DIMENSION ESTIMATES IN SMOOTH DYNAMICS: A SURVEY

LUIS BARREIRA AND KATRIN GELFERT

Abstract. We survey a collection of results in the dimension theory of dynamical systems, with emphasis on the study of repellers and hyperbolic sets of smooth dynamics. We discuss the most preeminent results in the area as well as the main difficulties in developing a general theory. Despite many interesting and nontrivial developments, only the case of conformal dynamics is completely understood. On the other hand, the study of the dimension of invariant sets of nonconformal maps unveiled several new phenomena, but it still lacks today a satisfactory general approach. Indeed, we have only a complete understanding of a few classes of invariant sets of nonconformal maps satisfying certain simplifying assumptions. For example, the assumptions may ensure that there is a clear separation between different Lyapunov directions or that number-theoretical properties do not influence the dimension.

Contents

1. Introduction 2
2. Dimension theory and thermodynamic formalism 4
2.1. Hausdorff and box dimensions 4
2.2. Topological pressure and variational principle 5
2.3. Nonadditive topological pressure 7
3. Dimension estimates for repellers 8
3.1. Notion of repeller and examples 8
3.2. Markov partitions and geometric constructions 9
3.3. A priori bounds for the dimension 10
3.4. Sharp upper dimension estimates 12
3.5. Self-affine repellers and nonlinear generalizations 13
3.6. Measures of full dimension 15
3.7. Nonuniformly expanding repellers 17
3.8. Upper bounds involving Lyapunov exponents 18
4. Dimension estimates for hyperbolic sets 20
4.1. Notion of hyperbolic set and examples 20
4.2. Additional structure 21
4.3. A priori bounds for the dimension 22
4.4. The conformal case 24
4.5. Measures of maximal dimension 25
4.6. Further developments 26

2000 *Mathematics Subject Classification.* Primary: 37Dxx.
Key words and phrases. dimension theory, hyperbolic dynamics, invariant sets.
L. Barreira was partially supported by FCT through CAMGSD, Lisbon. K. Gelfert was supported by the Alexander von Humboldt foundation.

References 27

1. Introduction

During the last two decades, the dimension theory of dynamical systems progressively developed into an independent field of research. Roughly speaking, its main objective is to measure the complexity from the dimensional point of view of the objects that remain invariant under the dynamics, such as the invariant sets and measures. The first monograph that clearly took this point of view was Pesin's book [74], which describes the state-of-the-art up to 1997. We refer to [4] for a description of many of the more recent results in the area. On purpose, we do not discuss in the survey results that are not of dynamical nature, of course independently of the importance that they may have in the general dimension theory.

Despite many interesting and nontrivial developments in the dimension theory of dynamical systems, only the case of *conformal* dynamics is completely understood. In the case of repellers it corresponds to assume that the derivative of the map is a multiple of an isometry at every point. In particular, this causes that there are no privileged directions, and that the elements of all Markov partitions (which are used to model the repeller) are essentially balls, in some precise sense. These crucial properties allowed Bowen [16] and Ruelle [80] to develop a complete theory for the dimension of repellers of conformal maps. We note that the thermodynamic formalism introduced by Ruelle in his seminal work [78] (see also [79]) played an important role in this development. Essentially, all known equations used to compute or estimate dimensions are appropriate versions of an equation introduced by Bowen in [16] in his study of quasi-circles, which involves the notion of topological pressure. Moreover, for any repeller of a conformal map there exists an ergodic invariant measure of full dimension, that is, a measure concentrated on the set having the same dimension as the repeller.

To a certain extent, the study of the dimension of hyperbolic sets is analogous. Indeed, assuming that along the stable and unstable directions the derivative of the map is a multiple of an isometry, starting with the work of McCluskey and Manning in [66] it was possible to develop a sufficiently complete corresponding theory. However, we note that there are nontrivial differences with respect to repellers. For example, unless some cohomology relations are satisfied, there are no invariant measures of full dimension concentrated on a given hyperbolic set.

On the other hand, the study of the dimension of invariant sets of *nonconformal* maps unveiled several new phenomena, but it still lacks today a satisfactory general approach, both for repellers and hyperbolic sets. Indeed, most authors make additional assumptions that essentially avoid two main types of difficulties. The first difficulty is the lack of a clear separation between different Lyapunov directions, together with a possible small regularity of the associated distributions. Typically these distributions are only Hölder continuous, which causes that in general it is not possible to add the dimensions along various distributions (thinking of the tangent bundle as a product of them). This strongly contrasts to what happens for hyperbolic sets of a dynamics that is conformal along the stable and unstable directions, in which case the stable and unstable holonomies are Lipschitz (which thus causes that the tangent bundle is taken by a Lipschitz map with Lipschitz inverse to the product of the stable and unstable distributions). The second difficulty is the existence of number-theoretical properties that may cause a variation of the Hausdorff dimension with respect to a certain typical value (such as the one obtained by Falconer in [25]; see Theorem 3.7). Other authors have obtained results not for a specific invariant set, but instead for almost all invariant sets in a given parameterized family. However, sometimes it is quite difficult to determine or it is even unknown what happens for each specific value of the parameters.

This causes that in the case of nonconformal maps we are often only able to establish estimates instead of giving formulas for the dimension of the invariant sets. Thus, sometimes the emphasis is on how to obtain sharp dimension estimates, starting essentially with the seminal work of Douady and Oesterlé [19] who devised an approach to cover the invariant set in a more optimal manner. We have made an effort to describe all the preeminent results using this approach, in several directions. As a rule, sharp lower dimension estimates are more difficult to obtain. In some cases they are yet unknown or may even be impossible to obtain for *all* sets in some specific classes of invariant sets of nonconformal dynamical systems.

We also discuss briefly some motivations for the study of dimension in the context of dynamical systems. In many situations the longtime behavior of dynamical systems such as those coming from delay differential equations or partial differential equations can essentially be described in terms of global attractors (see [1, 40, 88]). An important question, in particular in the context of infinite-dimensional systems, is how many degrees of freedom are actually necessary to specify the dynamics on the attractor. It turns out that in many situations the attractors have finite Hausdorff dimension or even finite box dimension and hence, the dynamics on the attractor is essentially finite-dimensional (see [1, 40, 88] for related discussions). In particular Mañé [63] obtained the following result.

Theorem 1.1. *Let $f\colon E \to E$ be a C^1 map of a Banach space such that at every point its derivative is the sum of a compact map and a contraction. Then every compact f-invariant set in E has finite upper box dimension.*

An analogous statement for the Hausdorff dimension was given by Mallet-Paret [62] in Hilbert spaces. Moreover, particularly in the experimental study of attractors one often considers their projection into an Euclidean space. It is also possible to give conditions for the invertibility of the projection. In particular, the following result is due to Mañé [63].

Theorem 1.2. *Let E be a Banach space and let $F \subset E$ be a p-dimensional subspace. For a residual set of the space of all continuous projections of E onto F (with respect to the operator norm topology) each projection is injective on a compact set $\Lambda \subset E$ provided that $\dim_H(\Lambda \times \Lambda) + 1 < p < \infty$.*

For an arbitrary projection of a compact subset of a Banach space Hunt and Kaloshin [50] showed that typically (in the sense of prevalence in [51]) the projection is injective and has Hölder continuous inverse. Earlier results concerning the Hölder continuity of the inverse are due to Ben-Artzi, Eden, Foias and Nikolaenko [11] in $\mathbb{R}^n$ and to Foias and Olson [33] in Hilbert spaces. These results estimate how much the dimension of the set may decrease under the projection.

2. Dimension theory and thermodynamic formalism

We briefly recall in a pragmatic manner the notions from dimension theory and the thermodynamic formalism that are needed in the survey. We refer to [4, 15, 74, 93] for more details.

2.1. **Hausdorff and box dimensions.** Let X be a separable metric space. Given a set $Z \subset X$ and a number $\alpha > 0$, we define the *α-dimensional Hausdorff measure* of Z by

$$m(Z,\alpha) = \lim_{\varepsilon\to 0} \inf_{\mathcal{U}} \sum_{U\in\mathcal{U}} (\operatorname{diam} U)^\alpha,$$

where the infimum is taken over all finite or countable covers $\mathcal{U}$ of Z by open sets U with diameter $\operatorname{diam} U \le \varepsilon$. The *Hausdorff dimension* of Z is defined by

$$\dim_H Z = \inf\big\{\alpha > 0 : m(Z,\alpha) = 0\big\}.$$

Moreover, the *lower* and *upper box dimensions* of Z are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{-\log\varepsilon} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{-\log\varepsilon},$$

where $N(Z,\varepsilon)$ denotes the number of balls of radius ε that are needed to cover Z. When these two numbers coincide we denote the common value by $\dim_B Z$, and we call it the *box dimension* of Z. One can easily show that

$$\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z.$$

We note that in general these inequalities may be strict.

Now let μ be a finite measure in X. The *Hausdorff dimension* and the *lower* and *upper box dimensions* of μ are defined respectively by

$$\dim_H \mu = \inf\big\{\dim_H Z : \mu(Z) = \mu(X)\big\},$$
$$\underline{\dim}_B \mu = \lim_{\delta\to 0} \inf\big\{\underline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\big\},$$
$$\overline{\dim}_B \mu = \lim_{\delta\to 0} \inf\big\{\overline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\big\}.$$

We emphasize that in general these three numbers do not coincide, respectively, with the Hausdorff dimension and the lower and upper box dimensions of the support of μ. Thus, they contain additional information about the distribution of the measure μ on its support. One can show that

$$\dim_H \mu \le \underline{\dim}_B \mu \le \overline{\dim}_B \mu.$$

Again, in general these inequalities may be strict. The following criterion for equality was established by Young in [96]: if there exists a constant $d \ge 0$ such that

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d \tag{1}$$

for μ-almost every $x \in X$, then

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d.$$

The limit in (1), when it exists, is called the *pointwise dimension* of μ at x.

2.2. **Topological pressure and variational principle.** There is a close relation between dimension theory and the thermodynamic formalism.

We first introduce the notion of topological pressure. Let $f \colon X \to X$ be a continuous transformation of a compact metric space. For each $n \in \mathbb{N}$ we define a distance ρ_n in X by

$$\rho_n(x, y) = \max\left\{\rho(f^k(x), f^k(y)) : k = 0, \ldots, n-1\right\}.$$

A set of points $\{x_i\}_{i \in I}$ in X is said to be (n, ε)-*separated* (with respect to f) if $\rho_n(x_i, x_j) \geq \varepsilon$ whenever $i \neq j$. The *topological pressure* of a continuous function $\varphi \colon X \to \mathbb{R}$ (with respect to f) is defined by

$$P(\varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup \sum_{i \in I} \exp \sum_{k=0}^{n-1} (\varphi \circ f^k)(x_i), \tag{2}$$

where the supremum is taken over all (n, ε)-separated sets $\{x_i\}_{i \in I}$. The *topological entropy* of f is defined by $h(f) = P(0)$. The notion of topological pressure was introduced by Ruelle in [78] for expansive transformations, and by Walters in [92] in the general case. We recall that f is said to be *expansive* if there exists $\delta > 0$ such that if

$$d(f^n(x), f^n(y)) < \delta \quad \text{for every} \quad n \in \mathbb{N},$$

then $x = y$ (when f is invertible we replace $\mathbb{N}$ by $\mathbb{Z}$ in the definition). For more details and further references see [15, 79, 93].

We illustrate with a simple example the relation between dimension theory and the thermodynamic formalism.

