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PESIN THEORY - An important branch of **dynamical systems** and of smooth **ergodic theory**, with many applications to non-linear dynamics. The name is due to the landmark work of Yakov B. Pesin in the mid-seventies [20, 21, 22]. Sometimes it is also referred to as the *theory of smooth dynamical systems with non-uniformly hyperbolic behavior*, or simply *theory of non-uniformly hyperbolic dynamical systems*.

A. Introduction. One of the paradigms of dynamical systems is that the local instability of trajectories influences the global behavior of the system, and paves the way to the existence of stochastic behavior. Mathematically, the instability of trajectories corresponds to some degree of hyperbolicity (cf. **Hyperbolic set**). The "strongest possible" hyperbolicity occurs in the important class of Anosov systems (also called **Y-systems**) [1]. These are only known to occur in some manifolds. Moreover there are several results of topological nature showing that some manifolds cannot carry Anosov systems.

Pesin Theory deals with a "weaker" hyperbolicity, a much more common property that is believed to be "typical": non-uniform hyperbolicity. Among the most important features due to hyperbolicity is the existence of invariant families of stable and unstable manifolds and their "absolute continuity". The combination of hyperbolicity with non-trivial recurrence produces a rich and complicated orbit structure. The theory also describes the ergodic properties of smooth dynamical systems possessing an absolutely continuous invariant measure in terms of the Lyapunov exponents. One of the most striking consequences is the Pesin entropy formula that expresses the metric entropy of the dynamical system through its Lyapunov exponents.

B. The Concept of Non-Uniform Hyperbolicity. Let $f\colon M \to M$ be a diffeomorphism of a compact manifold. It induces the discrete dynamical system (or **cascade**) composed of the powers $\{f^n : n \in \mathbb{Z}\}$. We fix a Riemannian metric on M. The trajectory $\{f^n x : n \in \mathbb{Z}\}$ of a point $x \in M$ is called *non-uniformly hyperbolic* if there are positive numbers $\lambda < 1 < \mu$ and splittings $T_{f^n x}M = E^u(f^n x) \oplus E^s(f^n x)$ for each $n \in \mathbb{Z}$, and if for all sufficiently small $\varepsilon > 0$ there is a positive function C_ε on the trajectory such that for every $k \in \mathbb{Z}$:

1. $C_\varepsilon(f^k x) \le e^{\varepsilon|k|} C_\varepsilon(x)$;
2. $Df^k E^u(x) = E^u(f^k x)$, $Df^k E^s(x) = E^s(f^k x)$;
3. if $v \in E^u(f^k x)$ and $m < 0$, then
$$\|Df^m v\| \le C_\varepsilon(f^{m+k}x)\mu^m \|v\|;$$
4. if $v \in E^s(f^k x)$ and $m > 0$, then
$$\|Df^m v\| \le C_\varepsilon(f^{m+k}x)\lambda^m \|v\|;$$
5. $\sphericalangle(E^u(f^k x), E^s(f^k x)) \ge C_\varepsilon(f^k x)^{-1}$.

(The indices "s" and "u" refer, respectively, to "stable" and "unstable".) We obtain the definition of *non-uniformly partially hyperbolic* trajectory if we substitute the inequality $\lambda < 1 < \mu$ by the weaker requirement that $\lambda < \mu$ and $\min\{\lambda, \mu^{-1}\} < 1$.

If $\lambda < 1 < \mu$ (resp. $\lambda < \mu$ and $\min\{\lambda, \mu^{-1}\} < 1$) and the conditions 1.-5. hold for $\varepsilon = 0$ (i.e., if one can choose C_ε = constant), the trajectory is called *uniformly hyperbolic* (resp. *uniformly partially hyperbolic*).

The term "non-uniformly" means that the estimates in 3. and 4. may differ from the "uniform" estimates μ^m and λ^m by at most slowly increasing terms along the trajectory, as in 1. (in the sense that the exponential rate ε in 1. is small compared to the number $\max\{\log\mu, -\log\lambda\}$); the term "partially" means that the hyperbolicity may hold only for a part of the tangent space.

