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MEASURES OF MAXIMAL DIMENSION FOR HYPERBOLIC DIFFEOMORPHISMS

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Abstract. We establish the existence of ergodic measures of maximal Hausdorff dimension for hyperbolic sets of surface diffeomorphisms. This is a dimension-theoretical version of the existence of ergodic measures of maximal entropy. The crucial difference is that while the entropy map is upper-semicontinuous, the map $\nu \mapsto \dim_H \nu$ is neither upper-semicontinuous nor lower-semicontinuous. This forces us to develop a new approach, which is based on the thermodynamic formalism. Remarkably, for a generic diffeomorphism with a hyperbolic set, there exists an ergodic measure of maximal Hausdorff dimension in a particular two-parameter family of equilibrium measures.

1. Introduction

In the theory of dynamical systems the study of dimension became popular around the late 70's and the beginning 80's. It appeared as a means to characterize the number of independent modes necessary to describe the "strange attractors" of many infinite-dimensional systems that are associated to natural phenomena. Furthermore, dimension is related to other invariants of a dynamical system which are associated to invariant sets and invariant measures. These invariants include topological and measure-theoretic entropies and Lyapunov exponents. We refer the reader to [6, 1, 11] for detailed accounts and references.

In particular, the measure-theoretic entropy describes the complexity of the dynamics from the point of view of a given invariant measure, and thus discards sets of small measure. On the other hand, topological entropy measures the complexity from the point of view of topological dynamics without discarding any component of the phase space. This indicates that measure-theoretic entropy cannot be larger than topological entropy, and this relation is made rigorous by the so-called variational principle of topological entropy.

Namely, let f be a homeomorphism of a compact metric space and denote by $h_{\text{top}}(f)$ the topological entropy and by $h_\nu(f)$ the measure-theoretic entropy of f (see [8] for the definitions; see also Section 2 below). Then

$$h_{\text{top}}(f) = \sup\{h_\nu(f) : \nu \in \mathcal{M}\}, \tag{1}$$

where $\mathcal{M}$ is the set of f-invariant probability measures. A measure $\nu \in \mathcal{M}$ at which the supremum in (1) is attained is called *measure of maximal*

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entropy. A priori there exists no natural measure of maximal entropy, that is, a natural measure of the size of sets from the point of view of entropy (which attains the maximal complexity in terms of entropy).

Nevertheless, when the map $\nu \mapsto h_\nu(f)$ is upper-semicontinuous (and therefore in particular when f is an expansive map of a compact metric space) there exist measures of maximal entropy. Furthermore, many transitive systems of hyperbolic nature (including topological Markov chains, hyperbolic sets, and repellers) possess a unique (and thus ergodic) measure of maximal entropy.

The main purpose of this paper is to discuss the corresponding questions in the case of Hausdorff dimension, for which the theory is poorly understood. Namely, let $\dim_H \nu$ denote the Hausdorff dimension of a measure ν (see Section 2 for the definition) and consider the quantity

$$\delta(f) = \sup\{\dim_H \nu : \nu \in \mathcal{M}\}. \tag{2}$$

A related quantity was introduced by Denker and Urbanski in [5] (with the supremum in (2) replaced by the supremum over the ergodic measures of positive entropy). In particular, this quantity has been intensively studied in one-dimensional complex dynamics (see [15] for further details).

A measure $\nu \in \mathcal{M}$ at which the supremum in (2) is attained is called *measure of maximal Hausdorff dimension* or simply *measure of maximal dimension.* This is the dimension-theoretical counterpart to a measure of maximal entropy, and each of these measures provides a natural measure of the size of sets from the point of view of dimension. Furthermore, it attains the maximal complexity in terms of dimension. It is thus of considerable interest to discuss the existence and ergodicity of measures of maximal dimension. The main outcome of this paper is a complete solution to this problem in the case of hyperbolic sets of diffeomorphisms on surfaces.

Our main result is the following (see Theorem 6 in Section 4):

Theorem 1. *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism, and let Λ be a compact locally maximal hyperbolic set such that $f|\Lambda$ is topologically mixing. Then there exists an ergodic f-invariant probability measure μ on Λ such that*

$$\dim_H \mu = \sup\{\dim_H \nu : \nu \in \mathcal{M} \textit{ is ergodic}\}. \tag{3}$$

It was established by Barreira, Pesin and Schmeling in [1] that for a $C^{1+\varepsilon}$ diffeomorphism, any finite invariant hyperbolic measure with compact support (and thus in particular any finite invariant measure supported on a compact hyperbolic set) possesses an "almost" local product structure. This means that the measure imitates (up to a small exponential error when compared to the Lyapunov exponents) the product structure of the invariant set. One consequence is that virtually all characteristics of dimension type of the measure (including the Hausdorff dimension, box dimensions, and information dimensions) coincide. In particular, the Hausdorff dimension can be replaced by any of these characteristics in our main result.

One consequence of Theorem 1 is the existence of measures of maximal dimension. This follows from the behavior of the Hausdorff dimension of an invariant measure under an ergodic decomposition.

Theorem 2 ([3])**.** *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism, and let Λ be a compact f-invariant locally maximal hyperbolic set. If μ is an f-invariant probability measure on Λ and τ is an ergodic decomposition of μ then*

$$\dim_H \mu = \operatorname{ess\,sup}\{\dim_H \nu : \nu \in \mathcal{M} \text{ is ergodic}\},$$

with the essential supremum taken with respect to τ.

An immediate consequence of Theorem 2 is that

$$\sup\{\dim_H \nu : \nu \in \mathcal{M}\} = \sup\{\dim_H \nu : \nu \in \mathcal{M} \text{ is ergodic}\}. \tag{4}$$

This identity allows us to reduce the study of measures of maximal dimension to the corresponding study for *ergodic* measures. Combining (4) with Theorem 1 we can finally establish the existence of ergodic measures of maximal dimension.

Corollary 3. *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism, and let Λ be a compact locally maximal hyperbolic set such that $f|\Lambda$ is topologically mixing. Then there exists an ergodic measure of maximal dimension.*

It should be noted that measures of maximal dimension are never unique. Namely, if μ is a measure of maximal dimension, then any linear combination of μ and any other measure is also a measure of maximal dimension. Nevertheless, it is possible to identify classes of diffeomorphisms under which the number of *ergodic* measures μ satisfying (3) is finite (see Section 5).

We now want to describe the main difficulties in the proof of Theorem 1. The crucial difference between entropy and dimension is that while the entropy map is upper-semicontinuous, the map $\nu \mapsto \dim_H \nu$ is neither upper- nor lower-semicontinuous. For the former one can consider the sequence $(\nu + (n-1)\nu_x)/n$ where $\dim_H \nu > 0$ and ν_x is supported on a periodic orbit, and for the latter consider a sequence of atomic measures converging to a measure of positive dimension (the existence of such a sequence follows from [14]).

On the other hand, it follows from work of Young in [17] and the upper-semicontinuity of the entropy map that $\nu \mapsto \dim_H \nu$ is upper-semicontinuous when *restricted to ergodic measures.* However, by a classical result of Sigmund in [14] the set of ergodic measures is a proper dense subset of $\mathcal{M}$ with respect to the weak$*$ topology. Thus the upper-semicontinuity on the set of ergodic measures does not imply the existence of a measure of maximal dimension (either ergodic or nonergodic).

These difficulties force us to develop a new approach, which is based on the thermodynamic formalism. We shall describe here one particular case that illustrates well the nature of our approach. Consider the functions

$$\phi_u = \log \|df|E^u\| \quad \text{and} \quad \phi_s = \log \|df|E^s\|, \tag{5}$$

where E^u and E^s are the unstable and stable distributions on Λ. We assume here that neither ϕ_u nor ϕ_s is cohomologous to a constant (see Section 2 for the definition). Our approach consists in the following:

1. consider a measure $\mu \in \mathcal{M}$ which is the limit of a sequence of ergodic measures $\nu_n \in \mathcal{M}$ satisfying

$$\lim_{n\to\infty} \dim_H \nu_n = \sup\{\dim_H \nu : \nu \in \mathcal{M} \text{ is ergodic}\};$$

2. construct curves $\gamma_u\colon \mathbb{R} \to \mathbb{R}$ and $\gamma_s\colon \mathbb{R} \to \mathbb{R}$ such that
$$\int_\Lambda \phi_u \, d\nu_{\gamma_u(q),q} = \int_\Lambda \phi_u \, d\mu \quad \text{and} \quad \int_\Lambda \phi_s \, d\nu_{p,\gamma_s(p)} = \int_\Lambda \phi_s \, d\mu,$$
where $\nu_{p,q}$ denotes the equilibrium measure of $-p\phi_u + q\phi_s$;
3. show that the two curves intersect, that is, there exists $(p, q) \in \mathbb{R}^2$ such that $(\gamma_u(q), q) = (p, \gamma_s(p))$;
4. show that for each intersection point $(p, q) \in \mathbb{R}^2$ the measure $\nu_{p,q}$ satisfies the identity
$$\dim_H \nu_{p,q} = \sup\{\dim_H \nu : \nu \in \mathcal{M} \text{ is ergodic}\}. \tag{6}$$

The proofs of these statements are based on the thermodynamic formalism. We note that the curves γ_u and γ_s correspond to level sets of measures $\nu_{p,q}$ having the same positive and negative values of the Lyapunov exponent.