Example 2.1. Consider constants $\lambda_1, \ldots, \lambda_p \in (0, 1)$, and disjoint closed intervals $\Delta_1, \ldots, \Delta_p \subset \mathbb{R}$ of lengths $\lambda_1, \ldots, \lambda_p$. For each $k = 1, \ldots, p$, we choose again p disjoint closed intervals $\Delta_{k1}, \ldots, \Delta_{kp} \subset \Delta_k$ of lengths respectively $\lambda_k\lambda_1, \ldots, \lambda_k\lambda_p$. Iterating this procedure, for each $n \in \mathbb{N}$ we obtain p^n disjoint closed intervals $\Delta_{i_1 \cdots i_n}$ of lengths $\prod_{k=1}^n \lambda_{i_k}$, and we consider the limit set

$$F = \bigcap_{n=1}^{\infty} \bigcup_{i_1 \cdots i_n} \Delta_{i_1 \cdots i_n}.$$

Moran showed in [68] that $\dim_H F = s$, where s is the unique real number such that

$$\sum_{k=1}^{p} {\lambda_k}^s = 1. \tag{3}$$

It is remarkable that the Hausdorff dimension of F does not depend on the location of the intervals $\Delta_{i_1 \cdots i_n}$ but only on their lengths. Now we consider the continuous function $\varphi \colon F \to \mathbb{R}$ given by $\varphi(x) = \log \lambda_i$ when $x \in F \cap \Delta_i$.

We obtain

$$\begin{aligned} P(s\varphi) &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp\left(s \sum_{k=1}^n \log \lambda_{i_k}\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \prod_{k=1}^n {\lambda_{i_k}}^s \\ &= \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i=1}^p {\lambda_i}^s\right)^n = \log \sum_{i=1}^p {\lambda_i}^s. \end{aligned}$$

Therefore, equation (3) is equivalent to

$$P(s\varphi) = 0. \tag{4}$$

Equation (4) is called Bowen's equation. It was introduced by Bowen in [16] in his study of quasi-circles. Virtually, all known equations used to compute or estimate dimensions are appropriate versions of equation (4).

Now let μ be an f-invariant probability measure in X. This means that $\mu(f^{-1}A) = \mu(A)$ for every measurable set $A \subset X$. A finite or countable family ξ of measurable subsets of X is called a *partition* of X if $\mu(\bigcup_{C\in\xi} C) = 1$, and $\mu(C \cap D) = 0$ for every $C, D \in \xi$ with $C \neq D$. We write

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. The *Kolmogorov–Sinai entropy* of f with respect to μ is defined by

$$h_\mu(f) = \sup_{H_\mu(\xi)<\infty} \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} f^{-k}\xi\right), \tag{5}$$

where $\bigvee_{k=0}^{n-1} f^{-k}\xi$ is the partition of X into the sets

$$C_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} C_{i_{k+1}}$$

with $C_{i_1}, \ldots, C_{i_n} \in \xi$. Indeed it can be shown that the limit in (5) exists. The topological pressure satisfies the *variational principle*

$$P(\varphi) = \sup_\mu \left\{ h_\mu(f) + \int_X \varphi \, d\mu \right\}, \tag{6}$$

with the supremum taken over all f-invariant probability measures in X. This was established by Ruelle in [78] for expansive maps, and by Walters in [92] in the general case. An f-invariant probability measure μ is said to be an *equilibrium measure* of φ (with respect to f) if

$$P(\varphi) = h_\mu(f) + \int_X \varphi \, d\mu,$$

that is, if the supremum in (6) is attained at μ. It follows immediately from (6) that when the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous, each continuous function φ has at least one equilibrium measure. For example, when f is expansive the entropy map is upper semi-continuous.

2.3. **Nonadditive topological pressure.** We also consider the nonadditive version of the topological pressure introduced by Barreira in [2].

Let $f\colon X \to X$ be a continuous transformation of a compact metric space. Let also $\mathcal{U}$ be a finite open cover of X. Given $n \in \mathbb{N}$, we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of n-tuples $U = (U_1, \ldots, U_n)$ with $U_1, \ldots, U_n \in \mathcal{U}$. For each $U \in \mathcal{W}_n(\mathcal{U})$ we set $m(U) = n$, and we define the open set

$$X(U) = \big\{x \in X\colon f^{k-1}(x) \in U_k \text{ for } k = 1, \ldots, m(U)\big\}.$$

We say that a collection $\Gamma \subset \bigcup_{n\in\mathbb{N}} \mathcal{W}_n(\mathcal{U})$ *covers* a set $Z \subset X$ provided that $\bigcup_{U\in\Gamma} X(U) \supset Z$. Now let $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions $\varphi_n\colon X \to \mathbb{R}$, and set

$$\gamma_n(\Phi, \mathcal{U}) = \sup\big\{|\varphi_n(x) - \varphi_n(y)|\colon x, y \in X(U) \text{ and } U \in \mathcal{W}_n(\mathcal{U})\big\}.$$

We assume that

$$\lim_{\operatorname{diam}\mathcal{U}\to 0} \limsup_{n\to\infty} \frac{1}{n}\gamma_n(\Phi, \mathcal{U}) = 0, \tag{7}$$

where $\operatorname{diam}\mathcal{U} = \sup_{U\in\mathcal{U}} \operatorname{diam} U$ is the *diameter* of the cover $\mathcal{U}$. We observe that (7) holds when Φ is an additive sequence, that is, when

$$\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k \tag{8}$$

for some continuous function $\varphi\colon X \to \mathbb{R}$ and all $n \in \mathbb{N}$. For each $U \in \mathcal{W}_n(\mathcal{U})$ we write

$$\varphi(U) = \begin{cases} \sup_{X(U)} \varphi_n & \text{if } X(U) \neq \varnothing, \\ -\infty & \text{if } X(U) = \varnothing. \end{cases}$$

Given $Z \subset X$ and $\alpha \in \mathbb{R}$, we set

$$M_Z(\alpha, \Phi, \mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp\left(-\alpha m(U) + \varphi(U)\right),$$

where the infimum is taken over all collections $\Gamma \subset \bigcup_{k\geq n} \mathcal{W}_k(\mathcal{U})$ covering Z. One can show that the function $\alpha \mapsto M_Z(\alpha, \Phi, \mathcal{U})$ jumps from $+\infty$ to 0 at a unique value of α, which we denote by

$$P_Z(\Phi, \mathcal{U}) = \inf\big\{\alpha \in \mathbb{R}\colon M_Z(\alpha, \Phi, \mathcal{U}) = 0\big\}.$$

Moreover, the limit

$$P_Z(\Phi) = \lim_{\operatorname{diam}\mathcal{U}\to 0} P_Z(\Phi, \mathcal{U})$$

exists (see [2] for details), and we call it the *nonadditive topological pressure* of the sequence Φ on the set Z (with respect to f). One can easily verify that if Φ is the (additive) sequence of functions in (8), then $P_X(\Phi)$ coincides with the topological pressure $P(\varphi)$ of the function φ (see (2)).

The nonadditive thermodynamic formalism developed in [2] contains as a particular case a new formulation of the subadditive thermodynamic formalism introduced by Falconer in [26]. It also includes a variational principle for the topological pressure, although with a restrictive assumption on the sequence Φ. Namely, if there exists a continuous function $\varphi\colon X \to \mathbb{R}$ such that

$$\varphi_{n+1} - \varphi_n \circ f \to \varphi \text{ uniformly when } n \to \infty, \tag{9}$$

then

$$P_X(\Phi) = \sup_\mu \left(h_\mu(f) + \int_X \varphi \, d\mu \right),$$

with the supremum taken over all f-invariant probability measures on X. The restrictive assumption in (9) caused that until recently there was no discussion of equilibrium and Gibbs measures, in the general context of the nonadditive thermodynamic formalism. On the other hand, it is well-known that equilibrium and Gibbs measures play a prominent role in dimension theory and in particular in the multifractal analysis of dynamical systems. This justifies the interest in the more general class of almost additive sequences, for which it is possible not only to establish a variational principle, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures, among other properties. We recall that a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *almost additive* if there is a constant $C > 0$ such that

$$-C + \varphi_n + \varphi_m \circ f^n \le \varphi_{n+m} \le C + \varphi_n + \varphi_m \circ f^n$$

for every $n, m \in \mathbb{N}$. Nontrivial examples of almost additive sequences occur for instance in the multifractal analysis of Lyapunov exponents for nonconformal maps in [6]. We refer to [5] for a survey of the existing results of the almost additive thermodynamic formalism, including the discussion of some recent developments and applications.

3. Dimension estimates for repellers

We discuss in this section the dimension of repellers, which are invariant sets of a hyperbolic noninvertible dynamics. After describing how Markov partitions can be used to model repellers, we present the existing results concerning dimension estimates, including the well-established case of conformal dynamics, as well as the most recent results for nonconformal dynamics and nonuniformly hyperbolic dynamics, among several other topics.

3.1. **Notion of repeller and examples.**

Let $f\colon M \to M$ be a differentiable transformation of a smooth manifold, and let $J \subset M$ be a compact f-invariant set (this means that $f^{-1}J = J$). We say that J is a *repeller* of f and that f is *expanding* on J if there exist constants $c > 0$ and $\beta > 1$ such that

$$\|d_x f^n v\| \ge c\beta^n \|v\| \tag{10}$$

for every $n \in \mathbb{N}$, $x \in J$, and $v \in T_xM$. We note that by (10) the linear transformation $d_x f^n$ is invertible for every $x \in J$ and $n \in \mathbb{N}$. In particular, f is a local diffeomorphism, that is, each $x \in J$ has an open neighborhood U_x such that $f|U_x\colon U_x \to f(U_x)$ is invertible and has differentiable inverse.

We describe two examples of expanding maps and repellers.

Example 3.1 (Rational maps)**.** Consider a rational map $f\colon S \to S$ of degree at least 2 on the Riemann sphere S. A n-periodic point x (this means that $f^n(x) = x$) is said to be *repelling* if $|(f^n)'(x)| > 1$. The *Julia set* $J \subset S$ of f is the closure of the set of repelling periodic points of f. In particular, J is compact, nonempty, and f-invariant. For example, the map $z \mapsto z^2 + c$ is expanding on its Julia set provided that $|c| < 1/4$ (see for example [30] for details), in which case J is a repeller of f.

Example 3.2 (One-dimensional Markov maps)**.** Consider disjoint closed intervals $\Delta_1, \ldots, \Delta_p \subset [0,1]$. We say that $f\colon [0,1] \to [0,1]$ is a *Markov map* provided that:

1. for every $j = 1, \ldots, p$ there is a set of indices $I(j)$ such that $f(\Delta_j) = \bigcup_{i \in I(j)} \Delta_i$;
2. at every point $x \in \Delta := \bigcup_{i=1}^p \operatorname{int} \Delta_i$ the derivative of f exists and satisfies $|f'(x)| \geq \gamma$ for some fixed $\gamma > 0$;
3. there exist $\lambda > 1$ and $n \in \mathbb{N}$ such that $|(f^n)'(x)| \geq \lambda$ whenever $f^k(x) \in \Delta$ for $0 \leq k \leq n$.

Then the set

$$J = \bigcap_{k=1}^{\infty} \overline{f^{-k}\Delta}$$

is a repeller of f.

3.2. **Markov partitions and geometric constructions.** Let J be a repeller of a differentiable transformation $f\colon M \to M$. A finite cover of J by nonempty closed sets $R_1, \ldots, R_p \subset J$ is called a *Markov partition* of J if:

1. $\overline{\operatorname{int} R_i} = R_i$ for each i;
2. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ whenever $i \neq j$;
3. $f(R_i) \supset R_j$ whenever $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing$.

We note that the interior of each set R_i is computed with respect to the induced topology on J. Any repeller has Markov partitions with arbitrarily small diameter (see [80]).

Now we describe how Markov partitions can be use to model repellers. Given a Markov partition, we define a $p \times p$ matrix $A = (a_{ij})$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing, \\ 0 & \text{if } f(\operatorname{int} R_i) \cap \operatorname{int} R_j = \varnothing. \end{cases} \tag{11}$$

We also consider the space of sequences $\Sigma_p^+ = \{1, \ldots, p\}^{\mathbb{N}}$ equipped with the distance

$$d(i_1 i_2 \cdots, j_1 j_2 \cdots) = \sum_{k=1}^{\infty} e^{-k} |i_k - j_k|, \tag{12}$$

and the shift map $\sigma\colon \Sigma_p^+ \to \Sigma_p^+$ satisfying $(\sigma\omega)_n = \omega_{n+1}$ for all $n \in \mathbb{N}$. The restriction of σ to the σ-invariant set

$$\Sigma_A^+ = \left\{ (i_1 i_2 \cdots) \in \Sigma_p^+ : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{N} \right\}$$

is called the *topological Markov chain* with *transition matrix* A.