In a similar way we define the corresponding notions for a **flow** (or **continuous-time dynamical system**) with $k \in \mathbb{Z}$ substituted by $k \in \mathbb{R}$, and the splitting of the tangent spaces substituted by $T_x M = E^u(x) \oplus E^s(x) \oplus X(x)$, where $X(x)$ is the one-dimensional subspace generated by the flow direction.

C. Stable and Unstable Manifolds. Let $\{f^n x : n \in \mathbb{Z}\}$ be a non-uniformly partially hyperbolic trajectory of a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism. We assume that $\lambda < 1$. Then there is a *local stable manifold* $V^s(x)$ such that $x \in V^s(x)$, $T_x V^s(x) = E^s(x)$, and for every $y \in V^s(x)$, $k \in \mathbb{Z}$, and $m > 0$,

$$d(f^{m+k}x, f^{m+k}y) \le K C_\varepsilon(f^k x)\lambda^m e^{\varepsilon m} d(f^k x, f^k y),$$

where d is the distance induced by the Riemannian metric and K is a positive constant. The size $r(x)$ of $V^s(x)$ can be chosen in such a way that $r(f^k x) \ge K' e^{-\varepsilon|k|} r(x)$ for every $k \in \mathbb{Z}$, where K' is a positive constant. If f is of class $C^{r+\alpha}$ ($\alpha > 0$), then $V^s(x)$ is of class C^r.

We define the *global stable manifold* of f at x by $W^s(x) = \bigcup_{k\in\mathbb{Z}} f^{-k}(V^s(f^k x))$; it is an immersed manifold with the same class of smoothness of $V^s(x)$. We have $W^s(x) \cap W^s(y) = \varnothing$ if $y \notin W^s(x)$, $W^s(x) = W^s(y)$ if $y \in W^s(x)$, and $f^n W^s(x) = W^s(f^n x)$ for every $n \in \mathbb{Z}$. The manifold $W^s(x)$ is independent of the particular size of the local stable manifolds $V^s(y)$.

In a similar way, when $\mu > 1$ we can define a *local* (resp. *global*) *unstable manifold* as a local (resp. global) stable manifold of f^{-1}.

D. Non-Uniformly Hyperbolic Dynamical Systems and Dynamical Systems with Non-Zero Lyapunov Exponents. Let $f\colon M \to M$ be a diffeomorphism and let ν be a (finite) Borel **f-invariant measure**. We call f *non-uniformly hyperbolic* (resp. *non-uniformly partially hyperbolic*) with respect to the measure ν if the set $\Lambda \subset M$ of points whose trajectories are *non-uniformly hyperbolic* (resp. *non-uniformly partially hyperbolic*) is such that $\nu(\Lambda) > 0$. In this case λ, μ, ε, and C_ε are replaced by measurable functions $\lambda(x)$, $\mu(x)$, $\varepsilon(x)$, and $C_\varepsilon(x)$.

The set Λ is f-invariant, i.e., it satisfies $f\Lambda = \Lambda$. Therefore, we can always assume that $\nu(\Lambda) = 1$ when $\nu(\Lambda) > 0$; this means that if $\nu(\Lambda) > 0$, then the measure $\widehat{\nu}$ on Λ defined by $\widehat{\nu}(B) = \nu(B)/\nu(\Lambda)$ is f-invariant and $\widehat{\nu}(\Lambda) = 1$.

For $(x, v) \in M \times T_xM$, we define the *forward upper Lyapunov exponent* of (x, v) (with respect to f) by

$$\chi(x, v) = \limsup_{m\to+\infty} \frac{1}{m} \log\|Df^m v\| \tag{1}$$

for each $v \neq 0$, and $\chi(x, 0) = -\infty$. For every $x \in M$, there exist a positive integer $s(x) \leq \dim M$ (the dimension of M), and collections of numbers $\chi_1(x) < \cdots < \chi_{s(x)}(x)$ and linear subspaces $E_1(x) \subset \cdots \subset E_{s(x)}(x) = T_xM$ such that for every $i = 1, \ldots, s(x)$,

$$E_i(x) = \{v \in T_xM : \chi(x, v) \leq \chi_i(x)\},$$

and if $v \in E_i(x) \setminus E_{i-1}(x)$, then $\chi(x, v) = \chi_i(x)$.

