

# Análise Numérica II - Licenciatura em Matemática Aplicada

## Formulário para o 1<sup>o</sup> exame

### I. Aproximação de funções

#### Interpolação polinomial

$$p_n(x) = \sum_{i=0}^n y_i l_i(x), \quad l_i(x) = \prod_{j=0, j \neq i}^n \left( \frac{x - x_j}{x_i - x_j} \right)$$

$$p_n(x) = D_0^0 + \sum_{i=1}^n D_0^i (x - x_i), \quad \begin{cases} D_j^0 = y_j, & j = 0, \dots, n \\ D_j^k = \frac{D_{j+1}^{k-1} - D_j^{k-1}}{x_{j+k} - x_j}, & j = 0, \dots, n - k, \quad k = 1, \dots, n \end{cases}$$

$$p_{2n+1}(x) = \sum_{i=0}^n y_i H_i^0(x) + \sum_{i=0}^n y_i' H_i^1(x), \quad H_i^0(x) = [1 - 2l_i'(x)(x - x_i)]l_i^2(x), \quad H_i^1(x) = (x - x_i)l_i^2(x)$$

$$f(x) - p_m(x) = \frac{f^{(m+1)}(\xi(x))}{(m+1)!} \prod_{j=0}^n (x - x_j)^{\alpha_j+1}, \quad m = \sum_{i=0}^n (\alpha_i + 1) - 1$$

$$f[x_0, x_1, \dots, x_n, x] = \frac{f^{(n)}(\eta(x))}{n!}$$

#### Interpolação trigonométrica

$$q_n(x) = \sum_{i=0}^{2n} y_i l_i(x), \quad l_i(x) = \prod_{j=0, j \neq i}^n \frac{\sin \frac{x-x_j}{2}}{\sin \frac{x_i-x_j}{2}}$$

#### Interpolação por splines

Splines cúbicos  $\left( K_j := s'(x_j), \quad \tau_i = \frac{1}{x_{i+1} - x_i} \right)$

$$\tau_{j-1} K_{j-1} + 2(\tau_{j-1} + \tau_j) K_j + \tau_j K_{j+1} = 3 \left[ \tau_{j-1}^2 (y_j - y_{j-1}) + \tau_j^2 (y_{j+1} - y_j) \right], \quad j = 1, \dots, n-1$$

- Condição da derivada nos extremos:  $K_0 := s'(x_0) = a, \quad K_n := s'(x_n) = b$
- Condição livre nos extremos:

$$s''(x_0) = 0, s''(x_n) = 0 \Leftrightarrow K_0 + \frac{1}{2} K_1 - \frac{3}{2} \tau_0 (y_1 - y_0) = 0, K_n + \frac{1}{2} K_{n-1} - \frac{3}{2} \tau_{n-1} (y_n - y_{n-1}) = 0$$

$$\max_{x \in [a, b]} |f(x) - s(x)| \leq \frac{h^2 \max_{t \in [a, b]} |f^{(2)}|}{8}, \quad s \in \mathcal{S}_1([a, b], \Delta_n)$$

$$\max_{x \in [a, b]} |f(x) - s(x)| \leq \frac{5h^4 \max_{t \in [a, b]} |f^{(4)}|}{384}, \quad s \in \mathcal{S}_3([a, b], \Delta_n)$$

#### Melhor aproximação no sentido dos mínimos quadrados

$$f^* = \sum_{i=0}^m c_i \varphi_i, \quad \sum_{i=0}^m c_i \langle \varphi_i, \varphi_j \rangle = \langle f, \varphi_j \rangle, \quad j = 0, \dots, m,$$

$$\langle \varphi, \psi \rangle = \int_a^b w(x) \varphi(x) \psi(x) dx, \quad \langle \varphi, \psi \rangle = \sum_{i=0}^n w_i \varphi(x_i) \psi(x_i)$$

**Polinômios ortogonais** (mônicos) em relação ao produto interno  $\langle \varphi, \psi \rangle = \int_a^b w(x)\varphi(x)\psi(x)dx$

$$\begin{cases} p_0(x) = 1; & p_1(x) = x - \langle 1, x \rangle / \langle 1, 1 \rangle; \\ p_n(x) = (x - \lambda_n)p_{n-1}(x) - \mu_n p_{n-2}(x), & \lambda_n = \langle xp_{n-1}, p_{n-1} \rangle / \|p_{n-1}\|^2, \quad \mu_n = \|p_{n-1}\|^2 / \|p_{n-2}\|^2 \end{cases}$$

- Polinômios de Chebychev:  $[a, b] = [-1, 1]$ ;  $w(x) = 1/(1-x^2)$

$$\langle T_0, T_0 \rangle = \pi, \quad \langle T_n, T_n \rangle = \pi/2, n \geq 1;$$

$$T_n(x) = \cos(n \arccos x), n = 0, 1, \dots; T_{n+1}(x_i) = 0 \Leftrightarrow x_i = \cos \frac{(2i+1)\pi}{2(n+1)}, i = 0, \dots, n;$$

$$\begin{cases} T_0(x) = 1, & T_1(x) = x, \\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), & n = 1, 2, \dots \end{cases}$$

- Polinômios de Legendre:  $[a, b] = [-1, 1]$ ;  $w(x) = 1$

$$\langle T_n, T_n \rangle = 2/(n+1), n \geq 0;$$

$$\begin{cases} T_0(x) = 1, & T_1(x) = x, \\ T_{n+1}(x) = \frac{1}{n+1}[(2n+1)xT_n(x) - nT_{n-1}(x)], & n = 1, 2, \dots \end{cases}$$