It is easy to show that one can define a map $\chi\colon \Sigma_A^+ \to J$ by

$$\chi(i_1 i_2 \cdots) = \bigcap_{k=0}^{\infty} f^{-k} R_{i_{k+1}}.$$

This map is surjective, Hölder continuous (with the distance in Σ_p^+ given by (12)), and it satisfies

$$\chi \circ \sigma = f \circ \chi. \tag{13}$$

In general χ is not invertible, although $\operatorname{card}\chi^{-1}x \le p^2$ for every $x \in \Sigma_A^+$. Nevertheless, identity (13) still allows one to see χ as a dictionary transferring the symbolic dynamics $\sigma|\Sigma_A^+$ (and often the results at this level) to the dynamics of f on J. For each $(i_1 i_2 \cdots) \in \Sigma_A^+$ and $n \in \mathbb{N}$ we set

$$\Delta_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} R_{i_{k+1}}. \tag{14}$$

Given a continuous function $\varphi\colon J \to \mathbb{R}$ we have

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \sup_{\Delta_{i_1\cdots i_n}} \sum_{k=0}^{n-1} (\varphi \circ f^k),$$

where the sum is taken over all n-tuples $i_1 \cdots i_n$ formed by the first n elements of some sequence in Σ_A^+.

More generally, one can consider constructions defined by an arbitrary symbolic dynamics. Namely, a *geometric construction* in $\mathbb{R}^m$ is defined by:

1. a compact shift-invariant set $Q \subset \Sigma_p^+$ for some positive integer p;
2. a decreasing sequence of closed sets $\Delta_{i_1\cdots i_n} \subset \mathbb{R}^m$ for each $(i_1 i_2 \cdots) \in Q$ and $n \in \mathbb{N}$, with $\operatorname{diam} \Delta_{i_1\cdots i_n} \to 0$ as $n \to \infty$.

We also say that the geometric construction is modeled by Q. The *limit set* of the construction is the compact set defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{i_1\cdots i_n} \Delta_{i_1\cdots i_n},$$

where the union is taken over all Q-admissible n-tuples, that is, all n-tuples $(i_1 \cdots i_n)$ such that

$$(j_1 \cdots j_n) = (i_1 \cdots i_n) \quad \text{for some} \quad (j_1 j_2 \cdots) \in Q.$$

In particular, each Markov partition of a repeller defines a geometric construction modeled by Σ_A^+, with the sets $\Delta_{i_1\cdots i_n}$ given by (14).

We recall that a transformation f is said to be *topologically mixing* on a set J if given open sets U and V with nonempty intersection with J there exists $n \in \mathbb{N}$ such that

$$f^m U \cap V \cap J \neq \varnothing \quad \text{for every} \quad m > n.$$

If f is topologically mixing on a repeller J, then for each transition matrix A associated to a Markov partition of J there exists $m \in \mathbb{N}$ such that A^m has only positive entries.

3.3. **A priori bounds for the dimension.** Our main aim in this section is to present the best possible dimension estimates of repellers using the least possible information about the dynamics. It is in this sense that we call them a priori bounds. When more geometric information is available one can often provide sharper estimates for the dimension or even compute its value (see Section 3.4 for a related discussion). However, this is done at the expense of a more elaborate approach that in particular may require the nonadditive version of the thermodynamic formalism or some appropriate substitution.

Given a repeller J of a differentiable transformation f, we consider the functions $\underline{\varphi}, \overline{\varphi}\colon J \to \mathbb{R}$ given by

$$\underline{\varphi}(x) = -\log\|d_x f\| \quad \text{and} \quad \overline{\varphi}(x) = \log\|(d_x f)^{-1}\|. \tag{15}$$

Since f is expanding, the functions

$$s \mapsto P(s\underline{\varphi}) \quad \text{and} \quad s \mapsto P(s\overline{\varphi})$$

are Lipschitz and strictly decreasing. This implies that there exist unique roots $\underline{s}$ and $\overline{s}$ of the equations

$$P(\underline{s}\underline{\varphi}) = 0 \quad \text{and} \quad P(\overline{s}\,\overline{\varphi}) = 0. \tag{16}$$

Theorem 3.3. *If J is a repeller of a C^1 map, then*

$$\underline{s} \le \dim_H J \le \underline{\dim}_B J \le \overline{\dim}_B J \le \overline{s}. \tag{17}$$

Theorem 3.3 is a special case of results of Barreira in [2]. The upper bound was obtained in [38] in the more general case of volume expanding maps. By the variational principle in (6), it follows from (17) that

$$\frac{h(f)}{\log \max_{x\in J}\|d_x f\|} \le \underline{s} \le \overline{s} \le \frac{h(f)}{-\log \max_{x\in J}\|(d_x f)^{-1}\|}.$$

In a related direction, Shafikov and Wolf showed in [82] that for a repeller J of a C^2 map $f\colon M \to M$ we have

$$\overline{\dim}_B J \le \dim M + \frac{P(-\log|\det df|)}{\lambda},$$

where

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in J}\|d_x f^n\|.$$

In the particular case of conformal maps the inequalities in Theorem 3.3 become identities. We recall that f is said to be *conformal* on J if $d_x f$ is a multiple of an isometry for every $x \in J$. In this case the two functions $\underline{\varphi}$ and $\overline{\varphi}$ in (15) coincide, and thus $\underline{s} = \overline{s}$. This yields equalities in (17). For examples of repellers of conformal maps see Examples 3.1 and 3.2.

Let us formulate an explicit statement. Consider the function $\varphi\colon J \to \mathbb{R}$ defined by

$$\varphi(x) = -\log\|d_x f\|.$$

Theorem 3.4. *If J is a repeller of a C^1 map f which is conformal on J, then*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s,$$

where s is the unique root of the equation $P(s\varphi) = 0$. Moreover, if f is of class $C^{1+\alpha}$ for some $\alpha \in (0,1)$ and is topologically mixing on J, then the unique equilibrium measure μ of $s\varphi$ satisfies

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = s.$$

Ruelle [80] showed that $\dim_H J = s$ when f is of class $C^{1+\alpha}$, assuming that f is topologically mixing. His proof consists in showing that the s-dimensional Hausdorff measure on J is equivalent to the equilibrium measure of $s\varphi$ (with Radon–Nikodym derivative bounded and bounded away from zero). The particular case of quasi-circles was earlier considered by Bowen in [16]. The coincidence between the Hausdorff and box dimensions was established by Falconer in [27]. The extension of these results to repellers of C^1 maps was obtained independently by Barreira [2] and Gatzouras and Peres [36], using different approaches. See [89] for a survey on the dimension theory of holomorphic endomorphisms.

3.4. **Sharp upper dimension estimates.** We describe in this section sharp upper dimension estimates for a repeller when more geometric information is available.

Given a linear map $L\colon \mathbb{R}^m \to \mathbb{R}^m$, let

$$\sigma_1(L) \geq \cdots \geq \sigma_n(L) \geq 0$$

be the *singular values* of L, that is, the eigenvalues of $(L^*L)^{1/2}$, counted with their multiplicities, where L^* denotes the transpose of L. These numbers coincide with the semiaxes of the ellipsoid which is the image of the unit ball in $\mathbb{R}^m$ under the map L. For each $s \in [0, n]$ we set

$$\omega_s(L) = \sigma_1(L) \cdots \sigma_{\lfloor s \rfloor}(L) \sigma_{\lfloor s \rfloor + 1}(L)^{s - \lfloor s \rfloor}, \tag{18}$$

where $\lfloor s \rfloor$ denotes the integer part of s.

In their seminal paper [19] Douady and Oesterlé obtained an upper bound for the Hausdorff dimension. Given a C^1 map $f\colon M \to M$, we consider the function $\psi^s\colon M \to \mathbb{R}$ defined by

$$\psi^s(x) = \log \omega_s(d_x f).$$

Moreover, given a subset $J \subset M$ we let

$$\dim_L(f, J) = \inf \left\{ s \in (0, \dim M] : \sup_{x \in J} \psi^s(x) < 0 \right\}.$$

Theorem 3.5 ([19])**.** *If f is a C^1 map and J is a compact f-invariant set, then*

$$\dim_H J \leq \dim_L(f, J). \tag{19}$$

Since J is f^m-invariant for every $m \in \mathbb{N}$, we also have

$$\dim_H J \leq \inf_{m \in \mathbb{N}} \dim_L(f^m, J), \tag{20}$$

which may sometimes give a better estimate than that in (19). It was shown by Hunt in [49] that we can replace $\dim_H J$ by $\overline{\dim}_B J$ in (19) for maps in $\mathbb{R}^n$. See [37] for a proof including the case of maps on manifolds. Leonov [57] established estimates for the Hausdorff dimension under weaker assumptions using Lyapunov-type functions. Namely, to show that $\dim_H J \leq s$ it suffices to show that

$$\sup_{x \in J} \left(\frac{\psi(f(x))}{\psi(x)} \omega_s(d_x f) \right) < 1,$$

where $\psi\colon J \to \mathbb{R}^+$ is some continuous function. This approach can be interpreted as a change of metric on the manifold, as studied for example by Noack and Reitmann in [70]. It may sometimes improve the estimate in Theorem 3.5. Indeed, while the Hausdorff dimension is invariant under smooth changes of the metric, the singular values may change, and thus the function corresponding to the new metric may be strictly smaller than φ_s.

Now we consider the sequences of functions $\Phi^s = (\varphi_n^s)_{n \in \mathbb{N}}$ defined by

$$\varphi_n^s(x) = \log \omega_s((d_x f^n)^{-1})$$

with ω_s as in (18). Using these functions Falconer [29] computed the Hausdorff dimension of a class of repellers of nonconformal transformations (building on his former work [26]). His main result can be reformulated as follows.

Theorem 3.6. *If J is a repeller of a C^2 map f which is topologically mixing on J, and*

$$\|(d_x f)^{-1}\|^2 \|d_x f\| < 1 \quad \textit{for every} \quad x \in J,$$

then

$$\underline{\dim}_B J = \overline{\dim}_B J \le s, \tag{21}$$

where s is the unique root of the equation $P_J(\Phi^s) = 0$.

Clearly, $\omega_s(L) \le \sigma_1(L)^s$, and thus,

$$\begin{aligned}\varphi_n^s(x) &\le s \log \sigma_1((d_x f^n)^{-1}) \\ &\le s \log \|(d_x f^n)^{-1}\| \le s \sum_{k=0}^{n-1} \overline{\varphi}(f^k(x)),\end{aligned}$$

with $\overline{\varphi}$ as in (15). This shows that $s \le \overline{s}$ (see (16)), and thus in general (21) may give a sharper estimate than that given by the upper bound in (17).

Under an additional geometric assumption, satisfied for example when J contains a nondifferentiable arc, the number s in Theorem 3.6 is equal to $\dim_H J$ (see [29]). Related results were obtained by Zhang in [97], and in the case of volume expanding maps by Gelfert in [38]. In another direction, Hu [47] computed the box dimension of a class of repellers of nonconformal transformations that leave invariant a strong unstable foliation. His formula for the box dimension is also expressed in terms of the topological pressure. Related results were obtained earlier by Bedford in [9] (see also [10]), for a class of self-similar sets that are graphs of continuous functions. We note that Manning and Simon [65] found a mistake in the related paper [3], which discusses dimension estimates using the sequences Φ^s. This causes that some statements in that paper do not hold (we refer to [65] for full details).

