POINCARÉ RECURRENCE: OLD AND NEW

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The classical theorem of Poincaré on recurrence only gives information of *qualitative* nature. On the other hand it is clearly a matter of intrinsic difficulty and not of lack of interest that less is known concerning the *quantitative* behavior of recurrence. Here we discuss recent developments that include the almost everywhere coincidence between the recurrence rate and the pointwise dimension in the case of hyperbolic dynamics. We also discuss the almost product structure of recurrence, which closely imitates the product structure provided by the families of stable and unstable manifolds as well as the almost product structure of hyperbolic measures.

1. Introduction

The notion of nontrivial recurrence goes back to Poincaré in his study of the three-body problem. He proved in his celebrated memoir [11] of 1890 that whenever a dynamical system preserves volume almost all trajectories return arbitrarily close to their initial position and that they do this an infinite number of times. More precisely, Poincaré established the following.

Theorem 1.1. *If a flow preserves volume and has only bounded orbits then for each open set there exist orbits that intersect the set infinitely often.*

This is Poincaré's recurrence theorem (the versions that we encounter today in the literature slightly differ from this original formulation). The memoir is the famous one that in its first version (printed in 1889, even having circulated shortly, and of which some copies still exist today) had the error that can be seen as the main cause for the study of chaotic behavior in the theory of dynamical systems. Incidentally, Poincaré's recurrence theorem was already present in the first printed version of the memoir as then again in [11].

We now present a somewhat more modern formulation of the theorem. We consider a measurable transformation $T\colon X \to X$ preserving a finite measure μ on X (this means that $\mu(T^{-1}A) = \mu(A)$ for every measurable set $A \subset X$). The following is an alternative version of Poincaré's recurrence theorem.

Theorem 1.2. *For each measurable set $A \subset X$ we have*

$$\mu(\{x \in A : T^n x \in A \text{ for infinitely many positive } n\text{'s}\}) = \mu(A).$$

In other words, the existence of a finite invariant measure guarantees that almost every orbit starting in a set A returns infinitely often to this set. When X is a metric space with distance d one can also establish the following version of the recurrence theorem.

Theorem 1.3. *For μ-almost every $x \in X$ we have*

$$\liminf_{n\to\infty} d(T^n x, x) = 0. \tag{1}$$

The identity (1) tells us that the orbit of μ-almost every point returns arbitrarily close to the initial point.

Poincaré's recurrence theorem is a basic but also fundamental result in the theory of dynamical systems. In particular, it tells us that the existence of a *finite* invariant measure causes a *nontrivial recurrence* in each set of positive measure. Unfortunately it only provides information of *qualitative* nature (in any of its alternative versions, such as those in Theorem 1.2 and Theorem 1.3). In particular it tells us nothing about the following two natural problems:

(1) with which frequency the orbit of a point visits a given set;
(2) with which rate the orbit of a point returns to an arbitrarily small neighborhood of the initial point.

Birkhoff's ergodic theorem gives a complete answer to the first problem. The second problem experienced a growing interest during the last decade, also in connection with other fields, including compression algorithms, and the numerical study of dynamical systems. Our main objective is to discuss several recent developments related to this problem, which pertains to the quantitative study of recurrence.

2. Quantitative recurrence

We assume in this section that $T\colon X \to X$ is a measurable transformation on a space X such that $X \subset \mathbb{R}^m$ for some positive integer m. We define the *return time* of a point $x \in X$ to the open ball $B(x,r)$ of radius r centered at x by

$$\tau_r(x) = \inf\{k > 0 : T^k x \in B(x,r)\}.$$

We also define the *lower* and *upper recurrence rates* of x by

$$\underline{R}(x) = \liminf_{r\to 0} \frac{\log \tau_r(x)}{-\log r} \quad \text{and} \quad \overline{R}(x) = \limsup_{r\to 0} \frac{\log \tau_r(x)}{-\log r}.$$