The numbers $\chi_i(x)$ are called the *forward upper Lyapunov exponents at* x, and the collection of linear subspaces $E_i(x)$ is called the *forward filtration at* x *associated to* f. The number $k_i(x) = \dim E_i(x) - \dim E_{i-1}(x)$ is the *forward multiplicity* of the exponent $\chi_i(x)$. We define the *forward spectrum of* f *at* x as the collection of pairs $(\chi_i(x), k_i(x))$ for $i = 1, \ldots, s(x)$. Let $\chi'_1(x) \leq \cdots \leq \chi'_{\dim M}(x)$ be the forward upper Lyapunov exponents at x counted with multiplicities, i.e., in such a way that the exponent $\chi_i(x)$ appears exactly a number $k_i(x)$ of times. The functions $s(x)$ and $\chi'_i(x)$ for $i = 1, \ldots, \dim M$ are measurable and f-invariant with respect to any f-invariant measure.

We define the *backward upper Lyapunov exponent* of (x, v) (with respect to f) by a similar expression to (1) with $m \to +\infty$ substituted by $m \to -\infty$, and consider the corresponding *backward spectrum*.

A Lyapunov regular trajectory $\{f^n x : n \in \mathbb{Z}\}$ (see, for example, [3, section 2]) is non-uniformly hyperbolic (resp. non-uniformly partially hyperbolic) if and only if $\chi(x, v) \neq 0$ for any $v \in T_xM$ (resp. $\chi(x, v) \neq 0$ for some $v \in T_xM$). For flows, a Lyapunov regular trajectory is non-uniformly hyperbolic if and only if $\chi(x, v) \neq 0$ for any $v \notin X(x)$.

The Oseledets' Multiplicative ergodic theorem [19] implies that ν-almost all points of M belong to a Lyapunov regular trajectory. Therefore, for a given diffeomorphism, we have $\chi(x, v) \neq 0$ for any $v \in T_xM$ (resp. $\chi(x, v) \neq 0$ for some $v \in T_xM$) on a set of positive ν-measure if and only if the diffeomorphism is non-uniformly hyperbolic (resp. non-uniformly partially hyperbolic). Hence, the non-uniformly hyperbolic diffeomorphisms (with respect to the measure ν) are precisely the diffeomorphisms with non-zero Lyapunov exponents (on a set of positive ν-measure).

Furthermore, for ν-almost every $x \in \Lambda$ there exist subspaces $H_j(x)$ for $j = 1, \ldots, s(x)$ such that for every $i = 1, \ldots, s(x)$, we have $E_i(x) = \bigoplus_{j=1}^{i} H_j(x)$,

$$\lim_{m\to\pm\infty} \frac{1}{m} \log\|Df^m v\| = \chi_i(x)$$

for every $v \in H_i(x) \setminus \{0\}$, and if $i \neq j$, then

$$\lim_{m\to\pm\infty} \frac{1}{m} \log|\sphericalangle(H_i(f^m x), H_j(f^m x))| = 0.$$

E. Pesin Sets. To a non-uniformly partially hyperbolic diffeomorphism one associates a filtration of measurable sets (not necessarily invariant) where the estimates 3.-5. are uniform.

Let f be a non-uniformly hyperbolic diffeomorphism and let $C(x) = C_{\varepsilon(x)}(x)$. Given $\ell > 0$, we define the measurable set Λ_ℓ by

$$\left\{ x \in \Lambda : C(x) \leq \ell, \lambda(x) \leq \frac{\ell - 1}{\ell} < \frac{\ell + 1}{\ell} \leq \mu(x) \right\}.$$

We have $\Lambda_\ell \subset \Lambda_L$ when $\ell \leq L$, and $\bigcup_{\ell>0} \Lambda_\ell = \Lambda$ (mod 0). Each set Λ_ℓ is closed but need not be f-invariant; for every $m \in \mathbb{Z}$ and $\ell > 0$ there exists $L = L(m, \ell)$ such that $f^m \Lambda_\ell \subset \Lambda_L$. The distribution $E^s(x)$ is in general only measurable on Λ but it is continuous on Λ_ℓ. The local stable manifolds $V^s(x)$ depend continuously on $x \in \Lambda_\ell$ and their sizes are uniformly bounded from below on Λ_ℓ. Each set Λ_ℓ is called a *Pesin set*.

