ALMOST ADDITIVE THERMODYNAMIC FORMALISM: SOME RECENT DEVELOPMENTS

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Abstract. This is a survey on recent developments concerning a thermodynamic formalism for almost additive sequences of functions. While the nonadditive thermodynamic formalism applies to much more general sequences, at the present stage of the theory there are no general results concerning for example a variational principle for the topological pressure or the existence of equilibrium or Gibbs measures, at least without further very restrictive assumptions. On the other hand, in the case of almost additive sequences it is possible to establish a variational principle and to discuss the existence and uniqueness of equilibrium and Gibbs measures, among several other results. After presenting in a self-contained manner the foundations of the theory, the survey includes the description of three applications of the almost additive thermodynamic formalism: a multifractal analysis of Lyapunov exponents for a class of nonconformal repellers; a conditional variational principle for limits of almost additive sequences; and the study of dimension spectra that consider simultaneously limits into the future and into the past.

Contents

1. Introduction

The point of departure for this survey is the nonadditive thermodynamic formalism developed in [1], having in mind certain applications to the dimension theory of dynamical systems, as detailed below. Our main aim is to survey some recent developments in the particular case of almost additive sequences of functions.

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During the last two decades, the dimension theory of dynamical systems progressively developed into an independent field of research, roughly speaking with the objective of measuring the complexity from the dimensional point of view of the objects that remain invariant under the dynamics, such as the invariant sets and measures. The first monograph that clearly took this point of view was Pesin's book [35], which describes the state-of-the-art up to 1997. We refer to our book [4] for a detailed description of many of the more recent results in the area.

The nonadditive thermodynamic formalism is a generalization of the classical thermodynamic formalism, in which the topological pressure $P(\varphi)$ of a continuous function φ (with respect to a given dynamics on a compact metric space), is replaced by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\varphi_n)_n$. The classical pressure $P(\varphi)$ was introduced by Ruelle in [38] for expansive maps (see also his book [39]), and by Walters in [45] in the general case. For arbitrary sets (not necessarily compact), the nonadditive topological pressure also generalizes (and imitates) the notion of topological pressure introduced by Pesin and Pitskel in [36], which is equivalent to the notion introduced earlier by Bowen in [13] (see [36]). The nonadditive thermodynamic formalism contains as a particular case a new formulation of the subadditive thermodynamic formalism earlier introduced by Falconer in [19].

The main motivation behind the nonadditive thermodynamic formalism is to allow certain applications to a more general class of invariant sets in the context of the dimension theory of dynamical systems. We first recall that the unique solution s of the equation

$$P(s\varphi) = 0, \tag{1}$$

where φ is a certain function associated to a given invariant set, is often related to the Hausdorff dimension of the set. Equation (1) was introduced by Bowen in [15] (in his study of quasi-circles) and is usually called Bowen's equation. It is also appropriate to call it Bowen–Ruelle's equation, taking into account the fundamental role of the thermodynamic formalism developed by Ruelle, and of his article [40]. Virtually all known equations used to compute or to estimate the dimension of invariant sets are particular cases of equation (1) or of an appropriate generalization. We recommend [41] for a quite detailed and informative related discussion.

On the other hand, in certain applications of dimension theory (we refer to the examples in [1, 4]) one is naturally led to consider sequences $\Phi = (\varphi_n)_n$ that may satisfy no additivity between the functions φ_n. The nonadditive topological pressure and its associated thermodynamic formalism allow us to consider these generalizations in a unified framework. In particular this allowed to establish in [1] sharp lower and upper dimension estimates for repellers and hyperbolic sets, including for a class of nondifferentiable maps, without further effort. The dimension estimates are obtained as solutions of appropriate generalizations of equation (1) now involving the nonadditive topological pressure.