We remark that a priori there may exist an ergodic measure of maximal dimension that is not among the measures $\nu_{p,q}$. Nevertheless, the number $\delta(f)$ can always be arbitrarily approximated by the dimension of the measures $\nu_{p,q}$, i.e.,
$$\delta(f) = \sup\{\dim_H \nu_{p,q} : p, q \in \mathbb{R}\}.$$
This is established in Section 4.

We now describe several applications of the measures of maximal dimension (actually not only of their existence but also of the properties established in Section 4). A first application concerns the dependence of $\delta(f)$ on the diffeomorphism f. Namely, by using the existence of the measures of maximal dimension we show, under reasonable assumptions (see Theorem 11), that $f \mapsto \delta(f)$ is C^{r-3} with respect to the C^r topology. This improves a result of McCluskey and Manning [10] showing that $f \mapsto \delta(f)$ is continuous.

A second application concerns the discussion of whether the measures of maximal dimension have the Gibbs property. We provide a very general condition (see Theorem 12) which implies that every ergodic measure of maximal dimension is an equilibrium measure of a Hölder continuous potential, and thus has the Gibbs property. This condition holds on an open subset of surface diffeomorphisms with a locally maximal hyperbolic set.

Another interesting application of Theorem 1 is the existence of measures of "maximal recurrence". It was proven by Barreira and Saussol in [2] that if ν is an equilibrium measure of a Hölder continuous potential of a $C^{1+\varepsilon}$ diffeomorphism f on a locally maximal hyperbolic set, then
$$\lim_{r\to 0} \frac{\log \inf\{k > 0 : f^k(x) \in B(x, r)\}}{-\log r} = \dim_H \nu \tag{7}$$
for ν-almost every point x, where $B(x, r)$ is the ball of radius r centered at x. When the limit in the left-hand side of (7) exists, it is called the recurrence rate at x. An immediate consequence of the above discussion is that any measure $\mu = \nu_{p,q}$ as in (6) satisfies (7) with the maximum possible value in the right-hand side, and thus is also a measure of "maximal recurrence" among all equilibrium measures of Hölder continuous potentials.

The paper is organized as follows. Section 2 defines the basic concepts and recalls several required results. In Section 3 we construct the curves γ_u and γ_s. Section 4 establishes Theorem 1 as well as several properties of measures of maximal dimension, including a complete characterization when neither ϕ_u nor ϕ_s is cohomologous to a constant. Section 5 provides further applications of Theorem 1 and of the properties established in Section 4. In particular, we discuss the regularity of the map $f \mapsto \delta(f)$ and the Gibbs property of the measures of maximal dimension.

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2. Preliminaries

Let $f\colon M \to M$ be a $C^{1+\varepsilon}$ diffeomorphism on a two-dimensional Riemannian manifold for some $\varepsilon \in (0,1]$, and $\Lambda \subset M$ a compact locally maximal hyperbolic set. This means that Λ is an f-invariant compact set and that there exists a continuous splitting of the tangent bundle $T_\Lambda M = E^u \oplus E^s$, and constants $c > 0$ and $\lambda \in (0,1)$ such that for each $x \in \Lambda$:

1. $d_x f(E^u_x) = E^u_{f(x)}$ and $d_x f(E^s_x) = E^s_{f(x)}$;
2. $\|d_x f^{-n} v\| \le c\lambda^n \|v\|$ whenever $v \in E^u_x$ and $n > 0$;
3. $\|d_x f^n v\| \le c\lambda^n \|v\|$ whenever $v \in E^s_x$ and $n > 0$.

Furthermore, there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U$. We shall always assume that the unstable and stable distributions E^u and E^s have dimension one, and that $f|\Lambda$ is topologically mixing.

We define functions $\phi_u\colon \Lambda \to \mathbb{R}$ and $\phi_s\colon \Lambda \to \mathbb{R}$ by (5). The unstable and stable distributions are Hölder continuous (in fact they are of class C^1 under our assumptions; see for example [8, Section 19.1]). Hence, the functions ϕ_u and ϕ_s are also Hölder continuous. Let now $\mathcal{M}$ be the family of f-invariant probability Borel measures on Λ equipped with the weak$*$ topology, and $\mathcal{M}_E \subset \mathcal{M}$ the subset of ergodic measures. This makes $\mathcal{M}$ a compact metrizable space. For each $\nu \in \mathcal{M}$ we define

$$\lambda_u(\nu) = \int_\Lambda \phi_u \, d\nu \quad \text{and} \quad \lambda_s(\nu) = \int_\Lambda \phi_s \, d\nu. \tag{8}$$

We denote by $\dim_H Z$ the Hausdorff dimension of the set Z, and define by

$$\dim_H \nu = \inf\{\dim_H Z : \nu(\Lambda \setminus Z) = 0\}$$

the Hausdorff dimension of the measure ν. Let $h_\nu(f)$ denote the measure-theoretic entropy of f with respect to ν (see for example [8] for the definition). For surface diffeomorphisms it was shown by Young in [17] that

$$\text{if } \nu \text{ is ergodic, then } \dim_H \nu = d(\nu), \tag{9}$$

where

$$d(\nu) \overset{\text{def}}{=} h_\nu(f)\left(\frac{1}{\lambda_u(\nu)} - \frac{1}{\lambda_s(\nu)}\right).$$

This result is crucial for our approach.

We also need the notion of topological pressure $P(\varphi)$ of a continuous function $\varphi\colon \Lambda \to \mathbb{R}$ with respect to $f|\Lambda$ (see [12, 8] for the definition and details). The topological pressure satisfies the following variational principle:

$$P(\varphi) = \sup_{\nu \in \mathcal{M}} \left(h_\nu(f) + \int_\Lambda \varphi \, d\nu \right). \tag{10}$$

Furthermore, the supremum in (10) can be replaced by the supremum over $\nu \in \mathcal{M}_E$. The number $h_{\mathrm{top}}(f) = P(0)$ is the topological entropy of $f|\Lambda$. If there exists a measure $\nu \in \mathcal{M}$ at which the supremum in (10), is attained it is called an equilibrium measure of φ. We recall that two functions φ, $\psi\colon \Lambda \to \mathbb{R}$ are said to be cohomologous if $\varphi - \psi = \eta - \eta \circ f$ for some continuous function $\eta\colon \Lambda \to \mathbb{R}$. In this case $P(\psi) = P(\varphi)$. Given $\alpha \in (0,1]$, let $C^\alpha(\Lambda)$ be the space of Hölder continuous functions $\varphi\colon \Lambda \to \mathbb{R}$ with Hölder exponent α. We now list several properties of the topological pressure which are needed later on (see [12] for details). Let $\alpha \in (0,1]$ be fixed. Then:

1. The map $\varphi \mapsto P(\varphi)$ is real-analytic on $C^\alpha(\Lambda)$.
2. Each function $\varphi \in C^\alpha(\Lambda)$ has a unique equilibrium measure $\nu_\varphi \in \mathcal{M}$; furthermore ν_φ is ergodic and given $\psi \in C^\alpha(\Lambda)$ we have
$$\frac{d}{dt} P(\varphi + t\psi)\Big|_{t=0} = \int_\Lambda \psi \, d\nu_\varphi. \tag{11}$$
3. For each φ, $\psi \in C^\alpha(\Lambda)$ we have $\nu_\varphi = \nu_\psi$ if and only if $\varphi - \psi$ is cohomologous to a constant.
4. For each φ, $\psi \in C^\alpha(\Lambda)$ and $t \in \mathbb{R}$ we have
$$\frac{d^2}{dt^2} P(\varphi + t\psi) \geq 0, \tag{12}$$
with equality if and only if ψ is cohomologous to a constant.

Our approach is based on the study of the topological pressure of the two-parameter family $(p,q) \mapsto -p\phi_u + q\phi_s$. More precisely, we consider the function $Q\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $Q(p,q) = P(-p\phi_u + q\phi_s)$. Since ϕ_u and ϕ_s are Hölder continuous, property 1 above implies that Q is real-analytic. Furthermore, property 2 implies that for each $(p,q) \in \mathbb{R}^2$ the function $-p\phi_u + q\phi_s$ has a unique equilibrium measure $\nu_{p,q} \in \mathcal{M}_E$. For simplicity, and since there is no danger of confusion, we shall use the notations

$$\lambda_u(p,q) = \lambda_u(\nu_{p,q}), \quad \lambda_s(p,q) = \lambda_s(\nu_{p,q}), \quad h(p,q) = h_{\nu_{p,q}}(f).$$

Accordingly, we also think of λ_u, λ_s, and h as functions in $\mathbb{R}^2$. Note that by the variational principle of the topological pressure (see (10)) we have

$$Q(p,q) = h(p,q) - p\lambda_u(p,q) + q\lambda_s(p,q). \tag{13}$$

We now briefly describe how these functions relate to dimension theory. It is straightforward to verify that there exist unique nonnegative numbers t_u and t_s satisfying $Q(t_u,0) = Q(0,t_s) = 0$. It follows from work of McCluskey and Manning in [10] that

$$\dim_H \Lambda = t_u + t_s \tag{14}$$

(for more details see [11] and the references therein). Since

$$0 = h(t_u,0) - t_u\lambda_u(t_u,0) \geq h_\nu(f) - t_u\lambda_u(\nu),$$

with strict inequality if and only if $\nu \neq \nu_{t_u,0}$ (together with an analogous property for the stable part), we have

$$t_u = \max_{\nu \in \mathcal{M}} \frac{h_\nu(f)}{\lambda_u(\nu)} = \frac{h(t_u,0)}{\lambda_u(t_u,0)} \quad \text{and} \quad t_s = \max_{\nu \in \mathcal{M}} \frac{h_\nu(f)}{-\lambda_s(\nu)} = -\frac{h(0,t_s)}{\lambda_s(0,t_s)}, \tag{15}$$

and the maxima are uniquely attained at the measures $\nu_{t_u,0}$ and ν_{0,t_s} respectively. We call a probability measure μ supported on Λ a *measure of full dimension* if $\dim_H \Lambda = \dim_H \mu$. Together with (9) and (14), the uniqueness of the maxima in (15) implies that there exists a measure $\mu \in \mathcal{M}_E$ of full dimension if and only if $\nu_{t_u,0} = \nu_{0,t_s}$, in which case $\mu = \nu_{t_u,0} = \nu_{0,t_s}$. In particular, if there is a measure of full dimension in $\mathcal{M}_E$, then it is unique.