In a similar way, one defines Pesin sets for arbitrary non-uniformly partially hyperbolic diffeomorphisms.

F. Lyapunov Metrics and Regular Neighborhoods. Let $\langle \cdot, \cdot \rangle$ be the Riemannian metric on TM. For each fixed $\varepsilon > 0$ and every $x \in \Lambda$, we define a *Lyapunov metric* on $H_i(x)$ by

$$\langle u, v \rangle'_x = \sum_{m\in\mathbb{Z}} \langle Df^m u, Df^m v \rangle_x e^{-2m\chi_i(x) - 2\varepsilon|m|},$$

for each $u, v \in H_i(x)$. We extend this metric to T_xM by declaring orthogonal the subspaces $H_i(x)$ for $i = 1, \ldots, s(x)$. The metric $\langle \cdot, \cdot \rangle'$ is continuous on Λ_ℓ. The sequence of weights $\{e^{-2m\chi_i(x) - 2\varepsilon|m|}\}_{m\in\mathbb{Z}}$ is called a *Pesin Tempering Kernel*. Any linear operator $L_\varepsilon(x)$ on T_xM such that

$$\langle u, v \rangle'_x = \langle L_\varepsilon(x)u, L_\varepsilon(x)v \rangle_x$$

is called a *Lyapunov change of coordinates*.

There exist a measurable function $q \colon \Lambda \to (0, 1]$ satisfying $e^{-\varepsilon} \leq q(fx)/q(x) \leq e^\varepsilon$, and for each $x \in \Lambda$ a collection of embeddings $\Psi_x \colon B(0, q(x)) \to M$, defined on the ball $B(0,q(x))$ of T_xM by $\Psi_x = \exp_x \circ L_\varepsilon(x)^{-1}$, such that if $f_x = \Psi_{fx}^{-1} \circ f \circ \Psi_x$, then:

1. the derivative $D_0 f_x$ of f_x at the point 0 has the *Lyapunov block form*

$$D_0 f_x = \begin{pmatrix} A_1(x) & & \\ & \ddots & \\ & & A_{s(x)}(x) \end{pmatrix},$$

where each $A_i(x)$ is an invertible linear operator between the $k_i(x)$-dimensional spaces $L_\varepsilon(x)H_i(x)$ and $L_\varepsilon(fx)H_i(fx)$, for $i = 1, \ldots, s(x)$;

2. for each $i = 1, \ldots, s(x)$,

$$e^{\chi_i(x)-\varepsilon} \le \|A_i(x)^{-1}\|^{-1},\ \|A_i(x)\| \le e^{\chi_i(x)+\varepsilon};$$

3. the C^1-distance between f_x and $d_0 f_x$ on the ball $B(0,q(x))$ is at most ε;

4. there exist a constant K and a measurable function $A\colon \Lambda \to \mathbb{R}$ satisfying $e^{-\varepsilon} \le A(fx)/A(x) \le e^{\varepsilon}$ such that for every $y, z \in B(0,q(x))$,

$$Kd(\Psi_x y, \Psi_x z) \le \|y - z\| \le A(x) d(\Psi_x y, \Psi_x z).$$

The function $A(x)$ is bounded on each Λ_ℓ. The set $\Psi_x(B(0,q(x))) \subset M$ is called a *regular neighborhood* of the point x.

G. Absolute Continuity. A property playing a crucial role in the study of the ergodic properties of (uniformly and non-uniformly) hyperbolic dynamical systems is the absolute continuity of the families of stable and unstable manifolds. It allows us to pass from the local properties of the system to the study of its global behavior.