Given a continuous function $\varphi\colon X \to \mathbb{R}$ in a compact metric space X, the classical topological pressure of φ, with respect to a given continuous map

$f\colon X \to X$, satisfies the variational principle

$$P(\varphi) = \sup_\mu \left(h_\mu(f) + \int_X \varphi \, d\mu \right),$$

where $h_\mu(f)$ is the Kolmogorov–Sinai entropy of f with respect to the measure μ, and where the supremum is taken over all f-invariant probability measures on X. The thermodynamic formalism developed in [1] also includes a variational principle for the topological pressure, although with a quite restrictive assumption on the sequence Φ. Namely, if there exists a continuous function $\varphi\colon X \to \mathbb{R}$ such that

$$\varphi_{n+1} - \varphi_n \circ f \to \varphi \text{ uniformly when } n \to \infty, \tag{2}$$

then

$$P(\Phi) = \sup_\mu \left(h_\mu(f) + \int_X \varphi \, d\mu \right),$$

again with the supremum taken over all f-invariant probability measures on X. The restrictive assumption in (2) caused that until recently there was no available discussion of equilibrium and Gibbs measures, in the general context of the nonadditive thermodynamic formalism. But it is well-known that equilibrium and Gibbs measures play a prominent role in dimension theory and in particular in the multifractal analysis of dynamical systems, in which the spectra are often obtained by providing equilibrium measures with the appropriate local entropy or the appropriate pointwise dimension. Equilibrium and Gibbs measures can also be for example measures of full topological entropy or full Hausdorff dimension. It is sometimes possible to develop the theory without a variational principle for the topological pressure, and thus without these measures, but the corresponding proofs tend to be more technical. Clearly, from the points of view of dimension theory and multifractal analysis, it is desirable to continue using equilibrium and Gibbs measures even when the classical thermodynamical formalism cannot be used.

The discussion above justifies the interest in looking for more general classes of sequences of functions, although perhaps not arbitrary sequences, for which it is still possible to establish a corresponding variational principle for the topological pressure, and to study the associated equilibrium and Gibbs measures, among several other results. This is precisely what happens with the so-called almost additive sequences, for which it is possible not only to establish a variational principle, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures. We recall that a sequence $\Phi = (\varphi_n)_n$ is said to be *almost additive* if there is a constant $C > 0$ such that

$$-C + \varphi_n + \varphi_m \circ f^n \le \varphi_{n+m} \le C + \varphi_n + \varphi_m \circ f^n \tag{3}$$

for every $n, m \in \mathbb{N}$. Clearly, for any function φ the sequence

$$\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$$

is almost additive, since in this case

$$\varphi_{n+m} = \varphi_n + \varphi_m \circ f^n$$

for every $n, m \in \mathbb{N}$. Nontrivial examples of almost additive sequences occur for example in the study of Lyapunov exponents for nonconformal maps by Barreira and Gelfert in [7] (see Section 7). Following [3], we consider in particular repellers and hyperbolic sets of C^1 transformations, and for an almost additive sequence Φ of continuous functions we describe several results towards the foundations of an almost additive thermodynamic formalism. This includes the formula

$$P(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x : f^n(x) = x} \exp \varphi_n(x)$$

for the topological pressure, for the class of almost additive sequences Φ with tempered variation. We also describe a variational principle for the topological pressure of an almost additive sequence, namely

$$P(\Phi) = \sup_{\mu} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu \right), \tag{4}$$

and we discuss the existence and uniqueness of equilibrium and invariant Gibbs measures, among several other results, for example concerning characterizations of unique equilibrium measures. Mummert [33] established independently identity (4), although under an additional assumption on the sequence Φ that can be removed by repeating verbatim arguments in [3]. Cao, Feng and Huang considered more recently in [16] the general class of subadditive sequences, and they also obtained the variational principle in (4), but they do not discuss the existence of equilibrium or Gibbs measures. Earlier results in this direction were obtained by Käenmäki in [29] for a particular class of subadditive sequences, while also discussing the existence of an equilibrium measure.

After presenting the foundations of the almost additive thermodynamic formalism, we describe three applications of the formalism.

The first application, following Barreira and Gelfert in [7], considers nonconformal repellers in $\mathbb{R}^2$ satisfying a cone condition. The main objective is to obtain a multifractal analysis for the level sets of the Lyapunov exponents. In particular we consider certain almost additive sequences related to the Lyapunov exponents to which one can apply the almost additive thermodynamic formalism. However, we emphasize that the results in [7] were obtained independently of the theory described in the survey. We also point out that the proofs of some results in Sections 4, 5, and 6 can be considered a distillation of arguments in that paper. We recall that a differentiable map f is said to be conformal on a given set provided that the differential $d_x f$ is a multiple of an isometry at every point x of the set. We emphasize that the dimension theory and the multifractal analysis of dynamical systems are only completely understood in the case of conformal uniformly hyperbolic dynamics, either invertible or noninvertible. This includes saddle-type hyperbolic diffeomorphisms on surfaces, and holomorphic maps in the complex plane with a hyperbolic Julia set. The study of the dimension of invariant sets of nonconformal transformations has proven to be much more delicate. The main difficulty is related with the possibility of existence of distinct Lyapunov exponents in different directions, which may change from point to point. Another difficulty is that certain number-theoretical properties may play an important role. Nevertheless, there exist several noteworthy results concerning the dimension theory of certain classes of invariant sets of nonconformal transformations, namely due to Falconer [18, 20], Bothe [12], Simon [43], and Simon and Solomyak [44]. We refer to [4] for a related discussion.