These quantities measure the rate with which the orbit of x returns to an arbitrarily small neighborhood of this point. The following result of Barreira and Saussol in [3] provides upper bounds for the lower and upper recurrence rates in terms of the *lower* and *upper pointwise dimensions* of μ at the point x. These are defined respectively by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Theorem 2.1. *If T preserves a finite measure μ on X, then for μ-almost every $x \in X$,*

$$\underline{R}(x) \le \underline{d}_\mu(x) \quad \text{and} \quad \overline{R}(x) \le \overline{d}_\mu(x). \tag{2}$$

It follows from Whitney's embedding theorem that if X is a subset of a finite-dimensional smooth manifold, then it can be smoothly embedded into $\mathbb{R}^m$ for any sufficiently large $m > 0$, and thus the inequalities in (2) also hold μ-almost everywhere in this situation.

The following example illustrates that without further hypotheses the inequalities in (2) may in general be strict on a set of positive μ-measure. On the other hand, Theorem 3.1 below shows that under certain reasonable additional assumptions the inequalities in (2) become identities on a set of full μ-measure. These assumptions are related to the existence of some hyperbolic behavior for the dynamical system.

Example 2.1. Consider a rotation of the circle by an irrational number ω which is well approximated by rational numbers, in the sense that there exists $\nu > 1$ such that $|\omega - p/q| < 1/q^{\nu+1}$ for infinitely many relatively prime integers p and q, say p_n and q_n for each positive integer n. Since $|q_n\omega - p_n| < 1/{q_n}^{\nu}$ for each n, we have

$$\tau_{1/{q_n}^{\nu}}(x) = \inf\{k > 0 : k\omega(\text{mod}1) < 1/{q_n}^{\nu}\} \leq q_n$$

for every x in the circle, and thus

$$\underline{R}(x) \leq \liminf_{n\to\infty} \frac{\log \tau_{1/{q_n}^{\nu}}(x)}{-\log(1/{q_n}^{\nu})} \leq \frac{1}{\nu} < 1.$$

On the other hand, irrational rotations have as unique invariant measure the Lebesgue measure m, for which $\underline{d}_m(x) = \overline{d}_m(x) = 1$ for every x. In particular $\underline{R}(x) < \underline{d}_m(x)$ for every x in the circle (and thus the first inequality in (2) is strict everywhere).

Boshernitzan proved earlier in [7] that if the α-dimensional Hausdorff measure m_α is σ-finite on X (that is, if X can be written as a countable union of sets X_i for $i = 1, 2, \ldots$ such that $m_\alpha(X_i) < \infty$ for all i), and T preserves a finite measure μ on X, then

$$\liminf_{n\to\infty}[n^{1/\alpha} d(T^n x, x)] < \infty$$

for μ-almost every $x \in X$. He also showed that if, in addition, $m_\alpha(X) = 0$, then

$$\liminf_{n\to\infty}[n^{1/\alpha} d(T^n x, x)] = 0. \tag{3}$$

for μ-almost every $x \in X$. This statement must be compared with (1). While (1) only tells us that some subsequence of the orbit of x converges to x, the identity (3) provides some information about the speed with which that subsequence converges to the point x.

The following statement uses the notion of *Hausdorff dimension* of a measure μ on X. This is defined by

$$\dim_H \mu = \inf\{\dim_H Z : \mu(Z) = \mu(X)\},$$

where $\dim_H Z$ denotes the Hausdorff dimension of the set $Z \subset X$. It can be shown (see for example Proposition 3 in [6]) that

$$\dim_H \mu = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in X\}. \tag{4}$$

Boshernitzan's results in [7] can be reformulated in the following manner (see [3] for details).

Theorem 2.2. *If T preserves a finite measure μ on X, then $\underline{R}(x) \leq \dim_H \mu$ for μ-almost every $x \in X$.*

One can also rephrase the first inequality in (2) in a form similar to (3) (see [3]).

Theorem 2.3. *If T preserves a finite measure μ on X, then (3) holds for μ-almost every $x \in X$ such that $\underline{d}_\mu(x) < \alpha$.*

In view of the identity (4), Theorem 2.3 may in general provide a stronger statement than that in (3), and the first inequality in (2) may be sharper than that in Theorem 2.2. This possibility indeed occurs, as the following example illustrates.