Let ν be an absolutely continuous f-invariant measure, i.e., an f-invariant measure which is absolutely continuous with respect to Lebesgue measure. For each $x \in \Lambda$ and $\ell > 0$ there exists a neighborhood $U(x)$ of x with size depending only on ℓ and with the following properties (see [21]). Choose $y \in \Lambda_\ell \cap U(x)$. Given two smooth manifolds W_1, $W_2 \subset U(x)$ transversal to the local stable manifolds in $U(x)$, we define

$$A_i = \{w \in W_i \cap V^s(z) : z \in \Lambda_\ell \cap U(x)\}$$

for $i = 1, 2$. Let $p\colon A_1 \to A_2$ be the correspondence that takes $w \in W_1$ into the point $p(w) \in W_2$ such that $w, p(w) \in V^s(z)$ for some z. If ν_i is the measure induced on W_i by the Riemannian metric, for $i = 1, 2$, then $p^*\nu_1$ is absolutely continuous with respect to ν_2 (if ℓ is sufficiently large, then $\nu_i(A_i) > 0$ for $i = 1, 2$).

This result has the following consequences (see [21]). For each measurable set $B \subset W_1 \cap \Lambda_\ell$, let $\widehat{B}$ be the union of all the sets $V^s(z) \cap U(x)$ such that $z \in \Lambda_\ell$ and $V^s(z) \cap B \ne \varnothing$. The partition of $\widehat{B}$ into the submanifolds $V^s(z)$ is a measurable partition (also called **measurable decomposition**), and the corresponding conditional measure of ν on $V^s(z)$ is absolutely continuous with respect to the measure ν_z induced on $V^s(z)$ by the Riemannian metric, for each $z \in \Lambda_\ell$ such that $V^s(z) \cap B \ne \varnothing$. In addition $\nu_z(V^s(z)) > 0$ for ν-almost all $z \in \widehat{B} \cap \Lambda_\ell$, and the measure $\widehat{\nu}$ on W_1 defined for each measurable set B by $\widehat{\nu}(B) = \nu(\widehat{B})$ is absolutely continuous with respect to ν_1.

H. Smooth Ergodic Theory. Let $f\colon M \to M$ be a non-uniformly hyperbolic $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism with respect to a *Sinai–Ruelle–Bowen-measure* ν, i.e., an f-invariant measure ν that has a non-zero Lyapunov exponent ν-almost everywhere and has absolutely continuous conditional measures on stable (or unstable) manifolds with respect to Lebesgue measure (in particular, this holds if ν is absolutely continuous with respect to Lebesgue measure and has no zero Lyapunov exponents [21]; see Section G). Then there is at most a countable number of disjoint f-invariant sets $\Lambda_0, \Lambda_1, \ldots$ (the ergodic components) such that [21, 11]:

1. $\bigcup_{i\ge 0} \Lambda_i = \Lambda$, $\nu(\Lambda_0) = 0$, and $\nu(\Lambda_i) > 0$ and $f|\Lambda_i$ is ergodic (see **ergodicity**) with respect to $\nu|\Lambda_i$ for every $i > 0$;

2. each set Λ_i is a disjoint union of sets $\Lambda_{i1}, \ldots, \Lambda_{in_i}$ such that $f(\Lambda_{ij}) = \Lambda_{i,j+1}$ for each $j < n_i$, and $f(\Lambda_{in_i}) = \Lambda_{i1}$;

3. for every i and j, there is a **metric isomorphism** between $f^{n_i}|\Lambda_{ij}$ and a **Bernoulli automorphism** (in particular, the map $f^{n_i}|\Lambda_{ij}$ is a K**-system**).

If ν is an absolutely continuous f-invariant measure and the foliation W^s (or W^u) of Λ is C^1-continuous (i.e., for each $x \in \Lambda$ there is a neighborhood of x in $W^s(x)$ which is the image of an injective C^1 map φ_x, defined on the ball of center at 0 and radius 1, and the map $x \mapsto \varphi_x$ from Λ into the family of C^1 maps is continuous), then any ergodic component of positive ν-measure is a (mod 0) open set; if in addition $f|\Lambda$ is topologically transitive, then $f|\Lambda$ is ergodic [21].

If $f|\Lambda$ is ergodic, then for Lebesgue almost every point $x \in M$ and every continuous function g, we have $\frac{1}{n}\sum_{k=0}^{n-1} g(f^k x) \to \int_M g\,d\nu$ as $n \to +\infty$.