The second application, following Barreira and Doutor in [5], has the objective of establishing a conditional variational principle for the multifractal spectra obtained from limits of almost additive sequences. This means that we consider the level sets

$$K_\alpha = \left\{ x \in X : \lim_{n\to\infty} \frac{\varphi_n(x)}{\psi_n(x)} = \alpha \right\},$$

where $(\varphi_n)_n$ and $(\psi_n)_n$ are almost additive sequences, and we give a description of their topological entropy or Hausdorff dimension in terms of a conditional variational principle. For example, in the case of the topological entropy the conditional variational principle takes the form

$$h(f|K_\alpha) = \max\left\{ h_\mu(f) : \lim_{n\to\infty} \frac{\int_X \varphi_n \, d\mu}{\int_X \psi_n \, d\mu} = \alpha \right\},$$

where $h(f|K_\alpha)$ denotes the topological entropy on K_α. It is also shown that the spectra, such as $\alpha \mapsto h(f|K_\alpha)$, are continuous, and that the associated irregular sets have full dimension. The approach in [5] builds on related arguments in former work of Barreira, Saussol and Schmeling in [9], although now for almost additive sequences. The multifractal analysis of dynamical systems can be considered a subfield of the dimension theory of dynamical systems, and it studies the complexity of the level sets of invariant local quantities obtained from a dynamical system. The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia and Shraiman in [26]. The first rigorous approach is due to Collet, Lebowitz and Porzio in [17] for a class of measures invariant under 1-dimensional Markov maps. In [31], Lopes considered the measure of maximal entropy for hyperbolic Julia sets, and in [37], Rand studied Gibbs measures for a class of repellers. We refer the reader to the books [4, 35] for details and further references.

The third application, following Barreira and Doutor in [6], is a complete description of the dimension spectra of limits of almost additive sequences on a hyperbolic set of a surface diffeomorphism. The main novelty is that we consider simultaneously limits into the future and into the past. More precisely, the spectra are obtained by computing the Hausdorff dimension of the level sets of limits of almost additive sequences both for positive and negative time. We emphasize that the description of the spectra is not a consequence of the results considering simply limits into the future (or into the past). The main difficulty is that although the local product structure provided by the intersection of stable and unstable manifolds is bi-Lipschitz equivalent to a product, the level sets are never compact (this causes that their box dimension is strictly larger than their Hausdorff dimension), and thus the product of level sets may have a dimension that need not be the sum of the dimensions of the sets. Instead we construct explicitly noninvariant measures concentrated on each product of level sets having the appropriate pointwise dimension. This approach builds on former work of Barreira and Valls in [11] in the additive case.

2. Nonadditive topological pressure

2.1. **General theory.** We recall in this section the notion of nonadditive topological pressure introduced by Barreira in [1]. The main idea is to replace each sequence of functions $\sum_{k=0}^{n-1}\varphi\circ f^k$ in the definition of topological pressure by an arbitrary sequence φ_n.

Let $f\colon X\to X$ be a continuous transformation of a compact metric space X. Given a finite open cover $\mathcal{U}$ of X, we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of vectors $U=(U_0,\dots,U_n)$ with $U_0,\dots,U_n\in\mathcal{U}$. For each $U\in\mathcal{W}_n(\mathcal{U})$, we write $m(U)=n$, and we consider the open set

$$X(U)=\bigcap_{k=0}^{n} f^{-k}U_k.$$

These sets can be thought of as cylinder sets. Now let Φ be a sequence of continuous functions $\varphi_n\colon X\to\mathbb{R}$ for each $n\in\mathbb{N}$. We define

$$\gamma_n(\Phi,\mathcal{U})=\sup\bigl\{|\varphi_n(x)-\varphi_n(y)| : x,y\in X(U) \text{ for some } U\in\mathcal{W}_n(\mathcal{U})\bigr\} \quad (5)$$

for each $n\in\mathbb{N}$, and we always assume that

$$\limsup_{\operatorname{diam}\mathcal{U}\to 0}\limsup_{n\to\infty}\frac{\gamma_n(\Phi,\mathcal{U})}{n}=0. \quad (6)$$