3.5. **Self-affine repellers and nonlinear generalizations.** In another direction, Falconer [25, 28] studied a class of limit sets of geometric constructions obtained from the composition of affine transformations that are not necessarily conformal. Consider affine transformations $f_i \colon \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, p$ given by $f_i(x) = A_i x + b_i$ for some linear contraction A_i and some vector $b_i \in \mathbb{R}^n$. Then there is a unique nonempty compact set $J \subset \mathbb{R}^n$ such that

$$J = \bigcup_{i=1}^{p} f_i(J) \tag{22}$$

(see [52]), and for every nonempty compact set $R \subset \mathbb{R}^n$ such that $f_i(R) \subset R$ for $i = 1, \ldots, p$ we have

$$J = \bigcap_{k=1}^{\infty} \bigcup_{i_1 \cdots i_k} (f_{i_1} \circ \cdots \circ f_{i_k})(R).$$

Set

$$s = \inf \left\{ d \in [0, n] : \sum_{k=1}^{\infty} \sum_{i_1 \cdots i_k} \omega_d(A_{i_1} \circ \cdots \circ A_{i_k}) < \infty \right\}$$

with ω_d as in (18). We emphasize that the number s does not depend on the vectors $b_1, \ldots, b_p$.

Theorem 3.7. *We have* $\overline{\dim}_B J \le s$. *In addition, if* $\|A_i\| < 1/2$ *for* $i = 1, \ldots, p$, *then for Lebesgue almost every* $(b_1, \ldots, b_p) \in (\mathbb{R}^n)^p$ *we have*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s. \tag{23}$$

The statement in Theorem 3.7 is due to Falconer [25] when $\|A_i\| < 1/3$ for $i = 1, \ldots, p$, and to Solomyak [86] in the general case.

The class of self-affine repellers can also be used to illustrate that to determine or even estimate the dimension of the limit set J, sometimes it is not sufficient to know the geometric shape of the sets $\Delta_{i_1 \cdots i_n}$, in strong contrast with what happens in Theorem 3.4. For example, the dimension can be affected by certain number-theoretical properties. Namely, consider a geometric construction in $\mathbb{R}^2$ modeled by Σ_2^+ such that the sets

$$\Delta_{i_1 \cdots i_n} = (f_{i_1} \circ \cdots \circ f_{i_n})([0,1] \times [0,1])$$

are rectangles with sides of length a^n and b^n, obtained from the composition of the functions

$$f_1(x,y) = (ax, by) \quad \text{and} \quad f_2(x,y) = (ax - a + 1, by - b + 1),$$

for some fixed constants $a \in (0,1)$ and $b \in (0,1/2)$. In particular, the projection of $\Delta_{i_1 \cdots i_n}$ on the horizontal axis is an interval with right endpoint given by

$$a^n + \sum_{k=0}^{n-1} j_k a^k, \tag{24}$$

where

$$j_k = \begin{cases} 0 & \text{if } i_k = 1, \\ 1 - a & \text{if } i_k = 2. \end{cases}$$

Now we assume that $a = (\sqrt{5} - 1)/2$. In this case we have $a^2 + a = 1$, and thus for each $n > 2$ there is more than one vector $(i_1 \cdots i_n)$ for which we obtain the same value in (24). This causes a larger concentration of the sets $\Delta_{i_1 \cdots i_n}$ in certain regions of the limit set J. Therefore, to compute the Hausdorff dimension, when we take an open cover of J it may happen that it is possible to replace, in the regions of larger concentration of the sets $\Delta_{i_1 \cdots i_n}$, several elements of the cover by a single element. This may cause J to have a Hausdorff dimension strictly smaller than the box dimension. It was established by Neunhäuserer in [69] that indeed this happens when $a = (\sqrt{5} - 1)/2$. See also [76] for former related results of Przytycki and Urbański. We note that the constant $(\sqrt{5} - 1)/2$ is only an example among many other possible values that lead to a similar phenomenon. McMullen [67] and Gatzouras and Lalley [35] also obtained different Hausdorff and box dimensions for particular classes of self-affine sets. Moreover, the s-dimensional Hausdorff measure with $s = \dim_H J$ need not be positive neither finite [35, 73]. Finally, for a smoothly parameterized family of self-affine sets the Hausdorff dimension may not vary continuously with the parameters [25, 35, 76].

It should be noted that Theorem 3.6 gives no indication of which sets J actually have dimension equal to s. In [48] Hueter and Lalley gave sufficient conditions for a given set $J \subset \mathbb{R}^2$ to satisfy (23). Let $f_1, \dots, f_p \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the affine transformations in Theorem 3.6, which we continue to write in the form $f_i(x) = A_i x + b_i$. Then the following statement holds.

Theorem 3.8 ([48]). *Assume that:*

1. $\|A_i\| < 1$ *and* $\sigma_1(A_i)^2 < \sigma_2(A_i)$ *for* $i = 1, \dots, p$*;*
2. *the sets* $A_1^{-1}Q, \dots, A_p^{-1}Q$ *are disjoint, where* Q *is the closed second quadrant;*
3. *the sets* $A_1 J, \dots, A_p J$ *are disjoint.*

Then (23) *holds. Moreover, the* s*-dimensional Hausdorff measure of* J *is finite, and there exists a unique ergodic measure of full dimension in* J*.*

We note that all the hypotheses in the theorem persist under sufficiently small perturbations of the entries of the matrices $A_1, \dots, A_p$.

A nonlinear extension of the work of Hueter and Lalley was obtained by Luzia in [59]. Let $f_1, \dots, f_p \colon \mathbb{R}^2 \to \mathbb{R}^2$ be C^2 diffeomorphisms such that:

1. $\sup_{x \in \mathbb{R}^2} \|d_x f_i\| < 1$ for $i = 1, \dots, p$;
2. there is a convex bounded open set U such that $f_1(\overline{U}), \dots, f_p(\overline{U})$ are pairwise disjoint subsets of U;
3. $d_x f_i P \subset \operatorname{int} P$ for every $x \in U$, where P is the union of the closed first and third quadrants;
4. $\|d_x f_i v\|^3 / |\det d_x f_i| < 1$ for every $x \in U$ and $v \in P$ with $\|v\| = 1$.

Then there exists a unique nonempty compact set satisfying (22). We also consider the function $\varphi \colon \{1, \dots, p\}^{\mathbb{N}} \to \mathbb{R}$ defined by

$$\varphi(i_1 i_2 \cdots) = \log \|d_{\pi(i_2 i_3 \cdots)} f_{i_1} | V\|,$$

where

$$\pi(i_2 i_3 \cdots) = \lim_{n \to \infty} (f_{i_2} \circ \cdots \circ f_{i_n})(\overline{U}),$$

and

$$V = \lim_{n \to \infty} d_{(f_{i_1} \circ \cdots \circ f_{i_n})^{-1} \pi(i_1 i_2 \cdots)} (f_{i_1} \circ \cdots \circ f_{i_n}) P.$$

Then the following statement holds.

Theorem 3.9 ([59]). *We have*

$$\dim_H J = \underline{\dim}_B J = \overline{\dim}_B J = s,$$

where s *is the unique root of the equation* $P(s\varphi) = 0$*. Moreover, the* s*-dimensional Hausdorff measure of* Λ *is finite, and there is a unique ergodic measure of full dimension in* Λ*.*

3.6. **Measures of full dimension.** Now we consider the related problem of the existence of measures of full dimension.

We say that an invariant measure μ in a repeller J is of *full dimension* if $\dim_H \mu = \dim_H J$. It was shown by Ruelle in [80] that for a map of class $C^{1+\alpha}$ which is conformal and topologically mixing on J, if μ is the unique equilibrium measure in Theorem 3.4, then μ is of full dimension. This follows from the equivalence between μ and the s-dimensional Hausdorff measure in J. The existence of an ergodic measure of full dimension for any repeller of a C^1 map was established by Gatzouras and Peres in [36].

The situation is much more complicated in the case of nonconformal dynamics, and there exist only some partial results. In particular, Gatzouras and Peres [36] considered maps of the form

$$f(x_1, x_2) = (f_1(x_1), f_2(x_2)),$$

where f_1 and f_2 are C^1 maps with repellers respectively J_1 and J_2 such that $f_1|J_1$ and $f_2|J_2$ are conformal. We note that $f|(J_1 \times J_2)$ is a factor of a topological Markov chain, and we denote the factor map by π. They obtained the following result.

Theorem 3.10. *If*

$$\min_{x_1 \in J_1} \|d_{x_1} f_1\| \geq \max_{x_2 \in J_2} \|d_{x_2} f_2\|,$$

then for any compact f-invariant set $J \subset J_1 \times J_2$ such that $\pi^{-1} J$ satisfies specification we have

$$\dim_H J = \sup_\mu \dim_H \mu,$$

where the supremum is taken over all ergodic f-invariant probability measures in J.

For piecewise linear maps f_i, Gatzouras and Lalley [35] proved earlier that certain invariant sets, corresponding to full shifts in the symbolic dynamics, carry an ergodic measure of full dimension. Kenyon and Peres [54] obtained the same result for linear maps f_i and arbitrary compact invariant sets. The proof of Theorem 3.10 uses the result in [35] and the idea from [54] of approximating arbitrary invariant sets by self-affine sets (corresponding to full shifts in their symbolic dynamics). Bedford and Urbański considered a particular class of self-affine sets in [10] and obtained conditions for the existence of a measure of full dimension.

Earlier related ideas appeared in work of Bedford [8] and McMullen [67]. We briefly describe their results. Given integers $m \leq n$ and a set $D \subset \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$, we define

$$J = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-k} & 0 \\ 0 & m^{-k} \end{pmatrix} d_k : d_k \in D \text{ for each } k \in \mathbb{N} \right\}.$$

Then J is called a *general Sierpiński carpet.* We note that viewed as a subset of the two-torus $\mathbb{T}^2$ the set J is invariant under the map $\left(\begin{smallmatrix} n & 0 \\ 0 & m \end{smallmatrix}\right)$. Bedford and McMullen obtained the following result independently.

Theorem 3.11. *There exists an ergodic measure of full dimension in J.*

The following higher-dimensional generalization was obtained by Kenyon and Peres [54].

Theorem 3.12. *If $J \subset \mathbb{T}^m$ is invariant under a toral endomorphism whose eigenvalues are roots of integers, then there exists an ergodic measure of full dimension in J.*

More recently, Yayama [95] considered general Sierpiński carpets modeled by arbitrary topological Markov chains, and she gave conditions for the existence of a unique measure of full dimension. In another direction, Luzia [60] considered expanding maps of the 2-torus of the form $f(x, y) = (a(x, y), b(y))$

that are C^2-perturbations of linear maps. He showed that if f is sufficiently C^2-close to a general Sierpiński carpet, then

$$\dim_H J = \sup_\mu \dim_H \mu, \tag{25}$$

where the supremum is taken over all ergodic f-invariant probability measures in J. He also showed in [61] that the supremum in (25) is attained.

3.7. **Nonuniformly expanding repellers.** Only a few related results exist in the case of transformations that are not uniformly expanding.

We first formulate a result established by Gelfert in [38]. For each $s \in [0, \dim M]$, let

$$\varphi^s(x) = \log \omega_s((d_x f)^{-1})$$

with ω_s as in (18).

Theorem 3.13. *If J is a compact invariant set of an expansive C^1 local diffeomorphism f, then*

$$\overline{\dim}_B J \le \inf\left\{ s \in (0, \dim M] \colon \sup_{x\in J} \varphi^s(x) < 1 \text{ and } P(\varphi^s) < 0 \right\}.$$

The proof is analogous to that of the upper bound in Theorem 3.3, and uses the characterization of the singular values as semiaxes of ellipsoids as well as covering techniques in the spirit of [19]. We note that the proof in [38] contains a mistake, although the statement remains true under the additional hypothesis of expansivity.