Example 2.2. In [10], Pesin and Weiss presented an example of a Hölder homeomorphism $T\colon X \to X$ on a closed subset X of $[0,1]$, preserving a probability measure μ such that: there exist disjoint sets A_1, $A_2 \subset [0,1]$ with positive μ-measure, and there exist positive constants c_1 and c_2 with $c_1 \neq c_2$ such that $\underline{d}_\mu(x) = \overline{d}_\mu(x) = c_i$ for μ-almost every $x \in A_i$ and $i = 1, 2$. Clearly $\dim_H \mu = \max\{c_1, c_2\}$ and thus $\underline{d}_\mu(x) < \dim_H \mu$ on a set of positive μ-measure (on the set A_i with i such that $c_i = \min\{c_1, c_2\}$).

This example illustrates that in general Theorem 2.3 provides a stronger statement then that in (3). Therefore, one can see the first inequality in Theorem 2.1 as a nontrivial generalization of Boshernitzan's results in [7]. In addition, Theorem 2.1 provides also an upper bound for the upper recurrence rate. Theorem 3.1 below shows that for a certain class of maps and a certain class of measures the inequalities in (2) are in fact identities on a set of full measure.

3. Hyperbolic dynamics

In the case of hyperbolic dynamics, the above results can be considerably strengthen. More precisely, we can obtain identities instead of inequalities in (2).

We first recall the notions of hyperbolic set and of equilibrium measure. Let $T\colon M \to M$ be a C^1 diffeomorphism on a smooth manifold. We say that a compact set $X \subset M$ is a *hyperbolic set* for T if it is T-invariant (i.e., $T^{-1}X = X$) and there exists a continuous splitting of the tangent bundle $T_X M = E^s \oplus E^u$, and constants $c > 0$ and $\lambda \in (0,1)$ such that for each $x \in X$:

(1) $d_x T(E^s(x)) = E^s(Tx)$ and $d_x T(E^u(x)) = E^u(Tx)$;
(2) $\|d_x T^n v\| \leq c\lambda^n \|v\|$ whenever $v \in E^s(x)$ and $n > 0$;
(3) $\|d_x T^{-n} v\| \leq c\lambda^n \|v\|$ whenever $v \in E^u(x)$ and $n > 0$.

We recall that a probability measure μ on X is called an *equilibrium measure* for the continuous function $\varphi\colon X \to \mathbb{R}$ if

$$P(\varphi) = h_\mu(T) + \int_X \varphi \, d\mu,$$

where $P(\varphi)$ is the topological pressure of φ and $h_\mu(T)$ is the Kolmogorov–Sinai entropy of T with respect to μ (see for example [1] for details).

We recall that a measure μ is called *ergodic* if $\mu(A) = 0$ or $\mu(X \setminus A) = 0$ whenever $T^{-1}A = A$. The following statement was established by Barreira and Saussol in [3].

Theorem 3.1. *For a $C^{1+\alpha}$ diffeomorphism with a compact hyperbolic set X, if μ is an ergodic equilibrium measure of a Hölder continuous function, then*

$$\lim_{r\to 0} \frac{\log \tau_r(x)}{-\log r} = \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \tag{5}$$

for μ-almost every point $x \in X$.

The existence for almost every point of the limit in the right-hand side of (5) is due to Barreira, Pesin and Schmeling in [2]. We remark that besides showing that the inequalities in (2) are in fact identities almost everywhere under the hypotheses of Theorem 3.1, the identity (5) also says that the lower and upper recurrence rates are equal almost everywhere. Thinking as if we could erase the limits in (5), we can say that Theorem 3.1 shows that

$$\inf\{k > 0 : f^k x \in B(x,r)\} \text{ is approximately equal to } 1/\mu(B(x,r))$$

when r is sufficiently small, that is, the time that the orbit of x takes to return to the ball $B(x,r)$ is approximately equal to $1/\mu(B(x,r))$. This should be compared to Kac's lemma: since μ is ergodic we have

$$\int_{B(x,r)} \tau_r(y,x)\, d\mu(y) = 1,$$

where

$$\tau_r(y,x) = \inf\{k > 0 : T^k y \in B(x,r)\}.$$

Hence, the average value of the return time to $B(x,r)$ is equal to $1/\mu(B(x,r))$. Therefore, Theorem 3.1 can be though of as a local version of Kac's lemma. It should also be noted that (5) relates two quantities of very different nature. In particular, only the left-hand side depends on the diffeomorphism and only the right-hand side depends on the measure.