We observe that condition (6) holds automatically when Φ is an additive sequence, that is, when

$$\varphi_n=\sum_{k=0}^{n-1}\varphi\circ f^k \quad (7)$$

for a given continuous function $\varphi\colon X\to\mathbb{R}$ and each $n\in\mathbb{N}$ (this is an immediate consequence of the uniform continuity of any continuous function in the compact metric space X). Now we proceed with the construction of the nonadditive topological pressure. For each $U\in\mathcal{W}_n(\mathcal{U})$ we write

$$\varphi(U)=\begin{cases}\sup_{X(U)}\varphi_n & \text{if } X(U)\ne\varnothing,\\ -\infty & \text{if } X(U)=\varnothing.\end{cases} \quad (8)$$

Given a set $Z\subset X$ and a number $\alpha\in\mathbb{R}$, we define the function

$$M(Z,\alpha,\Phi,\mathcal{U})=\lim_{n\to\infty}\inf_{\Gamma}\sum_{U\in\Gamma}\exp(-\alpha m(U)+\varphi(U)),$$

where the infimum is taken over all finite or countable collections $\Gamma\subset\bigcup_{k\ge n}\mathcal{W}_k(\mathcal{U})$ such that $\bigcup_{U\in\Gamma}X(U)\supset Z$ (in other words, such that the cylinder sets $X(U)$ cover the set Z). One can show that the function $\alpha\mapsto M(Z,\alpha,\Phi,\mathcal{U})$ jumps from $+\infty$ to 0 at a unique value of α, and thus we can define

$$P_Z(\Phi,\mathcal{U})=\inf\bigl\{\alpha\in\mathbb{R} : M(Z,\alpha,\Phi,\mathcal{U})=0\bigr\}.$$

Theorem 2.1 ([1]). *The following properties hold:*

1. *the limit*
$$P_Z(\Phi) := \lim_{\operatorname{diam} \mathfrak{U} \to 0} P_Z(\Phi, \mathfrak{U})$$
exists;
2. *if there exist constants* c_1, $c_2 < 0$ *such that* $c_1 n \le \varphi_n \le c_2 n$ *for every* $n \in \mathbb{N}$, *and the topological entropy* $h(f|X)$ *is finite, then there exists a unique number* $s \in \mathbb{R}$ *such that*
$$P_Z(s\Phi) = 0.$$

The number $P_Z(\Phi)$ is called the *nonadditive topological pressure* of the sequence of functions Φ (with respect to f on Z). We note that the set Z need not be compact nor f-invariant. For simplicity, when there is no danger of confusion, we simply refer to $P_Z(\Phi)$ as the *topological pressure* of Φ (with respect to f on Z). We also write $P(\Phi) = P_X(\Phi)$. One can easily verify that if Φ is the (additive) sequence of functions in (7), then $P(\Phi)$ coincides with the classical topological pressure of the function φ.

The number $h(f|Z) = P_Z(0)$ is called the *topological entropy* of f on Z. It coincides with the notion of topological entropy for noncompact sets introduced in [36], and is equivalent to the notion of topological entropy introduced earlier by Bowen in [13]. It can be described as follows. Given a set $Z \subset X$ and a number $\alpha \in \mathbb{R}$, we define the function
$$N(Z, \alpha, \mathfrak{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U \in \Gamma} \exp(-\alpha m(U)),$$
where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k \ge n} \mathcal{W}_k(\mathfrak{U})$ such that $\bigcup_{U \in \Gamma} X(U) \supset Z$. Then
$$h(f|Z) = \lim_{\operatorname{diam} \mathfrak{U} \to 0} \inf \big\{\alpha \in \mathbb{R} : N(Z, \alpha, \mathfrak{U}) = 0\big\}.$$

2.2. **Equilibrium measures for subadditive sequences.** As described in the introduction, the nonadditive thermodynamic formalism developed in [1] also includes a variational principle for the topological pressure, although with a quite restrictive assumption on the sequence Φ (see (2)). Nevertheless, it is still meaningful to consider some particular classes of dynamics and potentials, and to look for equilibrium and Gibbs measures.