For example, if f preserves volume then there exists an ergodic invariant measure of full dimension (and in this case $t_u = t_s$). To the best of our knowledge this was first explicitly observed by Friedland and Ochs in [7]. A short argument is the following. If $f^n(x) = x \in \Lambda$ and $k \in \mathbb{Z}$, then

$$1 = |\det d_x f^{kn}| = \exp[k\phi_u(f^n(x)) + k\phi_s(f^n(x))] \sin \angle(E^u(x), E^s(x)).$$

Letting $k \to \pm\infty$ yields $\phi_u(f^n(x)) + \phi_s(f^n(x)) = 0$ whenever $f^n(x) = x \in \Lambda$, and by Livshitz's theorem (see for example [8, Theorem 19.2.1]) $\phi_u + \phi_s$ is cohomologous to zero. This implies that $Q(t,0) = Q(0,t)$ for every $t \in \mathbb{R}$ and hence $t_u = t_s$. Thus $-t_u\phi_u$ is cohomologous to $t_s\phi_s$, and $\nu_{t_u,0} = \nu_{0,t_s}$ is the ergodic invariant measure of full dimension. In the case of hyperbolic polynomial automorphisms of $\mathbb{C}^2$ it is shown by Wolf in [16] that if there exists an ergodic invariant measure of full dimension, then either the map is volume preserving, or ϕ_u and ϕ_s are both cohomologous to a constant. In the latter case the ergodic invariant measure of full dimension obviously coincides with the measure of maximal entropy.

3. Preparatory results

We first introduce some notation. Since the maps $\nu \mapsto \lambda_u(\nu)$ and $\nu \mapsto \lambda_s(\nu)$ defined by (8) are continuous on $\mathcal{M}$, and $\mathcal{M}$ is compact, we can define

$$\lambda_u^{\min} = \min \lambda_u(\mathcal{M}), \quad \lambda_u^{\max} = \max \lambda_u(\mathcal{M}),$$

and

$$\lambda_s^{\min} = \min \lambda_s(\mathcal{M}), \quad \lambda_s^{\max} = \max \lambda_s(\mathcal{M}).$$

Set

$$I_u = (\lambda_u^{\min}, \lambda_u^{\max}) \quad \text{and} \quad I_s = (\lambda_s^{\min}, \lambda_s^{\max}).$$

Note that $I_u \neq \varnothing$ (respectively $I_s \neq \varnothing$) if and only if ϕ_u (respectively ϕ_s) is not cohomologous to a constant. We shall also consider the functions

$$d_u(p,q) = h(p,q)/\lambda_u(p,q) \quad \text{and} \quad d_s(p,q) = -h(p,q)/\lambda_s(p,q). \tag{16}$$

Recall that the function Q is real-analytic. It follows from (11) that

$$\partial_p Q = -\lambda_u \quad \text{and} \quad \partial_q Q = \lambda_s. \tag{17}$$

Therefore the functions λ_u and λ_s are also real-analytic. We conclude from (13) that h and hence also d_s and d_u are real-analytic.

Proposition 4. *The following properties hold:*

1. *if ϕ_u is not cohomologous to a constant and $q \in \mathbb{R}$, then:*
    a) *$\lambda_u(\cdot,q)$ is strictly decreasing and $\{\lambda_u(p,q) : p \in \mathbb{R}\} = I_u$;*

   b) *$h(\cdot,0)$ is strictly decreasing on $[0,\infty)$;*
   c) *$d_u(\cdot,0)$ is strictly increasing on $(-\infty,t_u]$ and strictly decreasing on $[t_u,\infty)$.*
2. *if ϕ_s is not cohomologous to a constant and $p\in\mathbb{R}$, then:*
   a) *$\lambda_s(p,\cdot)$ is strictly decreasing and $\{\lambda_s(p,q): q\in\mathbb{R}\}=I_s$;*
   b) *$h(0,\cdot)$ is strictly decreasing on $[0,\infty)$;*
   c) *$d_s(0,\cdot)$ is strictly increasing on $(-\infty,t_s]$ and strictly decreasing on $[t_s,\infty)$.*

Proof. Assume that ϕ_u is not cohomologous to a constant and fix $q\in\mathbb{R}$. By (12) and (17) we have

$$\partial_p\lambda_u=-\partial_p^2 Q<0 \tag{18}$$

and thus $\lambda_u(\cdot,q)$ is strictly decreasing. The continuity of $\lambda_u(\cdot,q)$ implies that $\{\lambda_u(p,q): p\in\mathbb{R}\}$ is an open interval. For property 1a we show that

$$\lim_{p\to\infty}\lambda_u(p,q)=\lambda_u^{\min} \quad\text{and}\quad \lim_{p\to-\infty}\lambda_u(p,q)=\lambda_u^{\max}. \tag{19}$$

Otherwise there would exist $\nu\in\mathcal{M}$ and $\varepsilon>0$ such that $\lambda_u(\nu)+\varepsilon<\lambda_u(p,q)$ for all $p\in\mathbb{R}$. Take $p>0$ satisfying

$$p\varepsilon>h_{\text{top}}(f)-q\lambda_s(\nu)+q\lambda_s(p,q)$$

(such a p always exists since the function $\lambda_s(\cdot,q)$ is bounded). We obtain

$$\begin{aligned}Q(p,q)&=h(p,q)-p\lambda_u(p,q)+q\lambda_s(p,q)\\&<h_{\text{top}}(f)-p(\lambda_u(\nu)+\varepsilon)+q\lambda_s(p,q)<h_\nu(f)-p\lambda_u(\nu)+q\lambda_s(\nu),\end{aligned}$$

which is a contradiction to the variational principle of the topological pressure. This establishes the first identity in (19). A similar argument establishes the second identity and property 1a holds.

It follows from (13) that

$$h(p,0)=Q(p,0)+p\lambda_u(p,0). \tag{20}$$

Using (17) and (18) it is straightforward to verify that

$$\partial_p h(p,0)=p\partial_p\lambda_u(p,0). \tag{21}$$

This establishes property 1b.

Using now (13) and (21) we obtain

$$\partial_p d_u(p,0)=\frac{p\partial_p\lambda_u(p,0)\lambda_u(p,0)-h(p,0)\partial_p\lambda_u(p,0)}{\lambda_u(p,0)^2}=-\partial_p\lambda_u(p,0)\frac{Q(p,0)}{\lambda_u(p,0)^2}.$$

It follows from the variational principle that $Q(\cdot,q)$ is strictly decreasing. This implies that $Q(p,0)>Q(t_u,0)=0$ for $p<t_u$ and $Q(p,0)<Q(t_u,0)=0$ for $p>t_u$. Property 1c follows now immediately from (18).

The proofs of the statements for the stable part are entirely analogous. □

Using Proposition 4 we introduce two curves, crucial for our approach.

Proposition 5. *The following properties hold:*

1. *for each $a\in I_u$ there exists a unique function $\gamma_u\colon\mathbb{R}\to\mathbb{R}$ satisfying $\lambda_u(\gamma_u(q),q)=a$ for all $q\in\mathbb{R}$, and γ_u is real-analytic;*
2. *for each $b\in I_s$ there exists a unique function $\gamma_s\colon\mathbb{R}\to\mathbb{R}$ satisfying $\lambda_s(p,\gamma_s(p))=b$ for all $p\in\mathbb{R}$, and γ_s is real-analytic.*

Proof. We shall prove the second statement. The proof of the first statement is analogous. Let $b \in I_s$. In particular $I_s \neq \varnothing$, and ϕ_s is not cohomologous to a constant. By statement 2a of Proposition 4 and (17), for each $p \in \mathbb{R}$ there exists a unique number $\gamma_s(p) \in \mathbb{R}$ such that $\partial_q Q(p, \gamma_s(p)) = \lambda_s(p, \gamma_s(p)) = b$. Since ϕ_s is not cohomologous to a constant, $\partial_q^2 Q(p,q) > 0$ for all $(p,q) \in \mathbb{R}^2$. The Implicit Function Theorem shows that $p \mapsto \gamma_s(p)$ is real-analytic. □

4. Measures of maximal dimension

4.1. **Existence.**

We now establish the existence of ergodic measures of maximal dimension on locally maximal hyperbolic sets. The following is our main result.