The proof of Theorem 3.1 combines new ideas with the study of hyperbolic measures by Barreira, Pesin and Schmeling in [2] and results and ideas of Saussol, Troubetzkoy and Vaienti in [12] and of Schmeling and Troubetzkoy in [14] (see also [13]). In view of work of Barreira and Wolf in [5], in the case of surface diffeomorphisms it always exists an ergodic measure of "maximal recurrence", i.e., a measure at which the supremum $\sup_\mu \dim_H \mu$ (over all finite invariant measures) is attained, and thus (in view of (4)) for which the left-hand side of (5) attains its maximal possible value almost everywhere.

A related result in the case of repellers was obtained by Barreira and Saussol in [4]. Let $T\colon M \to M$ be a differentiable map of a smooth manifold. Recall that a compact T-invariant set $X \subset M$ is called a *repeller* of T if there exist constants $c > 0$ and $\beta > 1$ such that

$$\|d_x T^n v\| \geq c\beta^n \|v\|$$

for each $n > 0$, $x \in X$ and $v \in T_x M$. It was established in [4] that the statement in Theorem 3.1 remains valid when one replaces "diffeomorphism with a hyperbolic set" by "differentiable map with a repeller" (see [4] for details). We briefly present two applications to number theory that can essentially be obtained from a direct application of this result. Let $x = 0.x_1 x_2 \cdots$ be the base-m representation of the point $x \in [0,1]$ (this representation is unique except for a countable set of points in $(0,1)$). By considering $[0,1]$ as a repeller of the transformation $x \mapsto mx \pmod 1$ one can show (see [4]) that

$$\inf\{n > 0\colon |0.x_n x_{n+1} \cdots - 0.x_1 x_2 \cdots| < r\} \sim \frac{1}{r} \text{ when } r \to 0$$

for Lebesgue-almost every $x \in [0,1]$, in the sense that

$$\lim_{r\to 0} \frac{\log \inf\{n > 0\colon |0.x_n x_{n+1} \cdots - 0.x_1 x_2 \cdots| < r\}}{-\log r} = 1$$

for Lebesgue-almost every $x \in [0,1]$. Another example is given by continued fractions. Writing each number $x \in (0,1)$ as a continued fraction

$$x = [m_1, m_2, \ldots] = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cdots}},$$

with $m_i = m_i(x) > 0$ for each i (again this representation is unique except for a countable set of points in $(0,1)$), one can show (see [4]) that for Lebesgue-almost every $x \in (0,1)$,

$$\inf\{n > 0 : |[m_n, m_{n+1}, \ldots] - [m_1, m_2, \ldots]| < r\} \sim \frac{1}{r} \text{ when } r \to 0.$$

4. Product structure and recurrence

We discuss in this section the product structure of recurrence on hyperbolic sets. It turns out that the recurrence also possesses an almost product structure, which closely imitates the product structure provided by the families of stable and unstable manifolds as well as the almost product structure of hyperbolic measures.

Let $T\colon M \to M$ be a C^1 diffeomorphism with a compact hyperbolic set $X \subset M$. We assume in this section that X is locally maximal. This means that there exists an open neighborhood $U \supset X$ such that $X = \bigcap_{n\in\mathbb{Z}} T^n U$. We also consider the *local stable* and *unstable manifolds* of (a sufficiently small) size ε of a point $x \in X$,

$$V^s_\varepsilon(x) = \{y \in M\colon d(T^n x, T^n y) < \varepsilon \text{ for every } n > 0\},$$
$$V^u_\varepsilon(x) = \{y \in M\colon d(T^n x, T^n y) < \varepsilon \text{ for every } n < 0\},$$

and we denote by d_s and d_u the distances induced by the distance d of M respectively on each stable and unstable manifold. We denote by $B^s(x,r) \subset V^s_\varepsilon(x)$ and $B^u(x,r) \subset V^u_\varepsilon(x)$ the corresponding open balls of radius r centered at x.