With this in mind we describe in this section results by Käenmäki [29] and by Feng and Käenmäki [24] concerning the construction of equilibrium measures for a class of subadditive sequences in the particular case of symbolic dynamics. These sequences are well adapted to the study of the dimension of a class of limit sets of iterated function systems (see [29]) and of the multifractal analysis of the top Lyapunov exponent of products of matrices (see [21, 23, 25]). We refer to the following sections for related results concerning the existence of equilibrium and Gibbs measures for other classes of dynamics and potentials.

We first introduce some notation to consider the particular case of symbolic dynamics. Given $p \in \mathbb{N}$, we write $\Sigma^n = \{1, \dots, p\}^n$ for each $n \in \mathbb{N}$ and $|\omega| = n$ for each $\omega \in \Sigma^n$. We also write
$$\Sigma = \{1, \dots, p\}^{\mathbb{N}} \quad \text{and} \quad \Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n,$$

and we consider the shift map $\sigma\colon \Sigma \to \Sigma$ by $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$. Given $t \geq 0$ and $\omega \in \Sigma^*$, let $\mathcal{C}$ be the class of all (parameterized) functions $\psi^t_\omega\colon \Sigma \to \mathbb{R}^+$ with $\psi^0_\omega = 1$ satisfying the following properties:

1. there exists $K_t > 0$ such that $\psi^t_\omega(\omega_1) \leq K\psi^t_\omega(\omega_2)$ for any $\omega_1, \omega_2 \in \Sigma$;
2. for every $\omega' \in \Sigma$ and $j \in [1, |\omega|] \cap \mathbb{N}$ we have
$$\psi^t_\omega(\omega') \leq \psi^t_{\omega|j}(\sigma^j(\omega)\omega')\psi^t_{\sigma^j(\omega)}(\omega'),$$
where $\omega|j$ are the first j elements ω, and where $\sigma^j(\omega)\omega'$ denotes the juxtaposition of the two sequences;
3. for each $\delta > 0$ there exist $a = a(\delta), b = b(\delta) \in (0, 1)$ depending only on δ, with $a(\delta) \nearrow 1$ and $b(\delta) \nearrow 1$ when $\delta \to 0$, such that
$$\psi^t_\omega(\omega')a^{|\omega|} \leq \psi^{t+\delta}_\omega(\omega') \leq \psi^t_\omega(\omega')b^{|\omega|}$$
for every $\omega' \in \Sigma$.

We note that this class of functions contains as particular examples several classes earlier considered by Falconer [18, 20] and by Barreira [2], in connection with the study of the dimension of repellers of nonconformal transformations.

For any function in the class $\mathcal{C}$, using the subadditivity it is shown in [29] that given $\omega' \in \Sigma$ and a σ-invariant probability measure μ in Σ, the limits
$$p(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\omega\in\Sigma^n} \psi^t_\omega(\omega') \tag{9}$$
and
$$s_\mu(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{\omega\in\Sigma^n} \mu(C_\omega) \log \psi^t_\omega(\omega')$$
exist, where $C_\omega \subset \Sigma$ is the set of sequences whose first n elements are equal to those of ω). Moreover, they are independent of ω'.

To verify that $p(t)$ is indeed a particular case of the nonadditive topological pressure, given $\omega' \in \Sigma$ and $n \in \mathbb{N}$ we define a sequence $\varphi_n\colon \Sigma \to \mathbb{R}$ by
$$\varphi^t_n(\omega) = \sup_{\omega''\in C_\omega} \log \psi^t_{\omega''}(\omega'). \tag{10}$$
Then the first condition on the class $\mathcal{C}$ ensures that (6) holds, and we can show that $p(t)$ coincides with the nonadditive topological pressure of the sequence $\Phi^t = (\varphi_n)_n$ for any ω'. This follows readily from results in [1] using the second condition on $\mathcal{C}$. Moreover, by the third condition we can readily apply Theorem 2.1 to conclude that there exists a unique $t \geq 0$ such that $p(t) = 0$ (the proof of this statement in [29] follows the same argument). This zero is often related to the dimension of certain classes of limit sets of iterated function systems and repellers (see for example [1, 2, 4, 18, 20]).

In addition, the following property holds.