There is a measurable partition η of M with the following properties:

1. for ν-almost every $x \in M$ the element $\eta(x)$ of η containing x is a (mod 0) open subset of $W^s(x)$;

2. $f\eta$ is a refinement of η, and $\bigvee_{k=0}^{\infty} f^k\eta$ is the partition of M into points;

3. $\bigwedge_{k=0}^{\infty} f^{-k}\eta$ coincides with the measurable hull of W^s, as well as with the maximal partition with zero entropy (the π-partition for f; see **entropy of a measurable partition**);

4. $h_\nu(f) = h_\nu(f,\eta)$ (cf. **entropy theory of a dynamical system**).

I. Pesin Entropy Formula. For a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism $f\colon M \to M$ of a compact manifold and an absolutely continuous f-invariant probability measure ν, the **metric entropy** $h_\nu(f)$ of f with respect to ν is given by the *Pesin Entropy Formula* [21]

$$h_\nu(f) = \int_M \sum_{i=1}^{s(x)} \chi_i^+(x) k_i(x)\, d\nu(x), \tag{2}$$

where $\chi_i^+(x) = \max\{\chi_i(x), 0\}$ and $(\chi_i(x), k_i(x))$ form the forward spectrum of f at x (see Section D).

For a C^1 diffeomorphism $f\colon M \to M$ of a compact manifold and an f-invariant probability measure ν, we have the *Ruelle Inequality* [25]

$$h_\nu(f) \le \int_M \sum_{i=1}^{s(x)} \chi_i^+(x) k_i(x)\, d\nu(x). \tag{3}$$

An important consequence of (3) is that any C^1 diffeomorphism with positive topological entropy has an f-invariant measure with at least one positive and one negative Lyapunov exponent; in particular, for surface diffeomorphisms there is an f-invariant measure with every exponent non-zero. For arbitrary invariant measures the inequality (3) may be strict [7].

The formula (2) was first established by Pesin [21]. A proof which does not use the theory of invariant manifolds and the absolute continuity was given by Mañé [17]. For C^2 diffeomorphisms, the formula (2) holds if and only if ν has absolutely continuous conditional measures on unstable manifolds [13, 12].

The formula (2) was extended by Ledrappier and Strelcyn to maps with singularities [12]. For C^2 diffeomorphisms and arbitrary invariant measures, results of Ledrappier and Young [14] show that the possible defect between the left and right-hand sides of (3) is due to the defects between $\dim E_i(x)$ and the Hausdorff dimension of ν "in the direction of $E_i(x)$" for each i.

J. Hyperbolic Measures. Let f be a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism and let ν be an f-invariant measure. We say that ν is *hyperbolic* (with respect to f) if $\chi_i(x) \neq 0$ for ν-almost every $x \in M$ and all $i = 1$, …, $s(x)$. The measure ν is hyperbolic (with respect to f) if and only if f is non-uniformly hyperbolic with respect to ν (and the set Λ has full ν-measure). The fundamental work of A. Katok revealed a rich and complicated orbit structure for diffeomorphisms possessing an hyperbolic measure.

Let ν be a hyperbolic measure. The support of ν is contained in the closure of the set of periodic points. If ν is ergodic and not concentrated on a periodic orbit, then [7, 9]:

1. the support of ν is contained in the closure of the set of hyperbolic periodic points possessing a transversal homoclinic point;

2. for every $\varepsilon > 0$ there exists a closed f-invariant hyperbolic set Γ such that: the restriction of f to Γ is topologically conjugate to a topological Markov chain with topological entropy $h(f|\Gamma) \ge h_\nu(f) - \varepsilon$, i.e., the entropy of a hyperbolic measure can be approximated by the topological entropies of invariant hyperbolic sets.