Under the above assumptions, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any x, $y \in X$ with $d(x,y) \le \delta$ the intersection $V^s_\varepsilon(x) \cap V^u_\varepsilon(y)$ contains exactly one point and we can define the map

$$[\cdot,\cdot]\colon \{(x,y) \in X \times X : d(x,y) \le \delta\} \to M$$

by $[x,y] = V^s_\varepsilon(x) \cap V^u_\varepsilon(y)$.

For each $\rho \le \delta$ we define the *stable* and *unstable return times* of $x \in X$ respectively into the stable and unstable strips of radius r by

$$\tau^s_r(x,\rho) = \inf\{n > 0 : d(T^{-n}x, x) \le \rho \text{ and } d_s([x, T^{-n}x], x) < r\},$$
$$\tau^u_r(x,\rho) = \inf\{n > 0 : d(T^n x, x) \le \rho \text{ and } d_u([T^n x, x], x) < r\}.$$

We note that the functions $\rho \mapsto \tau^s_r(x,\rho)$ and $\rho \mapsto \tau^u_r(x,\rho)$ are nondecreasing. We define the *lower* and *upper stable recurrence rates* of the point $x \in X$ by

$$\underline{R}^s(x) = \lim_{\rho\to 0} \underline{R}^s(x,\rho) \quad \text{and} \quad \overline{R}^s(x) = \lim_{\rho\to 0} \overline{R}^s(x,\rho),$$

and the *lower* and *upper stable recurrence rates* of the point $x \in X$ by

$$\underline{R}^u(x) = \lim_{\rho\to 0} \underline{R}^u(x,\rho) \quad \text{and} \quad \overline{R}^u(x) = \lim_{\rho\to 0} \overline{R}^u(x,\rho),$$

where

$$\underline{R}^s(x,\rho) = \liminf_{r\to 0} \frac{\log \tau_r^s(x,\rho)}{-\log r} \quad \text{and} \quad \overline{R}^s(x,\rho) = \limsup_{r\to 0} \frac{\log \tau_r^s(x,\rho)}{-\log r},$$

$$\underline{R}^u(x,\rho) = \liminf_{r\to 0} \frac{\log \tau_r^u(x,\rho)}{-\log r} \quad \text{and} \quad \overline{R}^u(x,\rho) = \limsup_{r\to 0} \frac{\log \tau_r^u(x,\rho)}{-\log r}.$$

Barreira and Saussol showed in [4] that for a $C^{1+\alpha}$ diffeomorphism that is topologically mixing on a locally maximal compact hyperbolic set X, and an equilibrium measure μ of a Hölder continuous function, we have

$$\underline{R}^s(x) = \overline{R}^s(x) = \lim_{r\to 0} \frac{\log \mu_x^s(B^s(x,r))}{\log r} \quad \text{and} \quad \underline{R}^u(x) = \overline{R}^u(x) = \lim_{r\to 0} \frac{\log \mu_x^s(B^u(x,r))}{\log r} \tag{6}$$

for μ-almost every $x \in X$, where μ_x^s and μ_x^u are the conditional measures induced by the measurable partitions ξ^s and ξ^u defined by the local stable and unstable manifolds. The existence for almost every point of the limits in (6) is due to Ledrappier and Young in [8].

Barreira, Pesin and Schmeling showed in [2] that measures supported on hyperbolic sets possess an almost product structure (the statement is also valid in the much more general case of hyperbolic measures; see [2] for details).

Theorem 4.1. *For a $C^{1+\alpha}$ diffeomorphism with a compact hyperbolic set X, if μ is a finite measure supported on X then given $\delta > 0$ there exists a set $Y \subset X$ with $\mu(Y) > \mu(X) - \delta$ such that for each $x \in Y$ we have*

$$r^\delta \le \frac{\mu(B(x,r))}{\mu_x^s(B^s(x,r))\mu_x^u(B^u(x,r))} \le r^{-\delta}$$

for all sufficiently small $r > 0$.