In a different direction, Horita and Viana [45] and Dysman [20] studied abstract models, called maps with holes, which include examples of nonuniform repellers. Namely, let $f \colon M \to M$ be a map such that there exist domains (that is, compact path-connected sets) $R_1, \dots, R_p \subset M$ with pairwise disjoint interiors, and such that $\overline{\dim}_B \partial R_i < \dim M$ for $i = 1, \dots, p$. We also assume that each restriction $f|R_i$ is a $C^{1+\alpha}$ diffeomorphism onto some domain W_i containing $R_1 \cup \cdots \cup R_p$. Moreover, we require that the holes

$$H_i = W_i \setminus (R_1 \cup \cdots \cup R_p)$$

have nonempty interior, and that the inner diameter (the supremum of the inner distances between any two points in the same connected component) of each set R_i is finite. The *repeller* of a map f with holes is the set of points whose forward orbit never falls into the holes, that is,

$$J = \left\{ x \in M \colon f^k(x) \in R_1 \cup \cdots \cup R_p \text{ for } k \in \mathbb{N} \right\}.$$

Given $n \in \mathbb{N}$ and $i_1, \dots, i_n \in \{1, \dots, p\}$ we set

$$C_{i_1 \cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} R_{i_{k+1}},$$

and

$$\varphi_n(i_1 \cdots i_n) = \frac{1}{n} \sum_{k=1}^{n} \inf_{x \in C_{i_1 \cdots i_k}} \log \| d_{f^k(x)} f^{-1} \|^{-1}.$$

Now we assume that there exist constants β, $\gamma > 0$ such that

$$\sum_{i_1\cdots i_n} m(C_{i_1\cdots i_n}) \le e^{-\beta n}$$

for any sufficiently large n, where the sum is taken over all n-tuples $i_1\cdots i_n$ such that $\varphi_n(i_1\cdots i_n) \le \gamma$, and where m denotes the Lebesgue measure. That is, although the expansion in J need not be uniform, the Lebesgue measure of the set of points remaining within a small neighborhood of J decreases exponentially fast with time. Under these assumptions, Dysman [20] showed that $\overline{\dim}_B J < \dim M$. This inequality generalizes a corresponding result for the Hausdorff dimension by Horita and Viana in [45] obtained under the same assumptions.

In [46] Horita and Viana investigate nonuniformly expanding repellers emerging from a perturbation of an Anosov diffeomorphism f of the 3-dimensional torus through a Hopf bifurcation. A saddle fixed point becomes an attractor, and the complement of its basin of attraction can be considered a repeller for the new diffeomorphism. We note that since the repeller contains an invariant circle produced by the bifurcation it is not uniformly expanding. Based on the above result of Dysman it is shown in [46] that the box dimension of the repeller is strictly less than 3 for all diffeomorphisms sufficiently C^5-close to f but different from f.

3.8. **Upper bounds involving Lyapunov exponents.** Now we describe briefly some upper bounds involving Lyapunov exponents. Related ideas go back to Kaplan and Yorke in [53]. The *local Lyapunov exponents* $\nu_1(x) \ge \cdots \ge \nu_n(x)$ of a smooth map $f\colon M \to M$ of a n-dimensional manifold are defined recursively by

$$\nu_1(x) + \cdots + \nu_j(x) = \limsup_{m\to\infty} \frac{1}{m} \log \omega_j(d_x f^m), \qquad j = 1, \dots, n.$$

Now we consider an ergodic f-invariant probability measure μ in M . Then the local Lyapunov exponents are constant μ-almost everywhere, and we denote them by $\nu_1(\mu) \ge \cdots \ge \nu_n(\mu)$. The *Lyapunov dimension* with respect to μ is defined by

$$\dim_L(f,\mu) = k + \frac{\nu_1(\mu) + \cdots + \nu_k(\mu)}{|\nu_{k+1}(\mu)|},$$

where $k < n$ is the smallest integer for which $\nu_1(\mu) + \cdots + \nu_{k+1}(\mu) < 0$. Ledrappier established the following statement in [55].

Theorem 3.14. *If J is a compact f-invariant set, then*

$$\sup_\mu \dim_L(f,\mu) = \inf_{m\in\mathbb{N}} \dim_L(f^m, J),$$

where the supremum is taken over all ergodic f-invariant probability measures in J.

Therefore, it follows from (20) that

$$\dim_H J \le \sup_\mu \dim_L(f,\mu).$$

There are examples where the supremum coincides with the Hausdorff dimension. Let us consider C^2 diffeomorphism f on a compact manifold and an ergodic f-invariant probability measure μ in M. It follows from work of Ledrappier and Young in [56] that if μ is an SRB measure, then

$$\dim_H J = \sup_{\mu} \dim_L(f, \mu). \tag{26}$$

However, as illustrated by Example 3.16 below in general (26) does not hold.

Now we consider the so-called *uniform Lyapunov exponents* $\nu_1^u \geq \cdots \geq \nu_n^u$ defined recursively by

$$\nu_1^u + \cdots + \nu_j^u = \lim_{m\to\infty} \frac{1}{m} \log \max_{x\in J} \omega_j(d_x f^m), \qquad j = 1, \ldots, n,$$

and we define

$$d^u(f, J) = k + \frac{\nu_1^u + \cdots + \nu_k^u}{|\nu_{k+1}^u|},$$

where $k < n$ is the smallest integer for which $\nu_1^u + \cdots + \nu_{k+1}^u < 0$. We note that

$$d^u(f, J) \leq \inf_{m\in\mathbb{N}} \dim_L(f^m, J).$$

Temam [88] obtained the following result.

Theorem 3.15. *If f is a C^1 map, and J is a compact f-invariant set, then*

$$\overline{\dim}_B \Lambda \leq d^u(f, J).$$

We also refer to [88] for generalizations to the infinite-dimensional setting, and for applications to attractors of many physical systems. See [21, 22, 23, 24] for further developments. Related problems were considered by Blinchevskaya and Ilyashenko [12] and Chapyzhov and Ilyin [17].

Leonov [58] developed techniques to compute the Lyapunov dimension involving Lyapunov functions. These can be interpreted as changes of Riemannian metrics. For more details see [37, 58]. We give an example.

Example 3.16. Given constants $\alpha > 0$, $\beta \in (0, 1)$, we consider the Hénon map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$(x, y) \mapsto (\alpha + \beta y - x^2, x).$$

This is conjugate to the map $(x, y) \mapsto (1+y-\alpha x^2, \beta x)$ by the linear transformation $L(x, y) = (\alpha x, \alpha\beta^{-1} y)$. We consider the function $\psi(x, y) = \gamma(x+\beta y)$, where

$$\gamma = \frac{1 - s}{(\beta - 1 - 2x_0)\sqrt{x_0^2 + \beta}}, \quad x_0 = (\beta - 1 - \sqrt{(\beta - 1)^2 + 4\alpha})/2,$$

and $s \in (0, 1)$ is a parameter. We introduce a metric g in $\mathbb{R}^2$ by

$$g_{(x,y)}(v, w) = \exp\left(\frac{1}{1+s}\psi(x, y)\right) \langle Av, w\rangle, \quad A = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix},$$

where the inner product $\langle\cdot, \cdot\rangle$ is induced by the Euclidean metric. The singular values of $d_{(x,y)}f$ with respect to the Riemannian structure are

$$\sigma_i(d_{(x,y)}f) = \exp\left(\frac{2}{1+s}(\psi(f(x, y)) - \psi(x, y))\right) \zeta_i(x, y),$$

where $\zeta_i(x,y)$ are the singular values of the matrix $\sqrt{A}d_{(x,y)}f\sqrt{A^{-1}}$. For

$$s = \frac{1}{1 - \log\beta/\log\zeta_1(x_0,x_0)}$$

one can show (see [58]) that

$$1 = \omega_{1+s}(d_{p_0}f) = \sup_{(x,y)\in\mathbb{R}^2} \omega_{1+s}(d_{(x,y)}f), \tag{27}$$

where $p_0 = (x_0,x_0)$ is a fixed point of f. For this fixed point we obtain

$$1 + s = \dim_L(f,p_0) \le d^u(f,J)$$

for any f-invariant compact set $J \subset \mathbb{R}^2$ containing p_0. Hence, it follows from (27) that

$$1 + s = d^u(f,J) = \lim_{m\to\infty} \dim_L(f^m,J) = \dim_L(f,J).$$

4. Dimension estimates for hyperbolic sets

We consider in this section the dimension of hyperbolic sets, and we describe a corresponding theory to the one in Section 3 in the case of repellers. In particular, besides describing how Markov partitions can be used to model hyperbolic sets and considering the well-understood case of conformal dynamics, we discuss the existing results for nonconformal and nonuniformly hyperbolic dynamics. We also consider the problem of the existence of measures of maximal dimension.

4.1. **Notion of hyperbolic set and examples.** Let $f\colon M \to M$ be a diffeomorphism, and let $\Lambda \subset M$ be a compact f-invariant set. We say that Λ is a *hyperbolic set* for f if for every point $x \in \Lambda$ there exists a decomposition of the tangent space

$$T_xM = E^s(x) \oplus E^u(x)$$

such that

$$d_xfE^s(x) = E^s(f(x)) \quad \text{and} \quad d_xfE^u(x) = E^u(f(x)),$$

and there exist constants $\lambda \in (0,1)$ and $c > 0$ such that

$$\|d_xf^n|E^s(x)\| \le c\lambda^n \quad \text{and} \quad \|d_xf^{-n}|E^u(x)\| \le c\lambda^n$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$.

Example 4.1 (Linear horseshoes in $\mathbb{R}^2$)**.** Consider a C^1 map contracting the unit square $Q = [0,1]^2 \subset \mathbb{R}^2$ horizontally by a factor $\lambda < 1/2$, expanding it vertically by a factor $1/\lambda$, and folding the resulting rectangle into a horseshoe. We assume that the set $f(Q) \cap Q$ has two vertical components V_1 and V_2, and that the set $Q \cap f^{-1}(Q)$ has two horizontal components H_1 and H_2. The compact f-invariant set $\Lambda \subset Q$ given by

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^nQ$$

is called a *Smale horseshoe*, and it is a hyperbolic set for f with respect to the decomposition of the tangent space into horizontal and vertical lines $T_x\mathbb{R}^2 = E^s(x) \oplus E^u(x)$, where $E^s(x) = \mathbb{R}$ is the horizontal line and $E^u(x) = \mathbb{R}$ is the vertical line, for each $x \in \Lambda$.

Example 4.2 (Solenoids)**.** Let D be the unit disc in $\mathbb{R}^2$ centered at the origin. Consider the map $f\colon S^1 \times D \to S^1 \times D$ given by

$$f(\theta, x, y) = (2\theta \bmod 1, \lambda x + \varepsilon \cos(2\pi\theta), \mu y + \varepsilon \sin(2\pi\theta)),$$

for some constants $\lambda, \mu < \min\{\varepsilon, 1/2\}$. The limit set

$$\Lambda = \bigcap_{n\in\mathbb{N}} f^n(S^1 \times D)$$

is called a *solenoid*. One can easily verify that Λ is a hyperbolic set for f. Furthermore, Λ is an attractor, in the sense that there exists an open neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$.

4.2. **Additional structure.** Any hyperbolic set Λ has a very rich structure. Given $\varepsilon > 0$, for each $x \in \Lambda$ we consider the sets

$$V^s(x) = \big\{y \in B(x, \varepsilon) : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \geq 0\big\},$$

and

$$V^u(x) = \big\{y \in B(x, \varepsilon) : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \leq 0\big\},$$

where d is the distance in M and $B(x, \varepsilon) \subset M$ is the open ball of radius ε centered at x. If Λ is a hyperbolic set for a C^1 diffeomorphism, then there exists $\varepsilon > 0$ such that for each $x \in \Lambda$ the sets $V^s(x)$ and $V^u(x)$ are smooth manifolds containing x, and satisfying

$$T_x V^s(x) = E^s(x) \quad \text{and} \quad T_x V^u(x) = E^u(x).$$

The manifolds $V^s(x)$ and $V^u(x)$ are called respectively *local stable manifold* and *local unstable manifold* at x (of size ε). Furthermore, there exists $\delta = \delta(\varepsilon) > 0$ such that if $d(x, y) < \delta$ for some x, $y \in \Lambda$, then the intersection $V^s(x) \cap V^u(y)$ has exactly one point. The function

$$[\cdot, \cdot]\colon \big\{(x, y) \in \Lambda \times \Lambda : d(x, y) < \delta\big\} \to M$$

defined by $[x, y] = V^s(x) \cap V^u(y)$ is called a *product structure*.