Theorem 2.2 ([29]). *We have*
$$p(t) \geq h_\mu(\sigma) + s_\mu(t). \tag{11}$$

By Kingman's subadditive ergodic theorem, we have
$$s_\mu(t) = \lim_{n\to\infty} \frac{1}{n} \int_\Sigma \varphi^t_n \, d\mu,$$

and thus, inequality (11) can be written in the form

$$P(\Phi^t) \geq h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int_\Sigma \varphi_n^t \, d\mu.$$

This inequality is due to Falconer [19] in the general case of arbitrary subadditive sequences (and not only for the sequences Φ^t) with a bounded distortion condition (which in the present context is given by the first condition on $\mathcal{C}$). Assuming a certain Lipschitz property for the elements of the sequence (more generally for topological Markov chains), he also obtained the variational principle

$$P(\Phi^t) = \sup_\mu \left(h_\mu(\sigma) + \lim_{n\to\infty} \frac{1}{n} \int_\Sigma \varphi_n^t \, d\mu \right). \tag{12}$$

In an analogous manner to that in the classical additive theory, we say that a σ-invariant probability measure μ in Σ is an *equilibrium measure* for the sequence Φ^t if it attains the supremum in (12). In the present context the existence of equilibrium measures was establish by Käenmäki.

Theorem 2.3 ([29])**.** *For each $t \geq 0$ there exists an equilibrium measure for the sequence Φ^t.*

The existence of these equilibrium measures is used in [29] to study the dimension of a class of limit sets of iterated function systems.

Now we consider a particular class of functions in $\mathcal{C}$ that are obtained from products of matrices. Given $p, m \in \mathbb{N}$, let $M_1, \ldots, M_p$ be $m \times m$ matrices. For each $t > 0$, $n \in \mathbb{N}$ and $\omega \in \Sigma^n$, we consider the constant function

$$\bar{\psi}_\omega^t = \|M_{i_1} \cdots M_{i_n}\|^t,$$

where $\omega = (i_1 \cdots i_n)$, and again we define a sequence $\bar{\Phi}^t$ as in (10), that is,

$$\bar{\varphi}_n^t(\omega) = \sup_{\omega'' \in C_\omega} \log \bar{\psi}_{\omega''}^t(\omega') = \sup_{\omega'' \in C_\omega} \log \|M_{i_1} \cdots M_{i_n}\|^t,$$

where $\omega'' = (i_1 \cdots i_n)$. One can easily verify that the functions $\bar{\psi}_\omega^t$ belong to the class $\mathcal{C}$, and that $p(t)$ in (9) is given by

$$p(t) = \lim_{n\to\infty} \frac{1}{n} \sum_{\omega \in \Sigma^n} \|M_{i_1} \cdots M_{i_n}\|^t.$$

Moreover, given a σ-invariant probability measure μ in Σ, we have

$$s_\mu(t) = t \lim_{n\to\infty} \frac{1}{n} \sum_{\omega \in \Sigma^n} \mu(C_\omega) \log \|M_{i_1} \cdots M_{i_n}\|,$$

and it follows from (12) (see also [16]) that

$$p(t) = \sup_\mu \left(h_\mu(\sigma) + s_\mu(t) \right).$$

The following result is due to Feng and Käenmäki.

Theorem 2.4 ([24])**.** *If for each $n \in \mathbb{N}$ there exist $i_1, \ldots, i_n \in \{1, \ldots, m\}$ such that $M_{i_1} \cdots M_{i_n} \neq 0$, then for each $t \geq 0$ there exist at most m ergodic equilibrium measures for the sequence $\bar{\Phi}^t$. If in addition the only proper vector space V such that $M_i V \subset V$ for $i = 1, \ldots, m$ is the origin, then for each $t \geq 0$ there exists a unique equilibrium measure for the sequence $\bar{\Phi}^t$.*

The irreducibility condition in Theorem 2.4 concerning the subspaces V is used in [23] to show that there exist $c > 0$ and $k \in \mathbb{N}$ such that for each $\omega, \omega' \in \Sigma^*$ there exists $\bar{\omega} \in \bigcup_{j=1}^k \Sigma^j$ for which

$$\|M_{\omega\bar{\omega}\omega'}\| \geq c\|M_\omega\| \cdot \|M_{\omega'}\|. \tag{13}$$

It is essentially this property that allows to establish the existence of a unique equilibrium measure in [24]. We note that property (13) ensures that the sequence $\bar{\Phi}^t$ is almost additive (see (3)), and thus the existence of a unique ergodic measure in Theorem 2.4 as well as its Gibbs property (also obtained in [24]) follow from general results in [3] for the class of almost additive sequences (compare with the results in Sections 4–5).