Theorem 6. *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism, and let Λ be a compact locally maximal hyperbolic set of f such that $f|\Lambda$ is topologically mixing. Then there exists a measure $\mu \in \mathcal{M}_E$ such that*

$$\dim_H \mu = \sup\{\dim_H \nu : \nu \in \mathcal{M}_E\}. \tag{22}$$

Proof. Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of measures in $\mathcal{M}_E$ such that

$$\lim_{n\to\infty} \dim_H \nu_n = \sup\{\dim_H \nu : \nu \in \mathcal{M}_E\}. \tag{23}$$

Since $\mathcal{M}$ is compact in the weak$*$ topology, we can also assume that $(\nu_n)_{n\in\mathbb{N}}$ converges to some measure $m \in \mathcal{M}$. Since the map $\mathcal{M} \ni \nu \mapsto h_\nu(f)$ is upper semi-continuous, it follows from (9) and the continuity of $\nu \mapsto \lambda_u(\nu)$ and $\nu \mapsto \lambda_s(\nu)$ that

$$\lim_{n\to\infty} \dim_H \nu_n \leq d(m). \tag{24}$$

Using (23) and (24) we obtain

$$\sup\{\dim_H \nu : \nu \in \mathcal{M}_E\} \leq d(m). \tag{25}$$

Therefore, in order to establish the existence of a measure $\mu \in \mathcal{M}_E$ satisfying (22), it is sufficient to show that there exists $\mu \in \mathcal{M}_E$ with

$$\dim_H \mu = d(m). \tag{26}$$

It is clear that any measure $\mu \in \mathcal{M}_E$ satisfying (26) also satisfies (22). We note that when m is ergodic, it follows from (9) that $\dim_H m = d(m)$, and hence (22) holds for m. However, m may not be ergodic.

Set $a = \lambda_u(m)$ and $b = \lambda_s(m)$. By Proposition 5, whenever $a \in I_u$ (respectively $b \in I_s$) we can consider the curve γ_u (respectively γ_s) associated to the number a (respectively b). We first prove some auxiliary statements.

Lemma 1. *If $\lambda_s(m) \in I_s$ then there exists $p \in [0, h_m(f)/\lambda_u(m)]$ such that $\lambda_u(p, \gamma_s(p)) = \lambda_u(m)$.*

Proof of the lemma. The assumption $\lambda_s(m) \in I_s$ guarantees that γ_s is well-defined. Since $\nu_{p,\gamma_s(p)}$ is the equilibrium measure of $-p\phi_u + \gamma_s(p)\phi_s$ we have

$$h(p,\gamma_s(p)) - p\lambda_u(p,\gamma_s(p)) + \gamma_s(p)\lambda_s(p,\gamma_s(p)) \geq h_m(f) - p\lambda_u(m) + \gamma_s(p)\lambda_s(m) \tag{27}$$

for all $p \in \mathbb{R}$. Note that $\lambda_u(p,\gamma_s(p)) > 0$. It is straightforward to verify that

$$\frac{h(p,\gamma_s(p))}{\lambda_u(p,\gamma_s(p))} - \frac{h_m(f)}{\lambda_u(m)} \geq \left(1 - \frac{\lambda_u(m)}{\lambda_u(p,\gamma_s(p))}\right)\left(p - \frac{h_m(f)}{\lambda_u(m)}\right). \tag{28}$$

Define $\kappa = h_m(f)/\lambda_u(m)$. Setting $p = \kappa$, it follows from (28) that

$$h(\kappa, \gamma_s(\kappa))/\lambda_u(\kappa, \gamma_s(\kappa)) \geq h_m(f)/\lambda_u(m). \tag{29}$$

Assume now that $\lambda_u(\kappa, \gamma_s(\kappa)) > \lambda_u(m)$. By (29), $h(\kappa, \gamma_s(\kappa)) > h_m(f)$. We conclude from (9) and (29) that $\dim_H \nu_{\kappa,\gamma_s(\kappa)} > d(m)$. This is a contradiction to (25) and thus we must have

$$\lambda_u(\kappa, \gamma_s(\kappa)) \leq \lambda_u(m). \tag{30}$$

On the other hand, it follows from (9) and (25) that

$$\frac{h(0, \gamma_s(0))}{\lambda_u(0, \gamma_s(0))} - \frac{h(0, \gamma_s(0))}{\lambda_s(m)} \leq \frac{h_m(f)}{\lambda_u(m)} - \frac{h_m(f)}{\lambda_s(m)}. \tag{31}$$

Setting $p = 0$ in (27) we obtain $h(0, \gamma_s(0)) \geq h_m(f)$. Therefore (31) yields

$$\lambda_u(0, \gamma_s(0)) \geq \lambda_u(m). \tag{32}$$

It follows from the continuity of $p \mapsto \lambda_u(p, \gamma_u(p))$ together with (30) and (32) that there exists $p \in [0, \kappa]$ for which $\lambda_u(p, \gamma_s(p)) = \lambda_u(m)$. This completes the proof of the lemma. $\square$

Lemma 2. *Assume that neither ϕ_u nor ϕ_s is cohomologous to a constant. Then $\lambda_u(m) \in I_u$ if and only if $\lambda_s(m) \in I_s$.*

Proof of the lemma. Assume that $\lambda_s(m) \in I_s$. By Lemma 1, there exists p for which $\lambda_u(p, \gamma_s(p)) = \lambda_u(m)$. Proposition 4 shows that $\lambda_u(p, \gamma_s(p)) \in I_u$ and hence $\lambda_u(m) \in I_u$. A similar argument and the corresponding version of Lemma 1 show that $\lambda_s(m) \in I_s$ whenever $\lambda_u(m) \in I_u$. $\square$

Lemma 2 indicates that it is enough to consider the following four cases:

1. $\lambda_s(m) \in I_s$ and $\lambda_u(m) \in I_u$;
2. $\lambda_s(m) \in I_s$ and ϕ_u is cohomologous to a constant;
3. $\lambda_u(m) \in I_u$ and ϕ_s is cohomologous to a constant;
4. $\lambda_s(m) \notin I_s$ and $\lambda_u(m) \notin I_u$.

We continue with an auxiliary statement.

Lemma 3. *If p, $q \in \mathbb{R}$ are such that $\lambda_u(p, q) = \lambda_u(m)$ and $\lambda_s(p, q) = \lambda_s(m)$, then $m = \nu_{p,q}$.*

Proof of the lemma. We have

$$\begin{aligned} h(p, q) + \int_\Lambda (-p\phi_u + q\phi_s)\, d\nu_{p,q} &= h(p, q) - p\lambda_u(m) + q\lambda_s(m) \\ &\geq h_m(f) + \int_\Lambda (-p\phi_u + q\phi_s)\, dm, \end{aligned}$$

and hence $h(p, q) \geq h_m(f)$, with equality if and only if $\nu_{p,q} = m$. On the other hand, combining (9) with (25) gives $h(p, q) \leq h_m(f)$. Therefore $h(p, q) = h_m(f)$ and $m = \nu_{p,q}$. $\square$

We now consider each of the above four cases.

Lemma 4. *If $\lambda_u(m) \in I_u$ and $\lambda_s(m) \in I_s$, then there exist p, $q \in \mathbb{R}$ such that $(p, \gamma_s(p)) = (\gamma_u(q), q)$ and $m = \nu_{p,q}$.*

Proof of the lemma. The hypotheses guarantee that γ_u and γ_s are well-defined. Since $\lambda_s(p, \gamma_s(p)) = \lambda_s(m)$, it follows from Lemma 1 and the uniqueness of γ_u that $(p, \gamma_s(p)) = (\gamma_u(q), q)$ for some $p, q \in \mathbb{R}$. In particular, $\lambda_u(p, q) = \lambda_u(m)$ and $\lambda_s(p, q) = \lambda_s(m)$. Lemma 3 implies that $m = \nu_{p,q}$. ☐

Lemma 5. *If $\lambda_s(m) \in I_s$ and ϕ_u is cohomologous to a constant, then there exist p, $q \in \mathbb{R}$ such that $m = \nu_{p,q}$.*

Proof of the lemma. Since $\lambda_s(m) \in I_s$, γ_s is well-defined, and $\lambda_s(p, \gamma_s(p)) = \lambda_s(m)$ for each p. On the other hand, the cohomological assumption ensures that $\lambda_u(p, \gamma_s(p)) = \lambda_u(m)$. Setting $q = \gamma_s(p)$ we obtain $\lambda_u(p, q) = \lambda_u(m)$ and $\lambda_s(p, q) = \lambda_s(m)$. Lemma 3 implies that $m = \nu_{p,q}$. ☐

An analogous argument establishes the following.

Lemma 6. *If $\lambda_u(m) \in I_u$ and ϕ_s is cohomologous to a constant, then there exist p, $q \in \mathbb{R}$ such that $m = \nu_{p,q}$.*

Finally we consider the fourth case.