We say that a set $\Lambda \subset M$ is *locally maximal* if there is an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n U.$$

In this case there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in \Lambda$ with $d(x, y) < \delta$ we have $\operatorname{card}[x, y] = 1$ and $[x, y] \in \Lambda$. A nonempty closed set $R \subset \Lambda$ is called a *rectangle* if $\operatorname{diam} R < \delta$, $R = \overline{\operatorname{int} R}$ (with the interior computed with respect to the induced topology on Λ), and $[x, y] \in R$ whenever $x, y \in R$. A finite cover of Λ by rectangles $R_1, \ldots, R_p$ is called a *Markov partition* of Λ (with respect to f) if:

1. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ whenever $i \neq j$;
2. if $x \in \operatorname{int} R_i \cap f^{-1}(\operatorname{int} R_j)$, then

$$f(V^s(x) \cap R_i) \subset V^s(f(x)) \cap R_j$$

and

$$V^u(f(x)) \cap R_j \subset f(V^u(x) \cap R_i).$$

Any locally maximal hyperbolic set has Markov partitions with arbitrarily small diameter (see [15]).

Markov partitions yield symbolic models for the hyperbolic set. Let Λ be a locally maximal hyperbolic set, and let $R_1, \dots, R_p$ be a Markov partition of Λ. We define a $p \times p$ matrix $A = (a_{ij})$ with entries a_{ij} as in (11). We also consider the space of two-sided sequences $\Sigma_p = \{1, \dots, p\}^{\mathbb{Z}}$, and the corresponding shift map $\sigma\colon \Sigma_p \to \Sigma_p$. The restriction of σ to the σ-invariant set

$$\Sigma_A = \big\{(\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma_p : a_{i_n i_{n+1}} = 1 \text{ for every } n \in \mathbb{Z}\big\}$$

is called the *(two-sided) topological Markov chain* with *transition matrix* A. In a similar manner to that in the case of repellers, one can define a map $\chi\colon \Sigma_A \to \Lambda$ by

$$\chi(\cdots i_{-1} i_0 i_1 \cdots) = \bigcap_{k\in\mathbb{Z}} f^{-k} R_{i_k}.$$

The map χ is surjective, Hölder continuous with respect to the distance

$$d(\cdots i_0 \cdots, \cdots j_0 \cdots) = \sum_{k\in\mathbb{Z}} e^{-|k|} |i_k - j_k|,$$

and it satisfies $\chi \circ \sigma = f \circ \chi$.

4.3. **A priori bounds for the dimension.** The following upper bound for the box dimension of a hyperbolic set was obtained by Fathi in [32]. Compared to the sharper dimension estimates discussed below it has the advantage of not using the topological pressure.

Theorem 4.3 ([32])**.** *If Λ is a hyperbolic set for a C^1 diffeomorphism f, then $\overline{\dim}_B \Lambda \le -2h(f)/\lambda$, where*

$$\lambda = \max\left\{\lim_{n\to\infty} \frac{1}{n} \log \max_{x\in\Lambda} \|d_x f^n | E^s\|, \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in\Lambda} \|(d_x f^n)^{-1} | E^u\|\right\}.$$

Now we describe sharper upper as well as lower dimension estimates. Given a locally maximal hyperbolic set Λ for a C^1 diffeomorphism f, we define functions $\underline{\varphi}_s, \overline{\varphi}_s\colon \Lambda \to \mathbb{R}$ by

$$\underline{\varphi}_s(x) = -\log\|(d_x f)^{-1} | E^s\| \quad \text{and} \quad \overline{\varphi}_s(x) = \log\|d_x f | E^s\|, \tag{28}$$

and $\underline{\varphi}_u, \overline{\varphi}_u\colon \Lambda \to \mathbb{R}$ by

$$\underline{\varphi}_u(x) = -\log\|d_x f | E^u\| \quad \text{and} \quad \overline{\varphi}_u(x) = \log\|(d_x f)^{-1} | E^u\|. \tag{29}$$

Let also $R_1, \dots, R_p$ be a Markov partition of Λ. Given a point $x \in \Lambda$ and an element R_{i_0} of the Markov partition containing x we set

$$A^s(x) = V^s(x) \cap R_{i_0} \quad \text{and} \quad A^u(x) = V^u(x) \cap R_{i_0}.$$

We denote by $\underline{t}_s(x)$ and $\overline{t}_s(x)$ the unique roots of the equations

$$P_{f|A^s(x)}(t\underline{\varphi}_s) = 0 \quad \text{and} \quad P_{f|A^s(x)}(t\overline{\varphi}_s) = 0,$$

and by $\underline{t}_u(x)$ and $\overline{t}_u(x)$ the unique roots of the equations

$$P_{f|A^u(x)}(t\underline{\varphi}_u) = 0 \quad \text{and} \quad P_{f|A^u(x)}(t\overline{\varphi}_u) = 0.$$

Here $P_Z(\varphi)$ denotes the nonadditive topological pressure on the set Z of the additive sequence of functions $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$. The following result was established by Barreira in [2].

Theorem 4.4. *If Λ is a locally maximal hyperbolic set of a topologically transitive C^1 diffeomorphism, then for every $x \in \Lambda$ we have*

$$\underline{t}_s(x) \le \dim_H(V^s(x)\cap\Lambda) \le \underline{\dim}_B(V^s(x)\cap\Lambda) \le \overline{\dim}_B(V^s(x)\cap\Lambda) \le \overline{t}_s(x),$$

and

$$\underline{t}_u(x) \le \dim_H(V^u(x)\cap\Lambda) \le \underline{\dim}_B(V^u(x)\cap\Lambda) \le \overline{\dim}_B(V^u(x)\cap\Lambda) \le \overline{t}_u(x).$$

The proof of Theorem 4.4 is an elaboration of the proof of Theorem 3.3.

We also describe a result of Franz in [34]. Let $\Lambda \subset M$ be a compact f-invariant set for a C^1 diffeomorphism f with an equivariant splitting $T_\Lambda M = E^1 \oplus E^2$. This means that

$$d_x f E^i(x) = E^i(f(x)), \quad i = 1, 2$$

for every $x \in \Lambda$. We note that Λ is not necessarily hyperbolic. We relabel the singular values

$$\sigma_1(d_x f|E^1), \ldots, \sigma_{\dim E^1}(d_x f|E^1), \sigma_1((d_y f|E^2)^{-1}), \ldots, \sigma_{\dim E^2}((d_y f|E^2)^{-1})$$

as $\sigma_1(x,y) \ge \cdots \ge \sigma_{\dim M}(x,y)$, and for each $s \in [0, \dim M]$ we write

$$\overline{\omega}_s(x,y) = \sigma_1(x,y)\cdots\sigma_{\lfloor s\rfloor}(x,y)\sigma_{\lfloor s\rfloor+1}(x,y)^{s-\lfloor s\rfloor}.$$

Franz showed in [34] that

$$\dim_H\Lambda \le \inf\left\{ s \in [0,\dim M]\colon\ \log \max_{x,y\in\Lambda} \overline{\omega}_s(x,y) < -2h(f|\Lambda) \right\},$$

where $h(f|\Lambda)$ denotes the topological entropy on Λ. This result generalizes work of Gu in [39] for hyperbolic sets of C^2 maps satisfying a certain pinching condition.

The following result requires the existence of an equivariant subbundle with respect to which f is volume-expanding. It was established by Gelfert in [38]. We recall the function ω_s in (18).

Theorem 4.5. *If Λ is a compact f-invariant set of an expansive C^1 local diffeomorphism f, and $E \subset T_\Lambda M$ is an equivariant bundle, then*

$$\overline{\dim}_B\Lambda \le \operatorname{codim} E + \inf\Big\{ s \in [0, \operatorname{rank} E]\colon \max_{x\in\Lambda} \omega_s((d_x f|E)^{-1}) < 1 \text{ and } P(\log\omega_s((d_x f|E)^{-1})) < 0 \Big\}$$

(with the convention $\inf\varnothing = \operatorname{rank} E$*).*

Shafikov and Wolf showed in [82] that if Λ is a hyperbolic set for a C^2 diffeomorphism $f\colon M \to M$, then

$$\dim_H\Lambda \le \dim M + \frac{P(-\log|\det df|E^u|)}{\lambda}, \tag{30}$$

where

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \max_{x\in\Lambda} \|d_x f^n\|.$$

See [90] for the case of axiom A endomorphisms. We note that (30) follows from Theorem 4.5.

4.4. **The conformal case.** We discuss in this section the particular case of hyperbolic sets of conformal maps, in which case the inequalities in Theorem 4.4 become identities.

We say that $f\colon M \to M$ is *conformal* on a hyperbolic set $\Lambda \subset M$ if $d_xf|E^s$ and $d_xf|E^u$ are multiples of isometries for every $x \in \Lambda$. For example, if M is a surface and $\dim E^s(x) = \dim E^u(x) = 1$ for every $x \in \Lambda$, then f is conformal on Λ. We note that in this case the functions in (28) and (29) satisfy

$$\varphi_s(x) := \underline{\varphi}_s(x) = \overline{\varphi}_s(x) = \log\|d_xf|E^s\|, \tag{31}$$

and

$$\varphi_u(x) := \underline{\varphi}_u(x) = \overline{\varphi}_u(x) = -\log\|d_xf|E^u\|. \tag{32}$$

The following result is thus a consequence of Theorem 4.4.

Theorem 4.6. *If Λ is a locally maximal hyperbolic set of a topologically transitive C^1 diffeomorphism which is conformal on Λ, then for every $x \in \Lambda$ we have*

$$\dim_H(V^s(x)\cap\Lambda) = \underline{\dim}_B(V^s(x)\cap\Lambda) = \overline{\dim}_B(V^s(x)\cap\Lambda) = t_s, \tag{33}$$

and

$$\dim_H(V^u(x)\cap\Lambda) = \underline{\dim}_B(V^u(x)\cap\Lambda) = \overline{\dim}_B(V^u(x)\cap\Lambda) = t_u, \tag{34}$$

where t_s and t_u are the unique real numbers such that

$$P(t_s\varphi_s) = P(t_u\varphi_u) = 0.$$

McCluskey and Manning showed in [66] that

$$\dim_H(V^s(x)\cap\Lambda) = t_s \quad \text{and} \quad \dim_H(V^u(x)\cap\Lambda) = t_u$$

for every $x \in \Lambda$. The equality between the Hausdorff dimension and the lower and upper box dimensions is due to Takens in [87] for C^2 diffeomorphisms (see also [71]) and to Palis and Viana in [72] in the general case. Barreira in [2] and Pesin in [74] presented alternative proofs based on the thermodynamic formalism. We emphasize that in higher dimensions the Hausdorff and box dimensions of a locally maximal hyperbolic set may not coincide.

Moreover we have the following statement.

Theorem 4.7. *If Λ is a locally maximal hyperbolic set of a topologically transitive C^1 diffeomorphism which is conformal on Λ, then*

$$\dim_H\Lambda = \underline{\dim}_B\Lambda = \overline{\dim}_B\Lambda = t_s + t_u.$$

The first complete argument establishing Theorem 4.7 in the case of surface diffeomorphisms was given by Barreira in [2]. The proof of Theorem 4.7 given by Pesin in [74] includes the general case of conformal diffeomorphisms on manifolds of arbitrary dimension (the statement can also be obtained from results in [2]). Under the assumptions in the theorem the product structure is a Lipschitz homeomorphism with Lipschitz inverse (see [41] for details), and thus Theorem 4.7 follows easily from Theorem 4.6 by adding the dimensions in the stable and unstable directions, which is possible due to the coincidence between the Hausdorff and box dimensions in (33) and (34) (otherwise the Lipschitz property would be of little help).

