Thermodynamic Formalism and Applications to Dimension Theory

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Preface

This monograph gives a unified exposition of the *thermodynamic formalism* and some of its main extensions, with emphasis on the relation to the *dimension theory* and the *multifractal analysis* of dynamical systems. Not only these are natural playgrounds for nontrivial applications of the thermodynamic formalism, but are also major sources of inspiration for further developments of the theory.

In particular, we present the main results and main techniques in the interplay between the thermodynamic formalism, symbolic dynamics, dimension theory, and multifractal analysis. We also discuss selected topics of current research interest that until now were scattered in the literature (incidentally, more than two thirds of the material appears here for the first time in book form). This includes the discussion of some of the most significant recent results in the area as well as some of its open problems, in particular concerning dimension estimates for repellers and hyperbolic sets, dimension estimates or even formulas for the dimension of limit sets of geometric constructions, and the multifractal analysis of entropy and dimension spectra, in particular associated to nonconformal repellers. Undoubtedly, this selection, although quite conscious, also reflects a personal taste.

The dimension theory and the multifractal analysis of dynamical systems progressively developed into an independent field of research during the last three decades. Nevertheless, despite a large number of interesting and nontrivial developments, only the case of *conformal* dynamics is completely understood. In the case of repellers this corresponds to assume that the derivative of the map is a multiple of an isometry at every point. This property allowed Bowen in 1979 (in the particular case of quasi-circles) and then Ruelle in 1982 (in full generality) to develop a fairly complete theory for the dimension of repellers of conformal maps. Their work is strongly based on the thermodynamic formalism, earlier developed by Ruelle in 1973 for expansive transformations, and then by Walters in 1976 in full generality.

On the other hand, the study of the dimension of invariant sets of *nonconformal* maps unveiled several new phenomena, but it still lacks today a satisfactory general approach. In particular, we are often only able to establish dimension estimates instead of giving formulas for the dimension of the invariant sets. Thus, sometimes the emphasis is on how to obtain sharp dimension estimates, starting essentially with the seminal work of Douady and Oesterlé in 1980, who devised an approach to cover an invariant set in a more optimal manner. Furthermore, it was early recognized, notably by Pesin and Pitskel' in 1984 (with the notion of topological pressure for noncompact sets) and by Falconer in 1988 (with his subadditive version of the thermodynamic formalism), that it would also be desirable to have an appropriate extension of the thermodynamic formalism in order to consider more general classes of invariant sets, and in particular invariant sets of nonconformal transformations. Most certainly, this is not foreign to the fact that virtually all known equations used to compute or estimate dimensions are appropriate versions of an equation introduced by Bowen in his study of quasi-circles that involves the topological pressure, which is the most basic notion of the thermodynamic formalism.

The exposition is organized in four parts. The first part gives an introduction to the classical thermodynamic formalism and its relations to symbolic dynamics. Although everything is proven, we develop the theory in a pragmatic manner, only as much as needed for the following parts. The remaining three parts consider three different versions of the thermodynamic formalism, namely nonadditive, subadditive, and almost additive. In each of these parts we detail generously not only the most significant results in the area, some of them quite recent, but also some of the most substantial applications of the corresponding thermodynamic formalism to the dimension theory and the multifractal analysis of dynamical systems.

The nonadditive thermodynamic formalism, which is a considerable extension of the classical thermodynamic formalism, provides the most general setting and has a unifying role. The subadditive and the almost additive formalisms successively consider more special situations. As always in mathematics, when one makes further hypotheses, one can often establish additional results. Thus, it is not surprising that the nonadditive, subadditive, and almost additive thermodynamic formalisms are progressively richer. On the other hand, and this is a major motivation for such developments, the new hypotheses are still sufficiently general to allow a large number of nontrivial applications. This includes dimension estimates for nonconformal repellers, nonconformal hyperbolic sets, and limit sets of geometric constructions, as well as a multifractal analysis of entropy and dimension spectra of a large class of nonconformal repellers.

The book is directed to researchers as well as graduate students who wish to have a global view of the main results and main techniques in the area. It can also be used for graduate courses on the thermodynamic formalism and its extensions, with the optional discussion of some applications to dimension theory and multifractal analysis, or for graduate courses on special topics of dimension theory and multifractal analysis, with the discussion of the strictly necessary material from the thermodynamic formalism. We emphasize that with the exception of a few sections of survey type, the text is self-contained and all the results are included with detailed proofs. In particular, it can also be used for independent study.

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