Lemma 7. *If $\lambda_u(m) \notin I_u$ and $\lambda_s(m) \notin I_s$ then:*

1. *$\lambda_u(m) = \lambda_u^{\min}$ and $\lambda_s(m) = \lambda_s^{\max}$;*
2. *there exists $\nu \in \mathcal{M}_E$ such that $\lambda_u(\nu) = \lambda_u(m)$, $\lambda_s(\nu) = \lambda_s(m)$, and $h_\nu(f) = h_m(f)$.*

Proof of the lemma. We first establish property 1. When $I_u = I_s = \varnothing$ (i.e., ϕ_u and ϕ_s are both cohomologous to constants), there is nothing to prove. Assume now that

$$I_u = \varnothing, \quad I_s \neq \varnothing, \quad \text{and} \quad \lambda_s(m) = \lambda_s^{\min}. \tag{33}$$

Since $\nu_{0,0}$ is the measure of maximal entropy we have $h(0, 0) \geq h_m(f)$. Therefore it follows from $\lambda_u(0, 0) = \lambda_u^{\min}$, statement 2a in Proposition 4, and (9) that $\dim_H \nu_{0,0} > d(m)$. But this contradicts (25). Thus (33) cannot occur. Analogously we can show that it is impossible to have $I_s = \varnothing$, $I_u \neq \varnothing$, and $\lambda_u(m) = \lambda_u^{\max}$.

In order to complete the proof of property 1 it remains to consider the case when $I_u \neq \varnothing$ and $I_s \neq \varnothing$. In this case $\lambda_u(m) \in \partial I_u = \{\lambda_u^{\min}, \lambda_u^{\max}\}$ and $\lambda_s(m) \in \partial I_s = \{\lambda_s^{\min}, \lambda_s^{\max}\}$. Assume first that

$$\lambda_u(m) = \lambda_u^{\max} \quad \text{and} \quad \lambda_s(m) = \lambda_s^{\min}. \tag{34}$$

Since $\nu_{0,0}$ is the measure of maximal entropy we have $h(0, 0) \geq h_m(f)$. On the other hand, Proposition 4 implies that $\lambda_u(0, 0) < \lambda_u(m)$ and $\lambda_s(0, 0) > \lambda_s(m)$. Using (9) we obtain $\dim_H \nu_{0,0} > d(m)$. This is again a contradiction to (25) and hence (34) cannot occur. Assume now that

$$\lambda_u(m) = \lambda_u^{\min} \quad \text{and} \quad \lambda_s(m) = \lambda_s^{\min}. \tag{35}$$

We claim that

$$h(p, 0) > h_m(f) \tag{36}$$

for all $p > 0$. Otherwise, if $h(p, 0) \leq h_m(f)$ for some $p > 0$, Proposition 4 would imply $h(p, 0) - p\lambda_u(p, 0) < h_m(f) - p\lambda_u(m)$. But this is impossible since $\nu_{p,0}$ is the equilibrium measure of $-p\phi_u$. We also claim that

$$d_u(p, 0) \geq h_m(f)/\lambda_u(m) \tag{37}$$

for all sufficiently large p (see (16) for the definition of d_u). Otherwise, Proposition 4 would guarantee the existence of $p_0 \in \mathbb{R}$ and $\varepsilon > 0$ such that $d_u(p,0) + \varepsilon < h_m(f)/\lambda_u(m)$ for all $p \geq p_0$. It would then follow from (19) that $h_m(f) > h(p,0)$ for all sufficiently large p. This contradicts to (36) and hence (37) holds for all sufficiently large p. It follows from (35)–(37) that

$$\dim_H \nu_{p,0} = d_u(p,0) + d_s(p,0) \geq \frac{h_m(f)}{\lambda_u(m)} - \frac{h(p,0)}{\lambda_s(p,0)} > d(m)$$

for all sufficiently large p. This contradicts (25) and hence (35) cannot occur. Analogously one can show that it is impossible to have $\lambda_u(m) = \lambda_u^{\max}$ and $\lambda_s(m) = \lambda_s^{\max}$. Therefore we must have $\lambda_u(m) = \lambda_u^{\min}$ and $\lambda_s(m) = \lambda_s^{\max}$, and property 1 is established.

To prove property 2 we consider an ergodic decomposition τ of m, i.e., a probability measure on the metrizable space $\mathcal{M}$ with $\tau(\mathcal{M}_E) = 1$ such that

$$\int_{\mathcal{M}} \int_{\Lambda} \varphi \, d\nu \, d\tau(\nu) = \int_{\Lambda} \varphi \, dm \tag{38}$$

for all $\varphi \in C(\Lambda, \mathbb{R})$. Applying (38) to ϕ_u yields

$$\lambda_u^{\min} = \lambda_u(m) = \int_{\mathcal{M}} \lambda_u(\nu) \, d\tau(\nu).$$

Since $\lambda_u(\nu) \geq \lambda_u^{\min}$ for all $\nu \in \mathcal{M}$ there exists $A_1 \subset \mathcal{M}_E$ with $\tau(A_1) = 1$ such that $\lambda_u(\nu) = \lambda_u^{\min}$ for all $\nu \in A_1$. Analogously there exists $A_2 \subset \mathcal{M}_E$ with $\tau(A_2) = 1$ such that $\lambda_s(\nu) = \lambda_s^{\max}$ for all $\nu \in A_2$. We conclude from (9) and (25) that $h_\nu(f) \leq h_m(f)$ for all $\nu \in A_1 \cap A_2$. On the other hand, since $\tau(A_1 \cap A_2) = 1$ and $h_m(f) = \int_{\mathcal{M}} h_\nu(f) \, d\tau(\nu)$ (see for example [4]), there exists $A \subset A_1 \cap A_2$ with $\tau(A) = 1$ such that $h_\nu(f) = h_m(f)$ for all $\nu \in A$. This completes the proof of the lemma. ☐

By Lemmas 4–7, in each of the four cases there exists a measure $\mu \in \mathcal{M}_E$ satisfying (26): each measure $\nu_{p,q}$ in the Lemmas 4–6, and each measure ν in statement 2 of Lemma 7. This completes the proof. ☐

As explained in the introduction each ergodic measure $\mu \in \mathcal{M}$ satisfying (22) is a measure of maximal dimension, and thus we can simply refer to it as an *ergodic measure of maximal dimension*.

For hyperbolic surface diffeomorphisms with constant Jacobian, the identity (22) follows from work of Wolf in [16] in the case of polynomial automorphisms of $\mathbb{C}^2$. We remark that because of the constant Jacobian assumption the methods in [16] cannot be used in our study.

We note that the compactness assumption in Theorem 6 is essential. Otherwise, we can consider a sequence of (linear) Smale horseshoes $\Lambda_n \subset \mathbb{R}^2$ such that $\dim_H \Lambda_n \nearrow 2$ as $n \to \infty$, and thus the (noncompact) hyperbolic set $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$ has Hausdorff dimension 2. It can be arranged that each of the horseshoes Λ_n has associated a natural ergodic invariant measure ν_n of full dimension. On the other hand, there exists no ergodic measure of maximal dimension. Note that it is nevertheless easy to find a nonergodic measure of maximal dimension: simply consider the measure $\mu = \sum_{n=1}^{\infty} (\nu_n / 2^n)$.

We now present several consequences of Theorem 6 and its proof. A first result pertains the connection with invariant measures of full dimension.

Corollary 7. *Under the hypotheses of Theorem 6, the following properties are equivalent:*

1. *there exists an invariant measure of full dimension on* Λ*;*
2. *there exists an ergodic invariant measure of full dimension on* Λ*;*
3. $\sup\{\dim_H \nu : \nu \in \mathcal{M}\} = \dim_H \Lambda$.

Proof. Clearly,

$$\sup\{\dim_H \nu : \nu \in \mathcal{M}_E\} \le \sup\{\dim_H \nu : \nu \in \mathcal{M}\} \le \dim_H \Lambda.$$

It is established in [3] (see also the discussion in the introduction) that the two suprema are equal. Theorem 6 shows that the common value is attained at an ergodic measure. This implies the desired statement. ☐

Let now $\mathcal{N} \subset \mathcal{M}$ be the set of measures that are accumulation points of some sequence $(\nu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}_E$ satisfying

$$\lim_{n\to\infty} \dim_H \nu_n = \sup\{\dim_H \nu : \nu \in \mathcal{M}_E\} \tag{39}$$

The following is another consequence of Theorem 6 and its proof.

Corollary 8. *Under the hypotheses of Theorem 6, the supremum in* (39) *is a maximum, and for each* $m \in \mathcal{N}$ *the following properties hold:*

1. $d(m) = \max\{\dim_H \nu : \nu \in \mathcal{M}_E\}$*;*
2. *if neither* ϕ_u *nor* ϕ_s *is cohomologous to a constant, then one of the following exclusive alternatives holds:*
   a) $\lambda_u(m) \in I_u$ *and* $\lambda_s(m) \in I_s$*, with* $m = \nu_{p,q}$ *for some* p*,* $q \in \mathbb{R}$*;*
   b) $\lambda_u(m) \notin I_u$ *and* $\lambda_s(m) \notin I_s$*, with* $\lambda_u(m) = \lambda_u^{\min}$ *and* $\lambda_s(m) = \lambda_s^{\max}$*;*
3. *there exists* $\nu \in \mathcal{M}_E$ *such that* $\lambda_u(\nu) = \lambda_u(m)$*,* $\lambda_s(\nu) = \lambda_s(m)$ *and* $h_\nu(f) = h_m(f)$*;*
4. *if* $(\nu_n)_{n\in\mathbb{N}}$ *as in* (39) *has limit* m *then* $h_{\nu_n}(f) \to h_m(f)$ *as* $n \to \infty$*.*

We note that under the assumption of statement 2b the measure m cannot be an equilibrium measure in the two-parameter family $\{\nu_{p,q} : p, q \in \mathbb{R}\}$. Nevertheless, somewhat surprisingly, the supremum of the Hausdorff dimension of ergodic invariant measures can be arbitrarily approximated by the dimension of the measures in this family.