If f possesses a hyperbolic measure, then f satisfies a *closing lemma*: given $\varepsilon > 0$, there exists $\delta = \delta(\ell, \varepsilon) > 0$ such that for each $x \in \Lambda_\ell$ and each integer m satisfying $f^m x \in \Lambda_\ell$ and $d(x, f^m x) < \delta$, there exists a point y such that $f^m y = y$, $d(f^k x, f^k y) < \varepsilon$ for every $k = 0$, …, m, and y is an hyperbolic periodic point [7]. The diffeomorphism f also satisfies a *shadowing lemma* (see [9]) and a *Livschitz-type theorem* [9]: if φ is a Hölder continuous function such that $\sum_{k=0}^{m-1} \varphi(f^k p) = 0$ for each periodic point p with $f^m p = p$, then there is a measurable function h such that $\varphi(x) = h(fx) - h(x)$ for ν-almost every x.

Let $P_n(f)$ be the number of periodic points of f with period n. If f possesses a hyperbolic measure or is a surface diffeomorphism, then

$$\limsup_{n\to+\infty} \frac{1}{n} \log^+ P_n(f) \ge h(f),$$

where $h(f)$ is the topological entropy of f [7].

Let ν be a hyperbolic ergodic measure. Barreira, Pesin and Schmeling [2] showed that there is a constant d such that for ν-almost every $x \in M$,

$$\lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} = d,$$

where $B(x, r)$ is the ball in M with center at x and radius r (this claim was known as the *Eckmann–Ruelle Conjecture*); this implies that the Hausdorff dimension of ν, and lower and upper box dimensions of ν coincide and are equal to d (see [2]). Ledrappier and Young [14] showed that if ν_x^s (resp. ν_x^u) are the conditional measures of ν with respect to the stable (resp. unstable) manifolds, then there are constants d^s and d^u such that for ν-almost every $x \in M$,

$$\lim_{r\to 0} \frac{\log \nu_x^s(B^s(x,r))}{\log r} = d^s, \quad \lim_{r\to 0} \frac{\log \nu_x^u(B^u(x,r))}{\log r} = d^u,$$

where $B^s(x, r)$ (resp. $B^u(x, r)$) is the ball in $V^s(x)$ (resp. $V^u(x)$) with center at x and radius r. Moreover $d = d^s + d^u$ [2] and ν has an "almost product structure" (see [2]).

K. Criteria for Non-Zero Lyapunov Exponents. We saw that non-uniformly hyperbolic dynamical systems possess strong ergodic properties, and many other important properties. Therefore, it is of primary interest to have verifiable methods to check the non-vanishing of Lyapunov exponents.

The following criterion is due to Katok and Burns [8]. A real-valued measurable function Q on the tangent bundle TM is called an *eventually strict Lyapunov function* if for ν-almost every $x \in M$:

1. the function $Q_x(v) = Q(x,v)$ is continuous, homogeneous of degree one, and takes both positive and negative values;

2. the maximal dimensions of the linear subspaces contained respectively in the sets $\{0\} \cup Q_x^{-1}(0,+\infty)$ and $\{0\} \cup Q_x^{-1}(-\infty,0)$ are constants $r^+(Q)$ and $r^-(Q)$, and $r^+(Q)+r^-(Q)$ is the dimension of M;

3. $Q_{fx}(Dfv) \geq Q_x(v)$ for all $v \in T_xM$;

4. there exists a positive integer $m = m(x)$ such that for all $v \in T_xM \setminus \{0\}$,

$$Q_{f^m x}(Df^m v) > Q_x(v),\ Q_{f^{-m}x}(Df^{-m}v) < Q_x(v).$$

If f possesses an eventually strict Lyapunov function, then there exist exactly $r^+(Q)$ positive Lyapunov exponents and $r^-(Q)$ negative ones [8] (see also the work of Wojtkowski [28]).

Another method to estimate the Lyapunov exponents was presented by Herman [6].

L. Generalizations. There are several natural and important generalizations of Pesin Theory. Namely: there are generalizations for non-invertible maps; the main results of Pesin's work were extended by Katok and Strelcyn [10] to maps with singularities, which include billiard systems and other physical models; there are infinite dimensional versions of the results in Section C by Ruelle [27] in Hilbert spaces and by Mañé [18] in Banach spaces, under certain compactness assumptions; some results were extended to random maps (see the work of Liu and Qian [15]).

Related results were obtained for products of random matrices (see [5] and the references therein).

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