3. Topological pressure for almost additive sequences

We introduce in this section the class of almost additive sequences, and we present formulas for the nonadditive topological pressure. For definiteness we consider only the case of functions defined on a repeller. We refer to the remaining sections for further developments.

3.1. Repellers and Markov partitions.

We recall in this section the notion of repeller and of Markov partition. Let $f\colon M \to M$ be a C^1 map, and let $\Lambda \subset M$ be a compact f-invariant set (this means that $f^{-1}\Lambda = \Lambda$). We say that f is *expanding* on Λ, and that Λ is a *repeller* of f if there exist constants $c > 0$ and $\beta > 1$ such that

$$\|d_x f^n v\| \geq c\beta^n \|v\|$$

for every $x \in \Lambda$, $n \in \mathbb{N}$, and $v \in T_x M$. In addition, we always assume in this presentation that there is an open set $U \supset \Lambda$ such that $\Lambda = \bigcap_{n\in\mathbb{N}} f^n U$, and that f is topologically mixing on Λ.

We recall that a collection of closed sets $R_1, \ldots, R_p \subset \Lambda$ is said to be a *Markov partition* of the repeller Λ if:

1. $\Lambda = \bigcup_{i=1}^p R_i$, and $\overline{\operatorname{int} R_i} = R_i$ for $i = 1, \ldots, p$;
2. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ whenever $i \neq j$;
3. $f(R_i) \supset R_j$ whenever $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing$.

We note that here the interior of each set R_i is computed with respect to the induced topology on Λ. Any repeller has Markov partitions with arbitrarily small diameter

$$\max\left\{ \operatorname{diam} R_i : i = 1, \ldots, p\right\} \tag{14}$$

(see [40]). Given a Markov partition $R_1, \ldots, R_p$ of Λ, we define a $p \times p$ matrix $A = (a_{ij})$ with entries

$$a_{ij} = \begin{cases} 1 & \text{if } f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \varnothing, \\ 0 & \text{if } f(\operatorname{int} R_i) \cap \operatorname{int} R_j = \varnothing, \end{cases} \tag{15}$$

and we consider the corresponding topological Markov chain $\sigma\colon \Sigma_A \to \Sigma_A$ defined by the shift map $\sigma(i_1 i_2 \cdots) = (i_2 i_3 \cdots)$ in the set

$$\Sigma_A = \left\{ (i_1 i_2 \cdots) \in \{1, \ldots, p\}^{\mathbb{N}} : a_{i_k i_{k+1}} = 1 \text{ for every } k \in \mathbb{N} \right\}. \tag{16}$$

We denote by $\Sigma_{A,n}$ the set of n-tuples $(i_1 \cdots i_n)$ for which there is a sequence $(j_1 j_2 \cdots) \in \Sigma_A$ such that $i_\ell = j_\ell$ for $\ell = 1, \ldots, n$. For each $(i_1 \cdots i_n) \in \Sigma_{A,n}$ we define

$$\Delta_{i_1 \cdots i_n} = \bigcap_{\ell=0}^{n-1} f^{-\ell} R_{i_{\ell+1}}, \tag{17}$$

and setting

$$\chi(i_1 i_2 \cdots) = \bigcap_{\ell=0}^{\infty} f^{-\ell} R_{i_{\ell+1}} = \bigcap_{n=1}^{\infty} \Delta_{i_1 \cdots i_n},$$

we obtain a coding map $\chi \colon \Sigma_A \to \Lambda$ for the repeller.

3.2. **Formulas for the topological pressure.** Now we introduce the class of almost additive sequences, and we describe corresponding formulas for the nonadditive topological pressure both using and avoiding Markov partitions.

We say that the sequence of functions $\Phi = (\varphi_n)_n$ with $\varphi_n \colon \Lambda \to \mathbb{R}$ for each $n \in \mathbb{N}$ is *almost additive* (with respect to f on Λ) if there exists a constant $C > 0$ such that for every $n, m \in \mathbb{N}$ and $x \in \Lambda$ we have

$$-C + \varphi_n(x) + \varphi_m(f^n(x)) \le \varphi_{n+m}(x) \le C + \varphi_n(x) + \varphi_m(f^n(x)). \tag{18}$$

Clearly, any additive sequence of functions $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$ is almost additive. Nontrivial examples of almost additive sequences occur naturally for example in the study of nonconformal repellers (see Section 7 for a detailed description).