Theorem 9. *Under the hypotheses of Theorem 6, the following holds:*

$$\sup\{\dim_H \nu_{p,q} : p, q \in \mathbb{R}\} = \sup\{\dim_H \nu : \nu \in \mathcal{M}_E\}. \tag{40}$$

Proof. Let $\mu \in \mathcal{M}_E$ be a measure of maximal dimension. Recall that the existence of μ is guaranteed by Theorem 6. We shall consider three cases:

1. $\lambda_s(\mu) \in I_s$ and $\lambda_u(\mu) \in I_u$;
2. $\lambda_u(\mu) \notin I_u$, $\lambda_s(\mu) \notin I_s$, and neither ϕ_u nor ϕ_s is cohomologous to a constant;
3. ϕ_u or ϕ_s is cohomologous to a constant.

We note that under the assumptions of case 1 neither ϕ_u nor ϕ_s is cohomologous to a constant. Therefore Corollary 8 implies that the above three cases cover all possibilities.

Under the assumptions of case 1 it follows from Corollary 8 that $\mu = \nu_{p,q}$ for some $p, q \in \mathbb{R}$; thus (40) holds.

In the second case, Corollary 8 implies that $\lambda_u(\mu) = \lambda_u^{\min}$ and $\lambda_s(\mu) = \lambda_s^{\max}$. Therefore (40) is a consequence of the following lemma.

Lemma 8. *If $\mu \in \mathcal{M}_E$ is a measure of maximal dimension such that $\lambda_u(\mu) = \lambda_u^{\min}$ and $\lambda_s(\mu) = \lambda_s^{\max}$, then $\dim_H \nu_{t,t} \to d(\mu)$ as $t \to \infty$.*

Proof of the lemma. Since $\nu_{t,t}$ is the equilibrium measure of $-t\phi_u + t\phi_s$,

$$h(t,t) - t\lambda_u(t,t) + t\lambda_s(t,t) \geq h_\mu(f) - t\lambda_u(\mu) + t\lambda_s(\mu). \tag{41}$$

By Proposition 4 we obtain

$$\lambda_u(t,t) > \lambda_u(\mu) \quad \text{and} \quad \lambda_s(t,t) < \lambda_s(\mu). \tag{42}$$

Therefore (41) implies that $h(t,t) > h_\mu(f)$ for all $t \geq 0$. Let now $\varepsilon > 0$ and $t > h_{\mathrm{top}}(f)/\varepsilon$. Using (41) we obtain

$$\begin{aligned} \lambda_u(t,t) - \lambda_s(t,t) &\leq \frac{h(t,t) - h_\mu(f)}{t} + \lambda_u(\mu) - \lambda_s(\mu) \\ &< \frac{\varepsilon h(t,t)}{h_{\mathrm{top}}(f)} + \lambda_u(\mu) - \lambda_s(\mu) \leq \varepsilon + \lambda_u(\mu) - \lambda_s(\mu). \end{aligned}$$

Therefore (42) yields $\lambda_u(t,t) \to \lambda_u(\mu)$ and $\lambda_s(t,t) \to \lambda_s(\mu)$ as $t \to \infty$. Since μ is an ergodic measure of maximal dimension, combining $h(t,t) > h_\mu(f)$ with (9) establishes the desired statement. ☐

Finally we consider the third case.

Lemma 9. *If ϕ_u or ϕ_s is cohomologous to a constant then $\mu = \nu_{p,q}$ for some $p, q \in \mathbb{R}$.*

Proof of the lemma. If ϕ_u and ϕ_s are both cohomologous to a constant, then $\lambda_u(\nu) = \lambda_u(\nu_{0,0})$ and $\lambda_s(\nu) = \lambda_s(\nu_{0,0})$ for all $\nu \in \mathcal{M}_E$. Setting $m = \mu$, it follows from Lemma 3 that $\mu = \nu_{0,0}$.

We now assume that only one of the functions ϕ_u and ϕ_s is cohomologous to a constant. Without loss of generality we shall only consider the case when ϕ_s is cohomologous to a constant. This implies that for all $\nu \in \mathcal{M}_E$,

$$\lambda_s(\nu) = \lambda_s(\nu_{0,0}). \tag{43}$$

Let us assume that $\lambda_u(\mu) \in I_u$. By Proposition 4 there exists $p \in \mathbb{R}$ such that $\lambda_u(p,0) = \lambda_u(\mu)$. Again Lemma 3 implies that $\mu = \nu_{p,0}$.

We now assume that $\lambda_u(\mu) \notin I_u$. If $\lambda_u(\mu) = \lambda_u^{\max}$, then the fact that $\nu_{0,0}$ is the measure of maximal entropy combined with (9) implies that $\dim_H \nu_{0,0} > \dim_H \mu$. Since μ is a measure of maximal dimension, we obtain a contradiction. Thus $\lambda_u(\mu) \neq \lambda_u^{\max}$.

To complete the proof of the lemma we shall show that $\lambda_u(\mu) \neq \lambda_u^{\min}$. Otherwise Lemma 8 implies that $\dim_H \nu_{t,t} \to \dim_H \mu$ as $t \to \infty$. Since ϕ_s is cohomologous to a constant it follows from the uniqueness of the equilibrium measure and (43) that $\nu_{t,t} = \nu_{t,0}$ for all $t \in \mathbb{R}$. Hence

$$\dim_H \nu_{t,0} \to \dim_H \mu \text{ as } t \to \infty. \tag{44}$$

On the other hand, Proposition 4 together with (9) and (43) imply that $d_u(\cdot, 0) + d_s(\cdot, 0)$ is strictly decreasing on $[t_u, \infty)$ and thus (44) is a contradiction to the fact that μ is a measure of maximal dimension. ☐

We have established (40) in the three cases. This completes the proof. ☐

An interesting consequence of Theorem 9 is that the number $\delta(f)$ (as defined in the introduction) can be arbitrarily approximated by the dimension of the measures $\nu_{p,q}$: it follows immediately from Theorem 9 and (4) that

$$\sup\{\dim_H \nu_{p,q} : p, q \in \mathbb{R}\} = \sup\{\dim_H \nu : \nu \in \mathcal{M}\}.$$

4.2. **Characterization.**

We now provide a characterization of the measures of maximal dimension under the assumption that neither ϕ_u nor ϕ_s is cohomologous to a constant.

Theorem 10. *Let f be a $C^{1+\varepsilon}$ surface diffeomorphism, and let Λ be a compact locally maximal hyperbolic set of f such that $f|\Lambda$ is topologically mixing, and neither ϕ_u nor ϕ_s is cohomologous to a constant.*

1. *If no nontrivial linear combination of ϕ_u and ϕ_s is cohomologous to a constant and $\mu = \nu_{p,q}$ is a measure of maximal dimension, then $0 \le p \le t_u$ and $0 \le q \le t_s$.*
2. *If some nontrivial linear combination $-\phi_u + \alpha\phi_s$ is cohomologous to a constant (in which case $\alpha > 0$) and $\mu \in \mathcal{M}_E$ is a measure of maximal dimension, then there exists*
$$t \in [\min\{t_u, \alpha^{-1}t_s\}, \max\{t_u, \alpha^{-1}t_s\}] \tag{45}$$
such that $\mu = \nu_{\kappa t,(1-\kappa)\alpha t}$ for every $\kappa \in [0, 1]$.

Proof. Assume first that no nontrivial linear combination of ϕ_u and ϕ_s is cohomologous to a constant and let ν_{p_0,q_0} be an ergodic measure of maximal dimension. In particular, ϕ_s is not cohomologous to a constant. Set $b = \lambda_s(p_0, q_0)$ and consider the curve γ_s given by Proposition 5. The uniqueness of γ_s implies that $q_0 = \gamma_s(p_0)$.