## II. Integração numérica

Fórmulas de Newton Côtes

$$\int_a^b f(x)dx - \sum_{j=0}^n A_{j,n} f(x_{j,n}) = \begin{cases} h^{n+3} \frac{f^{(n+2)}(\xi)}{(n+2)!} \int_0^n s \prod_{i=0}^n (s-i) ds, & n \text{ par} \\ h^{n+2} \frac{f^{(n+1)}(\eta)}{(n+1)!} \int_0^n \prod_{i=0}^n (s-i) ds, & n \text{ ímpar} \end{cases}$$

Fórmulas de Gauss

$$\int_a^b w(x)f(x)dx - \sum_{j=0}^n A_{j,n} f(x_{j,n}) = \frac{f^{(2n+2)}(\theta)}{(2n+2)!} \int_a^b w(x) \prod_{i=0}^n (x-x_i) dx$$

Regra dos Trapézios

$$T_n(f) = h \left[ \frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i) \right], \quad \int_a^b f(x)dx - T_n(f) = -\frac{h^2(b-a)}{12} f''(\xi)$$

Regra de Simpson

$$S_n(f) = \frac{h}{3} \left[ f(a) + f(b) + 4 \sum_{i=1}^{\frac{n}{2}} f(x_{2i-1}) + 2 \sum_{i=1}^{\frac{n}{2}-1} f(x_{2i}) \right], \quad \int_a^b f(x)dx - S_n(f) = -\frac{h^4(b-a)}{180} f^{(4)}(\eta)$$

### III. Resolução numérica de EDO's: problemas com valor inicial

$$\begin{cases} u'(x) = f(x, u(x)), & x \in ]a, b[, \\ u(a) = u_0 \end{cases}$$

- Métodos de Euler: explícito  $u_{j+1} = u_j + hf(x_j, u_j)$  ; implícito  $u_{j+1} = u_j + hf(x_{j+1}, u_{j+1})$
- Métodos de Taylor de ordem k:  $u_{j+1} = u_j + hf(x_j, u_j) + \dots + \frac{h^k}{k!} f^{(k-1)}(x_j, u_j)$
- Métodos de Runge-Kutta de ordem 2:  $u_{j+1} = u_j + \left(1 - \frac{1}{2\alpha}\right) hf(x_j, u_j) + \frac{1}{2\alpha} hf(x_j + \alpha h, u_j + \alpha hf(x_j, u_j))$   
 $\alpha = \frac{1}{2}$  - Método de Euler modificado;  $\alpha = 1$  - Método de Heun
- Método do trapézio:  $u_{j+1} = u_j + \frac{h}{2} [f(x_j, u_j) + f(x_{j+1}, u_{j+1})]$

Método com  $p + 1$  passos:  $u_{j+1} = \sum_{k=0}^p a_k u_{j-k} + h\Phi(x_{j+1}, x_j, \dots, x_{j-p}; u_{j+1}, u_j, \dots, u_{j-p}; h)$

$$\tau(x, h) = \frac{1}{h} [u(x+h) - \sum_{k=0}^p a_k u(x-kh)] - \Phi(x+h, x, \dots, x-ph; u(x+h), u(x), \dots, u(x-ph); h)$$

$$r(\lambda) = \lambda^{p+1} - \sum_{j=0}^p a_j \lambda^{p-j}$$

$$|u(x_j) - u_j| \leq K_1 \rho(h) + K_2 \tau(h), \quad \rho(h) = \max_{0 \leq k \leq p} |u(x_k) - u_k|$$

Para  $u_{j+1} = u_j + h\Phi(x_j, u_j; h)$ , a última desigualdade fica  $|u(x_j) - u_j| \leq \frac{e^{L(x_j-x_0)} - 1}{L} \tau(h)$

Métodos multipasso lineares:  $u_{j+1} = \sum_{k=0}^p a_k u_{j-k} + h \sum_{k=-1}^p b_k f(x_{j-k}, u_{j-k})$

$$c_0 = 1 - \sum_{j=0}^p a_j; \quad c_m = 1 + (-1)^{m-1} \sum_{j=1}^p j^m a_j + (-1)^m \sum_{j=-1}^p j^{m-1} b_j$$

### IV. Resolução numérica de EDO's: problemas com valores na fronteira

$$\begin{cases} u''(x) = p(x)u(x) + q(x)u'(x) + r(x), & x \in ]a, b[ \\ u(a) = \alpha, \quad u(b) = \beta \end{cases}$$

- método das diferenças finitas:

$$\begin{cases} \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = p(x_j)u_j + q(x_j) \frac{u_{j+1} - u_{j-1}}{2h} + r(x_j), & j = 1, \dots, N-1 \\ u_0 = \alpha, \quad u_N = \beta \end{cases}$$

- método dos elementos finitos ( $\alpha = 0, \beta = 0$ ):  $\sum_{i=1}^{N-1} u_i \mathbf{a}(\varphi_i, \varphi_j) = \mathbf{l}(\varphi_j), \quad j = 1, \dots, N-1,$

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}}, & x_{i-1} \leq x \leq x_i, \\ \frac{x - x_{i+1}}{x_i - x_{i+1}}, & x_i \leq x \leq x_{i+1}, \\ 0, & x \leq x_{i-1}, \quad x \geq x_{i+1} \end{cases}$$