We now formulate a result of Barreira and Saussol in [4] showing that recurrence also possesses an almost product structure, which imitates the product structure provided by the families of stable and unstable manifolds as well as the almost product structure of measures supported on hyperbolic sets in Theorem 4.1.

Theorem 4.2. *Let X be a locally maximal compact hyperbolic set of a $C^{1+\alpha}$ diffeomorphism that is topologically mixing on X, and μ an equilibrium measure of a Hölder continuous function. Then, for μ-almost every point $x \in X$ there exists $\rho(x) > 0$ such that for each $\rho < \rho(x)$ and $\varepsilon > 0$ there is $r(x,\rho,\varepsilon) > 0$ such that if $r < r(x,\rho,\varepsilon)$ then*

$$r^\varepsilon < \frac{\tau_r^s(x,\rho)\tau_r^u(x,\rho)}{\tau_r(x)} < r^{-\varepsilon}. \tag{7}$$

The identity (7) shows that the return time to a given set is approximately equal to the product of the return times to the stable and unstable directions, as if they were independent.

5. Relation to entropy

Ornstein and Weiss obtained related results in the special case of symbolic dynamics. Namely, they showed in [9] that if $\sigma^+ \colon \Sigma^+ \to \Sigma^+$ is a *one-sided* subshift and μ^+ is an ergodic σ^+-invariant probability measure on Σ^+, then for μ^+-almost every $(i_1 i_2 \cdots) \in \Sigma^+$,

$$\lim_{k\to\infty} \frac{\log \inf\{n > 0 : (i_{n+1} \cdots i_{n+k}) = (i_1 \cdots i_k)\}}{k} = h_{\mu^+}(\sigma) \tag{8}$$

They also showed in [9] that if $\sigma \colon \Sigma \to \Sigma$ is a *two-sided* subshift and μ is an ergodic σ-invariant probability measure on Σ, then for μ-almost every $(\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma$,

$$\lim_{k\to\infty} \frac{\log \inf\{n > 0 : (i_{n-k} \cdots i_{n+k}) = (i_{-k} \cdots i_k)\}}{2k+1} = h_{\mu}(\sigma) \tag{9}$$

Any two-sided shift $\sigma \colon \Sigma \to \Sigma$ has naturally associated two one-sided shifts $\sigma^+ \colon \Sigma^+ \to \Sigma^+$ and $\sigma^- \colon \Sigma^- \to \Sigma^-$ (related with future and past). Furthermore, any σ-invariant measure μ on Σ induces a σ^+-invariant measure μ^+ on Σ^+ and a σ^--invariant measure μ^- on Σ^- with $h_{\mu^+}(\sigma^+) = h_{\mu^-}(\sigma^-) = h_\mu(\sigma)$. For each $\omega = (\cdots i_{-1} i_0 i_1 \cdots) \in \Sigma$ and $k > 0$ we set

$$\begin{aligned}
\tau_k^+(\omega) &= \inf\{n > 0 : (i_{n+1} \cdots i_{n+k}) = (i_1 \cdots i_k)\},\\
\tau_k^-(\omega) &= \inf\{n > 0 : (i_{-n-k} \cdots i_{-n-1}) = (i_{-k} \cdots i_{-1})\},\\
\tau_k(\omega) &= \inf\{n > 0 : (i_{n-k} \cdots i_{n+k}) = (i_{-k} \cdots i_k)\}.
\end{aligned}$$

Let μ be an ergodic σ-invariant measure on Σ. It follows from (8) and (9) that for μ-almost every $\omega \in \Sigma$, given $\varepsilon > 0$, if $k > 0$ is sufficiently large then $e^{-k\varepsilon} \le \tau_k^+(\omega)\tau_k^-(\omega)/\tau_k(\omega) \le e^{k\varepsilon}$. Theorem 4.2 and the identities in (6) are versions of these statements.

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