Lemma 10. *The function $p \mapsto \lambda_u(p, \gamma_s(p))$ is strictly decreasing.*

Proof of the lemma. It follows from the definition of γ_s that

$$\partial_p\lambda_s(p, \gamma_s(p)) + \partial_q\lambda_s(p, \gamma_s(p))\gamma_s'(p) = 0,$$

and hence (17) implies that $\gamma_s'(p) = -\partial_p\partial_q Q/\partial_q^2 Q$, with the derivatives at the right-hand side computed at $(p, \gamma_s(p))$. Again applying (17) yields

$$\begin{aligned}\partial_p\lambda_u(p, \gamma_s(p)) + \partial_q\lambda_u(p, \gamma_s(p))\gamma_s'(p) &= -\partial_p^2 Q - \partial_p\partial_q Q(-\partial_p\partial_q Q/\partial_q^2 Q)\\ &= -[\partial_p^2 Q\partial_q^2 Q - (\partial_p\partial_q Q)^2]/\partial_q^2 Q.\end{aligned} \tag{46}$$

Consider now the bilinear form

$$A(\varphi_1, \varphi_2) = \partial_{t_1}\partial_{t_2}P(-p\phi_u + q\phi_s + t_1\varphi_1 + t_2\varphi_2)|_{t_1=t_2=0}.$$

Then $A(v\phi_u + w\phi_s, v\phi_u + w\phi_s)$ coincides with

$$\begin{pmatrix} v & w\end{pmatrix}\begin{pmatrix} A(\phi_u,\phi_u) & A(\phi_u,\phi_s)\\ A(\phi_s,\phi_u) & A(\phi_s,\phi_s)\end{pmatrix}\begin{pmatrix} v\\ w\end{pmatrix} = \begin{pmatrix} v & w\end{pmatrix} B \begin{pmatrix} v\\ w\end{pmatrix},$$

where

$$B = \begin{pmatrix} \partial_p^2 Q & -\partial_p\partial_q Q\\ -\partial_q\partial_p Q & \partial_q^2 Q\end{pmatrix}.$$

Since no nontrivial linear combination of ϕ_u and ϕ_s is cohomologous to a constant, if $(v, w) \ne 0$ then $A(v\phi_u + w\phi_s, v\phi_u + w\phi_s) > 0$ (see [12]) and hence B is positive definite. In particular $\det B$ (which coincides with the quantity in square brackets in (46)) is positive. Furthermore, since ϕ_s is not cohomologous to a constant we have $\partial_q^2 Q > 0$ and hence

$$\partial_p \lambda_u(p, \gamma_s(p)) + \partial_q \lambda_u(p, \gamma_s(p))\gamma_s'(p) < 0.$$

This establishes the desired statement. $\square$

Assume now that $p_0 < 0$. Lemma 10 implies that

$$\lambda_u(p_0, q_0) > \lambda_u(0, \gamma_s(0)). \tag{47}$$

Since $\nu_{0,\gamma_s(0)}$ is the equilibrium measure of $\gamma_s(0)\phi_s$ we have

$$h(0, \gamma_s(0)) + \gamma_s(0)\lambda_s(0, \gamma_s(0)) \geq h(p_0, q_0) + \gamma_s(0)\lambda_s(p_0, q_0).$$

Furthermore $\lambda_s(0, \gamma_s(0)) = \lambda_s(p_0, q_0)$ and hence,

$$h(0, \gamma_s(0)) \geq h(p_0, q_0). \tag{48}$$

Combining (47) and (48) with (9) yields

$$\frac{h(p_0, q_0)}{\lambda_u(p_0, q_0)} - \frac{h(p_0, q_0)}{\lambda_s(p_0, q_0)} < \frac{h(0, \gamma_s(0))}{\lambda_u(0, \gamma_s(0))} - \frac{h(0, \gamma_s(0))}{\lambda_s(0, \gamma_s(0))}.$$

But this contradicts the fact that ν_{p_0,q_0} is an ergodic measure of maximal dimension. This shows that $p_0 \geq 0$. Replacing m by ν_{p_0,q_0} and applying the same arguments as in the proof of Lemma 1 (see equations (27)–(32)) yields $p_0 \leq h(p_0, q_0)/\lambda_u(p_0, q_0)$. Therefore (15) implies $p_0 \leq t_u$. The number p in Lemma 1 is a priori not necessarily unique. However, the uniqueness follows immediately from Lemma 10. Similar arguments show that $0 \leq q_0 \leq t_s$.

Assume now that some nontrivial linear combination $-\phi_u + \alpha\phi_s$ is cohomologous to a constant. This implies that

$$\nu_{t,0} = \nu_{0,\alpha t} = \nu_{\kappa t,(1-\kappa)\alpha t} \tag{49}$$

for every $\kappa \in [0, 1]$. Therefore, the function $d_u + d_s$ is determined by the function $t \mapsto \dim_H \nu_{t,0}$. By Proposition 4, $d_u(\cdot, 0)$ is strictly increasing on $(-\infty, t_u]$ and strictly decreasing on $[t_u, \infty)$. Furthermore $d_s(0, \cdot)$ is strictly increasing on $(-\infty, t_s]$ and strictly decreasing on $[t_s, \infty)$. It follows from (49) that $t \mapsto d_u(t, 0) + d_s(t, 0)$ is strictly increasing on $(-\infty, \min\{t_u, \alpha^{-1}t_s\}]$ and strictly decreasing on $[\max\{t_u, \alpha^{-1}t_s\}, \infty)$. Therefore, its maximum can only be attained at some point t as in (45).

If μ is an ergodic measure of maximal dimension, it follows from Corollary 8, Lemma 8, and (49) that there exists t as in (45) such that $\mu = \nu_{t,0}$. This completes the proof. $\square$

We remark that an alternative proof of Lemma 10 can be obtained using the curves γ_u and γ_s. In the following we provide a brief sketch. Let $p_1, p_2 \in \mathbb{R}$ with $p_1 < p_2$ and denote by γ_u^1, γ_u^2 the unique real-analytic functions satisfying $\lambda_u(\gamma_u^k(q), q) = \lambda_u(p_k, \gamma_s(p_k))$ for all $q \in \mathbb{R}$ and $k = 1, 2$. Since no nontrivial linear combination of ϕ_u and ϕ_s is cohomologous to a constant, one can show that the curves γ_u^1 and γ_u^2 do not intersect in $\mathbb{R}^2$ and hence $p_1 < \gamma_u^2(\gamma_s(p_1))$. The result follows from the fact that $\lambda_u(\cdot, \gamma_s(p_1))$ is strictly decreasing (see Proposition 4).

The function $d_s(\cdot, 0) + d_u(\cdot, 0)$ is real-analytic. Therefore in the case when $d_s(\cdot, 0) + d_u(\cdot, 0)$ is not constant then it has at most finitely many maxima in $[\min\{t_u, \alpha^{-1}t_s\}, \max\{t_u, \alpha^{-1}t_s\}]$. Under the assumptions of Theorem 10 it is thus an immediate consequence that if ϕ_u and ϕ_s are linearly dependent as cohomology classes then there exist at most finitely many ergodic measures of maximal dimension. We note that an analogous result to that in statement 2 of Theorem 10 was obtained earlier in the case of hyperbolic polynomial automorphisms of $\mathbb{C}^2$ (see [16]).

5. Applications

We now provide several nontrivial applications of the results in the former section.

5.1. Regular dependence on the diffeomorphism.

In this section we discuss the dependence of the Hausdorff dimension of the measure of maximal dimension (given by Theorem 6) on the diffeomorphism. Namely, we are interested how the quantity $\delta(f) = \sup\{\dim_H \nu : \nu \in \mathcal{M}\}$ varies with f (restricted to the locally maximal hyperbolic set Λ).

More precisely, consider an open neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U$. Then (see for example [8]) there exists a neighborhood $\mathcal{U}$ of f in the space of C^r diffeomorphisms (with respect to the C^r topology) such that $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n U$ is a locally maximal hyperbolic set of g for all $g \in \mathcal{U}$. Moreover, $g|\Lambda_g$ is topologically mixing. We are interested in the regularity of the map $g \mapsto \delta(g)$. One might think that the regularity of this map could be determined from that of the Hausdorff dimension of the set Λ_g. In fact it was proven by Mañé in [9] that the map $g \mapsto \dim_H \Lambda_g$ is of class C^{r-1}. On the other hand, it follows from work of McCluskey and Manning in [10] that the identity $\delta(f) = \dim_H \Lambda$ fails whenever $-t_u\phi_u$ and $t_s\phi_s$ are not cohomologous, and thus one can show that for every g in an open and dense subset of $\mathcal{U}$ (with respect to the C^r topology) we have $\delta(g) < \dim_H \Lambda$. This indicates that a priori one cannot study the regularity of the map $g \mapsto \delta(g)$ using information about the regularity of the map $g \mapsto \dim_H \Lambda_g$. It is shown in [10] that $g \mapsto \delta(g)$ is continuous.

We shall discuss the higher regularity of the map $g \mapsto \delta(g)$ under some reasonable assumptions. It is well-known that there exists a neighborhood $\mathcal{U}$ of f and $\alpha \in (0, 1]$ such that for all $g \in \mathcal{U}$ there exists a unique α-Hölder homeomorphism $h_g \colon \Lambda \to \Lambda_g$ satisfying $g \circ h_g = h_g \circ f$ (see for example [8]). For a given $g \in \mathcal{U}$ we denote by $\mathcal{M}_g$ the space of g-invariant probability measures on Λ_g and by $\mathcal{M}_{E,g}$ the subset of ergodic measures. For each $g \in \mathcal{U}$ we define a map $T_g \colon \mathcal{M} \to \mathcal{M}_g$ by $T_g(\nu) = (h_g)_*\nu$ where $((h_g)_*\nu)(A) = \nu(h_g^{-1}(A))$ for all Borel sets $A \subset \Lambda_g$. It is easy to see that T_g is a homeomorphism. Moreover it follows directly from the definition of T_g that $(f|\Lambda, \nu)$ and $(g|\Lambda_g, T_g(\nu))$ are measure-theoretically isomorphic, in particular $h_{T_g(\nu)}(g) = h_\nu(f)$ for all $\nu \in \mathcal{M}$ and $g \in \mathcal{U}$.

