider the problem of obtaining confidence intervals for a threshold parameter in the context of two models developed for analysing toxicity tests data. The models are:
(I) $y_j = l - m\,(x_j - c)\, I\,(x_j - c) + \varepsilon_j, \quad \varepsilon_j \sim N\left(0, \sigma^2\right)$
(II) $y_j \sim Poisson\,(\lambda_j) \quad$ with $\quad \lambda_j = l \exp\left[-m\,(x_j - c)\, I\,(x_j - c)\right]$
where: (x_j, y_j) are the pairs of observations (x is usually a concentration of a toxic substance); l, m and c are parameters, c, the threshold parameter, is of special interest; $I(x)$ is an indicator function ($I(x) = 1$, if $x \geq 0$; $I(x) = 0$, if $x < 0$).

2 Results and conclusions

Point estimators of the parameters are easy to obtain using the ML method with grid search for the treshold parameter. In what concerns the interval estimation for the parameter c (given the type of model, a theoretical derivation of the sampling distribuition of $\hat{c}$ does not seem feasible), two approaches are possible: to use a (non-parametric) bootstrap method, or to rely on the asymptotic distribution of $\hat{c}$ as a maximum likelihood estimator. The resampling schemes considered were: (1)paired bootstrap; (2)bootstrap separately on each level of x (if the experiment has such a design); (3)bootstrap based on residuals. When applied to some simulated data sets we verified that the three schemes give similar distributions for $\hat{c}$. In the simulation study described next only scheme (3) is used. Three types of bootstrap confidence intervals are considered (Shao and Tu, 1995, Chapter 4): (BP) the bootstrap percentile; (BC) the bootstrap bias corrected percentile, with the transformation $\Psi = \Phi$; (BT) the bootstrap-t, with the standard deviation of $\hat{c}$ estimated by the standard jackknife method $\left(\hat{\sigma}_{(1)}\right)$; and also three types of asymptotic intervals: (PL) the profile likelihood based interval, that is the solution of $2 \log L\left(c, \hat{l}\,(c)\,, \hat{m}\,(c)\right) \geq 2 \log L\left(\hat{c}, \hat{l}, \hat{m}\right) - \chi^2_{1;0.95}$ (for 95% confidence); (NJ1) assuming a normal distribution for $\hat{c}$ and standard deviation estimated as in the (BT), that is $\hat{c} \pm 1.96 \hat{\sigma}_{(1)}$; (NJ2) same as (NJ1) but with the standard deviation estimated by the delete-2 jackknife (Shao and Tu, 1995, Section 2.3), that is $\hat{c} \pm 1.96 \hat{\sigma}_{(2)}$ (for 95% confidence).

To compare the confidence intervals produced by the various methods a simulation study based on a 1000 samples, generated from Models I and II,

Table 1. Results of the confidence interval simulation

		c=0.005			c=0.01		
Model	Method	CP	AL×10^2	SL×10^2	CP	AL×10^2	SL×10^2
I	BP	0.999	1.487	0.329	0.998	1.669	0.309
	BC	0.994	1.502	0.334	0.994	1.616	0.344
	BT	0.955	1.868	1.954	0.980	2.747	2.364
	PL	0.949	0.807	0.204	0.946	0.922	0.262
	NJ1	0.961	0.879	0.604	0.959	1.042	0.671
	NJ2	0.952	0.863	0.537	0.952	1.021	0.611
II	BP	1.000	1.153	0.156	1.000	1.738	0.078
	BC	1.000	1.183	0.186	1.000	1.477	0.204
	BT	1.000	1.130	0.354	1.000	1.910	0.695
	PL	0.945	0.350	0.074	0.948	0.450	0.100
	NJ1	0.871	0.539	0.585	0.898	0.717	0.595
	NJ2	0.942	0.529	0.360	0.941	0.696	0.447

was performed. The parameters used were $c = 0.005; 0.01$, $l = 32.5$, $m = 670$ (I), $m = 95$ (II) and $\sigma = 4$ (I). Each sample consists of 50 observations with 10 observations on each of five possible values of the concentration (x =0; 0.005; 0.01; 0.02; 0.05). These conditions were chosen in order to reflect a real experimental situation. The number of bootstrap replicates was set on 500.

The results of the simulation study are presented in Table 1. CP is an estimate of the coverage probability of the given interval (target= 0.95), AL is the average length of the intervals and SL the corresponding standard deviation.

It is clear, from these results, that the best interval is (PL): it meets the specified coverage probability with the shortest AL and shortest SL in all cases. (NJ2) performs almost as well in terms of CP but has worse AL and SL. The bootstrap intervals have a very poor behavior. (PL) is also the simplest interval to compute as it does not involve resampling and the profile likelihood has already been computed to obtain the point estimate. In terms of computational effort the worst method is (BT), requiring nb estimation cycles, where b is the number of bootstrap samples, while (NJ1) requires n and (NJ2) $n(n-1)/2$. In similar simulation exercises, not reported here, for smaller n ($n = 25$) and different combinations of the parameters the conclusions were the same.

A simple explanation for the failure of the normal based intervals is the non-differentiability of the profile log likelihood at some points. As a consequence a second order polynomial may be a very poor approximation (to the profile log likelihood). We can also infer that it is not worth trying other Wald-type confidence intervals. The same fact may also explain the poor behavior of the bootstrap intervals because the consistency of the bootstrap distribution estimator requires some smoothness conditions that are almost the same as those required for asymptotic normality (Shao and Tu, 1995, Chapter 3, page 128).

An important conclusion from this work is the failure of the resampling methods, particularly of the bootstrap. This case constitutes thus a warning against the blind use of resampling methods.

References

Shao, J. and Tu, D. (1995). *The Jackknife and Bootstrap.* New York: Springer.