Palis and Viana in [72] established the continuous dependence of the dimension on the diffeomorphism. Let Λ be a locally maximal hyperbolic set of a C^1 surface diffeomorphism f with $\dim E^s = \dim E^u = 1$. Then there is a neighborhood $\mathcal{U}$ of f in the space of C^1 diffeomorphisms and a continuous map $\mathcal{U} \ni g \mapsto h_g$ to the space of all continuous embeddings $\Lambda_f \to M$ satisfying $h_f = \mathrm{id}$, such that $\Lambda_g = h_g(\Lambda_f)$ is a locally maximal hyperbolic set for g, with $f|\Lambda_f$ and $g|\Lambda_g$ topologically conjugate.

Theorem 4.8 ([72])**.** *The function $g \mapsto \dim_H \Lambda_g$ is continuous.*

Mañé [64] obtained even higher regularity for the dimension.

Theorem 4.9. *Let Λ be a locally maximal and totally disconnected hyperbolic set of a C^r surface diffeomorphism f, $r \geq 2$ with $\dim E^s = \dim E^u = 1$. Then there is a C^r neighborhood $\mathcal{U}$ of f and a C^r map $\mathcal{U} \ni g \mapsto h_g$ to the space of continuous embeddings $\Lambda_f \to M$ such that $g \mapsto \dim_H h_g(\Lambda_f)$ is of class C^{r-1}.*

In higher-dimensional manifolds (and thus in the nonconformal case) the Hausdorff dimension of hyperbolic sets may vary discontinuously. Examples were given by Pollicott and Weiss in [75] followed by Bonatti, Díaz, and Viana in [13]. Díaz and Viana [18] studied a one-parameter family of diffeomorphisms on the 2-torus bifurcating from an Anosov map to a DA map. They showed that for an open set of these families the Hausdorff and box dimensions of the nonwandering set are discontinuous across the bifurcation.

4.5. **Measures of maximal dimension.** We also address the question of the existence of measures of full dimension for hyperbolic sets. It turns out that the answer is almost always negative. More precisely, by results of McCluskey and Manning in [66] there exists an invariant measure of full dimension if and only if there exists a continuous function $\psi\colon \Lambda \to \mathbb{R}$ such that

$$t_s\varphi_s - t_u\varphi_u = \psi \circ f - f$$

on Λ, with φ_s and φ_u as in (31) and (32). By Livschitz's theorem, this happens if and only if

$$\|d_x f|E^s(x)\|^{t_s} \|d_x f|E^u(x)\|^{t_u} = 1$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$ such that $f^n(x) = x$.

Instead one can ask whether the supremum

$$\delta(f) = \sup\left\{\dim_H \mu : \mu \text{ is an } f\text{-invariant measure on } \Lambda\right\}$$

is attained. Any f-invariant measure on Λ with $\dim_H \nu = \delta(f)$ is called a measure of *maximal dimension*. The main difficulty of this problem is that the map $\mu \mapsto \dim_H \mu$ is not upper semi-continuous: simply consider the sequence $(\mu + (n-1)\delta)/n$ where $\dim_H \mu > 0$ and δ is an atomic measure. The following result was obtained by Barreira and Wolf in [7].

Theorem 4.10. *Let Λ be a locally maximal hyperbolic set for a $C^{1+\alpha}$ surface diffeomorphism which is topologically mixing on Λ. Then there exists an ergodic measure of maximal dimension.*

See [94] for a related result of Wolf in the particular case of polynomial automorphisms of $\mathbb{C}^2$. It was shown by Rams in [77] that a unique ergodic measure of maximal dimension does not exist in general, even in the case of linear horseshoes (more precisely, he showed that there exists a one-parameter family of Bernoulli measures of maximal dimension).

4.6. **Further developments.** Related ideas to those in the proof of Theorem 3.7 were applied by Simon and Solomyak in [85] to compute the Hausdorff dimension of a class of horseshoes in $\mathbb{R}^3$, obtained from $C^{1+\alpha}$ transformations of the form

$$(x, y, z) \mapsto (\varphi(x, z) + a_k, \psi(y, z) + b_k, \zeta(z)). \tag{35}$$

Assuming that $|\partial\varphi/\partial x|$, $|\partial\psi/\partial y| < 1/2$ and$|\zeta'| > 1$, it is shown in [85] how to compute the Hausdorff dimension of the horseshoe for Lebesgue almost every $(a, b) \in \mathbb{R}^2$. Again, the dimension is expressed implicitly in terms of the topological pressure.

More precisely, we assume that there are disjoint closed intervals $I_1, \dots, I_p$ whose union is a proper subset of $[0, 1]$ such that letting $\Delta_k = [0, 1]^2 \times I_k$ we have

$$f\Big(\bigcup_{k=1}^{p} \Delta_k\Big) \subset (0, 1)^2 \times [0, 1],$$

with f given by (35) in Δ_k. The limit set

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n([0, 1]^3)$$

is the horseshoe. Now let s_1, s_2, r_1, and r_2 be the unique solutions of the equations

$$P\Big(s_1 \log\Big|\frac{\partial\varphi}{\partial x}\Big|\Big) = 0, \quad P\Big(s_2 \log\Big|\frac{\partial\psi}{\partial y}\Big|\Big) = 0,$$
$$P\Big(\log\Big(\Big|\frac{\partial\varphi}{\partial x}\Big| \cdot \Big|\frac{\partial\psi}{\partial y}\Big|^{r_1}\Big)\Big) = 0, \quad P\Big(\log\Big(\Big|\frac{\partial\psi}{\partial y}\Big| \cdot \Big|\frac{\partial\varphi}{\partial x}\Big|^{r_2}\Big)\Big) = 0.$$

We also set $s = \max\{s_1, s_2\}$ and $r = \max\{r_1, r_2\}$. Finally, let t be the unique solution of the equation

$$P(-t \log|\zeta'|) = 0.$$

Then we have the following statement.

Theorem 4.11 ([85])**.** *For Lebesgue almost every vector* $(a_1, b_1, \dots, a_p, b_p)$ *we have:*

(i) *if* $s \le 1$ *then*

$$\dim_H\Lambda = \underline{\dim}_B\Lambda = \overline{\dim}_B\Lambda = t + s;$$

(ii) *if* $s > 1$ *then*

$$\dim_H\Lambda = \underline{\dim}_B\Lambda = \overline{\dim}_B\Lambda = t + 1 + \min\{1, r\}.$$

In particular, the dimension does not depend on the vector.

Finally, we consider more general solenoids from those in Example 4.2. We recall that a *solenoid* is a hyperbolic set $\Lambda = \bigcap_{n=1}^{\infty} f^n T$, where $T \subset \mathbb{R}^3$ is diffeomorphic to a solid torus $S^1 \times D$ for some closed disk $D \subset \mathbb{R}^2$, and

$f\colon T \to T$ extends to a diffeomorphism in some open neighborhood of T such that for each $x \in S^1$ the *section*

$$\Lambda_x = f(T) \cap (\{x\} \times D)$$

is a disjoint union of a fixed number of sets homeomorphic to a closed disk. Bothe [14] and then Simon [84] (also using his methods in [83] for noninvertible transformations) studied the dimension of solenoids (see [74, 81] for a related discussion). In particular, it is shown in [14] that under certain conditions on the diffeomorphism the map $x \mapsto \dim_H \Lambda_x$ is constant (even though the holonomies are typically not Lipschitz). More recently, Hasselblatt and Schmeling conjectured in [43] (see also [42]) that, in spite of the difficulties due to the possible low regularity of the holonomies, the Hausdorff dimension of hyperbolic sets can be computed by adding the dimensions of the stable and unstable sections. They prove this conjecture for a class of solenoids, by showing that the Hausdorff dimension of the sections is in fact independent of the section.

For some related results in the case of nonuniformly hyperbolic invariant sets we mention the works of Hirayama [44] with an upper bound for the Hausdorff dimension of the stable set of the set of typical points for a hyperbolic measure, Fan, Jiang and Wu [31] with the study of the dimension of the maximal invariant set of an asymptotically nonhyperbolic family, and Urbánski and Wolf [91] who considered horseshoe maps that are uniformly hyperbolic except at a parabolic point, in particular establishing a dimension formula for the horseshoe.

References

1. A. Babin and M. Vishik, *Attractors of Evolution Equations*, Studies in Mathematics and its Applications 25. North-Holland, 1992.
2. L. Barreira, *A non-additive thermodynamic formalism and applications to dimension theory of hyperbolic dynamical systems*, Ergodic Theory Dynam. Systems **16** (1996), 871–927.
3. L. Barreira, *Dimension estimates in nonconformal hyperbolic dynamics*, Nonlinearity **16** (2003), 1657–1672.
4. L. Barreira, *Dimension and Recurrence in Hyperbolic Dynamics*, Progress in Mathematics 272, Birkhäuser, 2008.
5. L. Barreira, *Almost additive thermodynamic formalism: some recent developments*, preprint.
6. L. Barreira and K. Gelfert, *Multifractal analysis for Lyapunov exponents on nonconformal repellers*, Comm. Math. Phys. 267 (2006), 393–418.
7. L. Barreira and C. Wolf, *Measures of maximal dimension for hyperbolic diffeomorphisms*, Comm. Math. Phys. **239** (2003), 93–113.
8. T. Bedford, *Crinkly curves, Markov partitions and box dimension of self-similar sets*, Ph.D. Thesis, University of Warwick, 1984.
9. T. Bedford, *The box dimension of self-affine graphs and repellers*, Nonlinearity **2** (1989), 53–71.
10. T. Bedford and M. Urbański, *The box and Hausdorff dimension of self-affine sets*, Ergodic Theory Dynam. Systems **10** (1990), 627–644.
11. A. Ben-Artzi, A. Eden, C. Foias and B. Nicolaenko, *Hölder continuity for the inverse of Mañé's projection*, J. Math. Anal. Appl. **178** (1993), 22–29.
12. M. Blinchevskaya and Yu. Ilyashenko, *Estimates for the entropy dimension of the maximal attractor for k-contracting systems in an infinite-dimensional space*, Russ. J. Math. Phys. **6** (1999), 20–26.