Theorem 11. *Let f be C^r surface diffeomorphism, for some $r \geq 4$, and let Λ be a compact locally maximal hyperbolic set of f such that $f|\Lambda$ topologically mixing. Assume that neither ϕ_u nor ϕ_s is cohomologous to a constant and that f admits a unique measure $\mu \in \mathcal{M}_E$ of maximal dimension. Assume furthermore that $\mu = \nu_{p_0,q_0}$ for some p_0, $q_0 \in \mathbb{R}$, and that $D^2(d_u+d_s)(p_0, q_0)$ is invertible. Then $g \mapsto \delta(g)$ is of class C^{r-3} in a neighborhood $\mathcal{U}$ of f.*

Proof. Since $\delta(g) = \sup\{\dim_H \nu : \nu \in \mathcal{M}_{E,g}\}$, it is sufficient to restrict ourselves to ergodic measures. Consider a neighborhood $\mathcal{U}$ of f such that for each $g \in \mathcal{U}$ neither ϕ_u nor ϕ_s (defined with respect to g) is cohomologous to a constant. The existence of such a neighborhood is a simple consequence of work of McCluskey and Manning in [10] combined with Livshitz's theorem. We now need the following lemma.

Lemma 11. *Let V be an open neighborhood of ν_{p_0,q_0} in $\mathcal{M}_E$. Then there exists a neighborhood $\mathcal{U}$ of f such that each measure $\nu \in \mathcal{M}_{E,g}$ of maximal dimension satisfies $\nu \in T_g(V)$.*

Proof of the lemma. It follows from work of McCluskey and Manning in [10] that if $\mathcal{U}$ is small, then the functions $\delta_V(g) = \sup\{\dim_H \nu : \nu \in T_g(V)\}$ and

$$\delta_{\mathcal{M}_E \setminus V}(g) = \sup\{\dim_H \nu : \nu \in T_g(\mathcal{M}_E \setminus V)\}$$

are continuous in g. Since $\mathcal{M}_E \setminus V$ does not contain an ergodic measure of maximal dimension we may conclude from Corollary 8 that $\delta_{\mathcal{M}_E \setminus V}(f) < \delta(f)$. In order to see this, notice that otherwise there would exist a sequence of measures $\nu_n \in \mathcal{M}_E \setminus V$ with $\dim_H \nu_n \to \delta(f)$ as $n \to \infty$. For any accumulation point $m \in \mathcal{N}$ (see (39)) of this sequence, it follows from statement 2 of Corollary 8 and the uniqueness of the measure ν_{p_0,q_0} that $\lambda_u(\mu) \notin I_u$ and $\lambda_s(\mu) \notin I_s$. By statement 3 in the same theorem there exists an ergodic measure of maximal dimension in $\mathcal{M}_E \setminus V$, but this violates the uniqueness of ν_{p_0,q_0}. By making $\mathcal{U}$ smaller if necessary, it follows from the continuous dependence of the quantities $\delta_V(g)$ and $\delta_{\mathcal{M}_E \setminus V}(g)$ on g that $\delta_{\mathcal{M}_E \setminus V}(g) < \delta(g)$ for all $g \in \mathcal{U}$ which proves the lemma. ☐

Define a function $\mathcal{Q} \colon \mathcal{U} \times \mathbb{R}^2 \to \mathbb{R}$ by $\mathcal{Q}(g, p, q) = P(g, -p\phi_u + q\phi_s)$, where $P(g, \cdot)$ denotes the topological pressure of $g|\Lambda_g$. It follows from work of Mañé in [9] that $\mathcal{Q}$ is of class C^{r-1}. Therefore (9), (17), and (20) imply that the function $\Delta \colon \mathcal{U} \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\Delta(g, p, q) = \dim_H \nu_{p,q}(g)$ is of class C^{r-2}. Here $\nu_{p,q}(g)$ denotes the equilibrium measure of $-p\phi_u + q\phi_s$ with respect to $g|\Lambda_g$ (with ϕ_u and ϕ_s defined with respect to g). Since neither ϕ_u nor ϕ_s is cohomologous to a constant we may conclude from Proposition 4 that $\lambda_u(p_0, q_0) \in I_u$ and $\lambda_s(p_0, q_0) \in I_s$. Let now V be an open neighborhood of ν_{p_0,q_0} in $\mathcal{M}_E$ such that $\lambda_u(\nu) \in I_u$ and $\lambda_s(\nu) \in I_s$ for all $\nu \in V$. Without loss of generality we may assume that $\mathcal{U}$ is the same as in Lemma 11.

Since T_g is a homeomorphism and I_u and I_s are open intervals for all $g \in \mathcal{U}$, it follows that $\lambda_u(\nu) \in I_u$ and $\lambda_s(\nu) \in I_s$ for all $\nu \in T_g(V)$. Therefore Corollary 8 and Lemma 11 imply that each measure $\mu(g) \in \mathcal{M}_{E,g}$ of maximal dimension coincides with $\nu_{p,q}(g)$ for some p, $q \in \mathbb{R}$. The map $(p, q) \mapsto \nu_{p,q}$ is real-analytic and thus in particular continuous (see [9]). This implies that every neighborhood V of ν_{p_0,q_0} in $\mathcal{M}_E$ also contains a neighborhood of ν_{p_0,q_0} in $\{\nu_{p,q} : p, q \in \mathbb{R}\}$. By making V and $\mathcal{U}$ smaller if necessary, using the fact that Δ is of class C^{r-2} we may assume that $\Delta(g, \cdot)$ has a unique extremum

$$\nu_{p(g),q(g)}(g) \in T_g(V) \cap \{\nu_{p,q}(g) : p, q \in \mathbb{R}\}$$

and this must be a maximum. We conclude that $\delta(g)$ coincides with the Hausdorff dimension of the unique measure $\nu_{p(g),q(g)}(g)$ such that

$$(\partial_p \Delta, \partial_q \Delta)(g, p(g), q(g)) = 0. \tag{50}$$

Observe that $(\partial_p\Delta, \partial_q\Delta)$ is of class C^{r-3}. Applying the Implicit Function Theorem to equation (50) completes the proof of the theorem. □

We remark that the assumptions of Theorem 11 for example hold if f preserves volume (provided that neither ϕ_u nor ϕ_s is cohomologous to a constant), and more generally when f admits an ergodic invariant measure of full dimension (see Section 2 for the definition).

5.2. **Gibbs property of measures of maximal dimension.** We now provide conditions which imply that every ergodic measure of maximal dimension is an equilibrium measure of a Hölder continuous function, and thus has the Gibbs property. We continue to use the notation of Section 5.1.

Theorem 12. *Let f be C^2 surface diffeomorphism, and let Λ be a compact locally maximal hyperbolic set of f such that $f|\Lambda$ is topologically mixing. Assume that no measure of maximal dimension m satisfies $\lambda_u(m) = \lambda_u^{\min}$ and $\lambda_s(m) = \lambda_s^{\max}$. Then there exists an open neighborhood $\mathcal{U}$ of f in the C^2 topology such that for every $g \in \mathcal{U}$ each ergodic measure of maximal dimension μ_g of $g|\Lambda_g$ is an equilibrium measure of a Hölder continuous function.*

Proof. We first consider the map f. The assumptions imply that ϕ_u and ϕ_s cannot be simultaneously cohomologous to a constant. This follows from Theorem 6. We define

$$\delta_b(f) = \sup\{d(\nu) : \lambda_u(\nu) = \lambda_u^{\min} \text{ and } \lambda_s(\nu) = \lambda_s^{\max}\}$$

(with "b" standing for boundary case). In a similar way to that in the proof of Lemma 7 we are able to show that there exists $\nu \in \mathcal{M}_E$ with $\lambda_u(\nu) = \lambda_u^{\min}$ and $\lambda_s(\nu) = \lambda_s^{\max}$ such that $\dim_H \nu = \delta_b(f)$. By Theorem 6 there exists an ergodic measure μ of maximal dimension for $f|\Lambda$, and therefore the hypothesis in the theorem implies that $\delta_b(f) < \delta(f)$. By Corollary 8 we conclude that $\mu = \nu_{p,q}$ for some p, $q \in \mathbb{R}$.

McCluskey and Manning [10] showed that $g \mapsto \delta(g)$ is continuous in the C^2 topology. Moreover, one can modify their proof to show that $g \mapsto \delta_b(g)$ is also continuous. Therefore, there exists an open neighborhood $\mathcal{U}$ of f in the C^2 topology such that $\delta_b(g) < \delta(g)$ for all $g \in \mathcal{U}$. In particular, if $g \in \mathcal{U}$ then no measure m_g of maximal dimension for $g|\Lambda_g$ satisfies $\lambda_u(m_g) = \lambda_u^{\min}$ and $\lambda_s(m_g) = \lambda_s^{\max}$. Arguing as above (but now for $g \in \mathcal{U}$ instead of f), we conclude that every ergodic measure of maximal dimension for $g|\Lambda_g$ is an equilibrium measure of a Hölder continuous function. □

It follows immediately from the proof of Theorem 6 that if ϕ_u and ϕ_s are both cohomologous to a constant, then the measure of maximal entropy is the unique ergodic measure of maximal dimension. To our best knowledge, all known examples of hyperbolic surface diffeomorphisms satisfy one of the following exclusive alternatives:

1. no measure of maximal dimension m satisfies $\lambda_u(m) = \lambda_u^{\min}$ and $\lambda_s(m) = \lambda_s^{\max}$;
2. ϕ_u and ϕ_s are cohomologous to a constant.

Conjecturally these two cases cover all possibilities. This would imply that every ergodic measure of maximal dimension has the Gibbs property.

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