13. C. Bonatti, L. Díaz and M. Viana, *Discontinuity of the Hausdorff dimension of hyperbolic sets*, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), 713–718.
14. H. Bothe, *The Hausdorff dimension of certain solenoids*, Ergodic Theory Dynam. Systems **15** (1995), 449–474.
15. R. Bowen, *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*, Lect. Notes in Math. 470, Springer, 1975.
16. R. Bowen, *Hausdorff dimension of quasi-circles*, Inst. Hautes Études Sci. Publ. Math. **50** (1979), 259–273.
17. V. Chepyzhov and A. Ilyin, *A note on the fractal dimension of attractors of dissipative dynamical systems*, Nonlinear Anal., Theory Methods Appl. **44A** (2001), 811–819.
18. L. Díaz and M. Viana, *Discontinuity of Hausdorff dimension and limit capacity on arcs of diffeomorphisms*, Ergodic Theory Dynam. Systems **9** (1989), 403–425.
19. A. Douady and J. Oesterlé, *Dimension de Hausdorff des attracteurs*, C. R. Acad. Sc. Paris **290** (1980), 1135–1138.
20. M. Dysman, *Fractal dimension for repellers of maps with holes*, J. Stat. Phys. **120** (2005), 479–509.
21. A. Eden, *Local estimates for the Hausdorff dimension of an attractor*, J. Math. Anal. Appl. **150** (1990), 100–119.
22. A. Eden, C. Foias and R. Temam, *Local and global Lyapunov exponents*, J. Dyn. Differ. Equations **3** (1991), 133–177.
23. A. Eden, C. Foias and B. Nicolaenko, *Exponential attractors of optimal Lyapunov dimension for Navier-Stokes equations*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), 1211–1215.
24. A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Research in Applied Mathematics 37, John Wiley & Sons, 1994.
25. K. Falconer, *The Hausdorff dimension of self-affine fractals*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 339–350.
26. K. Falconer, *A subadditive thermodynamic formalism for mixing repellers*, J. Phys. A **21** (1988), L737–L742.
27. K. Falconer, *Dimensions and measures of quasi self-similar sets*, Proc. Amer. Math. Soc. **106** (1989), 543–554.
28. K. Falconer, *The dimension of self-affine fractals. II*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 169–179.
29. K. Falconer, *Bounded distortion and dimension for non-conformal repellers*, Math. Proc. Cambridge Philos. Soc. **115** (1994), 315–334.
30. K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, Wiley, 2003.
31. A. Fan, Y. Jiang and J. Wu, *Asymptotic Hausdorff dimensions of Cantor sets associated with an asymptotically non-hyperbolic family*, Ergodic Theory Dynam. Systems **25** (2005), 1799–1808.
32. A. Fathi, *Expansiveness, hyperbolicity, and Hausdorff dimension*, Comm. Math. Phys. **126** (1989), 249–262.
33. C. Foias and E. Olson, *Finite fractal dimension and Hölder-Lipschitz parametrization*, Indiana Univ. Math. J. **45** (1996), 603–616.
34. A. Franz, *Hausdorff dimension estimates for invariant sets with an equivariant tangent bundle splitting*, Nonlinearity **11** (1998), 1063–1074.
35. D. Gatzouras and S. Lalley, *Hausdorff and box dimensions of certain self-affine fractals*, Indiana Univ. Math. J. **41** (1992), 533–568
36. D. Gatzouras and Y. Peres, *Invariant measures of full dimension for some expanding maps*, Ergodic Theory Dynam. Systems **17** (1997), 147–167.
37. K. Gelfert, *Maximal local Lyapunov dimension bounds the box dimension. Direct proof for invariant sets on Riemannian manifolds*, Z. Anal. Anwend. **22** (2003), 553–568.
38. K. Gelfert, *Dimension estimates beyond conformal and hyperbolic dynamics*, Dyn. Syst. **20** (2005), 267–280.
39. X. Gu, *An upper bound for the Hausdorff dimension of a hyperbolic set*, Nonlinearity **4** (1991), 927–934.

40. J. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs 25, Amer. Math. Soc., 1988.
41. B. Hasselblatt, *Regularity of the Anosov splitting and of horospheric foliations*, Ergodic Theory Dynam. Systems **14** (1994), 645–666.
42. B. Hasselblatt and J. Schmeling, *Dimension product structure of hyperbolic sets*, Electron. Res. Announc. Amer. Math. Soc. **10** (2004), 88–96.
43. B. Hasselblatt and J. Schmeling, *Dimension product structure of hyperbolic sets*, in Modern Dynamical Systems and Applications, Cambridge Univ. Press, Cambridge, 2004, pp. 331–345.
44. M. Hirayama, *An upper estimate of the Hausdorff dimension of stable sets*, Ergodic Theory Dynam. Systems **24** (2004), 1109–1125.
45. V. Horita and M. Viana, *Hausdorff dimension of non-hyperbolic repellers. I. Maps with holes*, J. Statist. Phys. **105** (2001), 835–862.
46. V. Horita and M. Viana, *Hausdorff dimension for non-hyperbolic repellers. II: DA diffeomorphisms*, Discrete Contin. Dyn. Syst. **13** (2005), 1125–1152.
47. H. Hu, *Box dimensions and topological pressure for some expanding maps*, Comm. Math. Phys. **191** (1998), 397–407.
48. I. Hueter and S. Lalley, *Falconer's formula for the Hausdorff dimension of a self-affine set in* $\mathbb{R}^2$, Ergodic Theory Dynam. Systems **15** (1995), 77–97.
49. B. Hunt, *Maximum local Lyapunov dimension bounds the box dimension of chaotic attractors*, Nonlinearity **9** (1996), 845–852.
50. B. Hunt and V. Kaloshin, *Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces*, Nonlinearity **12** (1999), 1263–1275.
51. B. Hunt, T. Sauer and J. Yorke, *Prevalence: a translation-invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 217–238.
52. J. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.
53. J. Kaplan and J. Yorke, *Chaotic behavior of multidimensional difference equations*, in Functional Differential Equations and Approximation of Fixed Points, Proc. Summer School and Conf. (Bonn, 1978), edited by H.-O. Peitgen and H.-O. Walther, Lect. Notes in Math. 730, Springer, 1979, pp. 204–227.
54. R. Kenyon and Y. Peres, *Measures of full dimension on affine-invariant sets*, Ergodic Theory Dynam. Systems **16** (1996), 307–323.
55. F. Ledrappier, *Some relations between dimension and Lyapunov exponents*, Comm. Math. Phys. **81** (1981), 229–238.
56. F. Ledrappier and L.-S. Young. *The metric entropy of diffeomorphisms. II: Relations between entropy, exponents and dimension*, Ann. of Math. (2) **122** (1985), 540–574.
57. G. Leonov, *On estimates of the Hausdorff dimension of attractors*, Vestn. Leningr. Univ., Ser. I **3** (1991), 41–44.
58. G. Leonov, *Lyapunov dimensions formulas for Hénon and Lorenz attractors*, Alg. & Anal. **13** (2001), 453–464.
59. N. Luzia, *Hausdorff dimension for an open class of repellers in* $\mathbb{R}^2$, Nonlinearity **19** (2006), 2895–2908.
60. N. Luzia, *A variational principle for the dimension for a class of non-conformal repellers*, Ergodic Theory Dynam. Systems **26** (2006), 821–845.
61. N. Luzia, *Measure of full dimension for some nonconformal repellers*, Discrete Contin. Dyn. Syst. **26** (2010), 291–302.
62. J. Mallet-Paret, *Negatively invariant sets of compact maps and an extension of a theorem of Cartwright*, J. Differential Equations **22** (1976), 331–348.
63. R. Mañé, *On the dimension of the compact invariant sets of certain non-linear maps*, in Dynamical Systems and Turbulence (Warwick, 1979/80), Lect. Notes in Math. 898, Springer, 1981, pp. 230–242.
64. R. Mañé, *The Hausdorff dimension of horseshoes of diffeomorphisms of surfaces*, Bol. Soc. Brasil. Mat. (N.S.) **20** (1990), 1–24.
65. A. Manning and K. Simon, *Subadditive pressure for triangular maps*, Nonlinearity **20** (2007), 133–149.
66. H. McCluskey and A. Manning, *Hausdorff dimension for horseshoes*, Ergodic Theory Dynam. Systems **3** (1983), 251–260.

67. C. McMullen, *The Hausdorff dimension of general Sierpiński carpets*, Nagoya Math. J. **96** (1984), 1–9.
68. P. Moran, *Additive functions of intervals and Hausdorff measure*, Proc. Cambridge Philos. Soc. **42** (1946), 15–23.
69. J. Neunhäuserer, *Number theoretical peculiarities in the dimension theory of dynamical systems*, Israel J. Math. **128** (2002), 267–283.
70. A. Noack and V. Reitmann, *Hausdorff dimension estimates for invariant sets of time-dependent vector fields*, Z. Anal. Anw. **15** (1996), 457–473.
71. J. Palis and F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations*, Cambridge University Press, 1993.
72. J. Palis and M. Viana, *On the continuity of Hausdorff dimension and limit capacity for horseshoes*, in Dynamical Systems (Valparaiso, 1986), edited by R. Bamón, R. Labarca and J. Palis, Lect. Notes in Math. 1331, Springer, 1988, pp. 150–160.
73. Y. Peres, *The self-affine carpets of McMullen and Bedford have infinite Hausdorff measure*, Math. Proc. Cambridge Philos. Soc. **116** (1994), 513–526.
74. Ya. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, Chicago Lectures in Math., Chicago University Press, 1997.
75. M. Pollicott and H. Weiss, *The dimensions of some self-affine limit sets in the plane and hyperbolic sets*, J. Statist. Phys. **77** (1994), 841–866.
76. F. Przytycki and M. Urbański, *On the Hausdorff dimension of some fractal sets*, Studia Math. **93** (1989), 155–186.
77. M. Rams, *Measures of maximal dimension for linear horseshoes*, Real Anal. Exchange **31** (2005/06), 55–62.
78. D. Ruelle, *Statistical mechanics on a compact set with $\mathbb{Z}^\nu$ action satisfying expansiveness and specification*, Trans. Amer. Math. Soc. **185** (1973), 237–251.
79. D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Mathematics and Its Applications 5, Addison-Wesley, 1978.
80. D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory Dynam. Systems **2** (1982), 99–107.
81. J. Schmeling and H. Weiss, *An overview of the dimension theory of dynamical systems*, in Smooth Ergodic Theory and Its Applications (Seattle, 1999), edited by A. Katok, R. de la Llave, Ya. Pesin and H. Weiss, Proceedings of Symposia in Pure Mathematics 69, Amer. Math. Soc., 2001, pp. 429–488.
82. R. Shafikov and C. Wolf, *Stable sets, hyperbolicity and dimension*, Discrete Contin. Dyn. Syst. **12** (2005), 403–412.
83. K. Simon, *Hausdorff dimension for noninvertible maps*, Ergodic Theory Dynam. Systems **13** (1993), 199–212.
84. K. Simon, *The Hausdorff dimension of the Smale–Williams solenoid with different contraction coefficients*, Proc. Amer. Math. Soc. **125** (1997), 1221–1228.
85. K. Simon and B. Solomyak, *Hausdorff dimension for horseshoes in $\mathbb{R}^3$*, Ergodic Theory Dynam. Systems **19** (1999), 1343–1363.
86. B. Solomyak, *Measure and dimension for some fractal families*, Math. Proc. Cambridge Philos. Soc. **124** (1998), 531–546.
87. F. Takens, *Limit capacity and Hausdorff dimension of dynamically defined Cantor sets*, in Dynamical Systems (Valparaiso, 1986), edited by R. Bamón, R. Labarca and J. Palis, Lect. Notes in Math. 1331, Springer, 1988, pp. 196–212.
88. R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences 68, Springer, 1988.
89. M. Urbański, *Measures and dimensions in conformal dynamics*, Bull. Amer. Math. Soc. **40** (2003), 281–321.
90. M. Urbański and C. Wolf, *SRB measures for Axiom A endomorphisms*, Math. Res. Letters **11** (2004), 785–797.
91. M. Urbański and C. Wolf, *Ergodic theory of parabolic horseshoes*, Comm. Math. Phys. **281** (2008), 711–751.
92. P. Walters, *A variational principle for the pressure of continuous transformations*, Amer. J. Math. **97** (1976), 937–971.

93. P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics 79, Springer, 1981.
94. C. Wolf, *On measures of maximal and full dimension for polynomial automorphisms of* $\mathbb{C}^2$, Trans. Amer. Math. Soc. **355** (2003), 3227–3239.
95. Y. Yayama, *Dimensions of compact invariant sets of some expanding maps*, Ergodic Theory Dynam. Systems **29** (2009), 281–315.
96. L.-S. Young, *Dimension, entropy and Lyapunov exponents*, Ergodic Theory Dynam. Systems **2** (1982), 109–124.
97. Y. Zhang, *Dynamical upper bounds for Hausdorff dimension of invariant sets*, Ergodic Theory Dyn. Syst. **17** (1997), 739–756.

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal
E-mail address: luis.barreira@math.ist.utl.pt
URL: http://www.math.ist.utl.pt/~barreira/

IMPA, Estrada Dona Castorina 110, Rio de Janeiro / Brasil 22460-320, Brazil
E-mail address: katrin.gelfert@googlemail.com
URL: http://www.pks.mpg.de/~gelfert/