Now let Λ be a repeller of f, and let $\Delta_{i_1 \cdots i_n}$ be the sets in (17) obtained from a given Markov partition. We write

$$\gamma_n(\Phi) = \sup \left\{ |\varphi_n(x) - \varphi_n(y)| : x, y \in \Delta_{i_1 \cdots i_n} \text{ and } (i_1 \cdots i_n) \in \Sigma_{A,n} \right\}. \tag{19}$$

One can easily verify that $\gamma_n(\Phi)$ coincides with $\gamma_n(\Phi, \mathcal{U})$ in (5) for the open cover $\mathcal{U}$ of Λ formed by the elements $R_1, \ldots, R_p$ of the Markov partition (with respect to the induced topology on Λ). We say that Φ has *tempered variation* if $\gamma_n(\Phi)/n \to 0$ as $n \to \infty$. Clearly, any sequence with tempered variation satisfies condition (6).

The following result provides a formula for the topological pressure of an almost additive sequence with tempered variation.

Theorem 3.1 ([7, Proposition 3]). *Let Λ be a repeller of a C^1 map, and let $\Phi = (\varphi_n)_n$ be an almost additive sequence of continuous functions on Λ with tempered variation. Then*

$$P(\Phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i_1 \cdots i_n} \exp \varphi_n(x_{i_1 \cdots i_n}) \tag{20}$$

for any points $x_{i_1 \cdots i_n} \in \Delta_{i_1 \cdots i_n}$, for each $(i_1 \cdots i_n) \in \Sigma_{A,n}$ and $n \in \mathbb{N}$.

The statement in Theorem 3.1 was first established by Barreira and Gelfert in [7], and was then extended by Barreira in [3] to other classes of transformations (see Sections 5 and 6). We emphasize that identity (20) ensures not only that the nonadditive topological pressure of an almost additive sequence is a limit, but also that the limit is independent of the particular Markov partition used to define it.

For a continuous function $\varphi\colon \Lambda \to \mathbb{R}$, we recall that the (classical) *topological pressure* of φ (with respect to f on Λ) is given by

$$P(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp \max_{x\in\Delta_{i_1\cdots i_n}} \sum_{k=0}^{n-1} \varphi(f^k(x)), \tag{21}$$

where $\Delta_{i_1\cdots i_n}$ are the sets in (17) obtained from any given Markov partition. One can easily verify that the limit in (20) exists (by showing that the first sum defines a submultiplicative sequence). Furthermore, the limit is independent of the particular Markov partition used to define it (see [35, 46] for details). We note that identity (20) includes identity (21) (which is often taken as the definition of topological pressure) as a particular case.

We have also the following alternative characterization of the topological pressure. It has the advantage of avoiding Markov partitions and the associated symbolic dynamics. Let

$$\mathrm{Fix}(f) = \big\{x \in \Lambda : f(x) = x\big\}$$

be the set of fixed points of f in Λ.

Theorem 3.2 ([3])**.** *Let Λ be a repeller of a C^1 map, and let $\Phi = (\varphi_n)_n$ be an almost additive sequence of continuous functions on Λ with tempered variation. Then*

$$P(\Phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x\in\mathrm{Fix}(f^n)} \exp \varphi_n(x). \tag{22}$$

4. Results for repellers

We describe in this section several results of the almost additive thermodynamic formalism, again for definiteness in the particular case of functions defined in a repeller. In particular, we describe a variational principle for the topological pressure. We also introduce, for almost additive sequences, the notions of equilibrium measure and of Gibbs measure, and we consider the problem of existence and uniqueness of these measures.

4.1. Variational principle for the topological pressure.

To formulate the variational principle for the topological pressure, we first recall the notion of Kolmogorov–Sinai entropy. Given a measurable transformation $f\colon \Lambda \to \Lambda$, we denote by $\mathcal{M}$ the family of f-invariant probability measures in Λ. We recall that a measure μ in Λ is said to be *f-invariant* if $\mu(f^{-1}A) = \mu(A)$ for every measurable set $A \subset \Lambda$. Given a measure $\mu \in \mathcal{M}$ and a partition ξ of Λ into measurable subsets, we define

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. The *Kolmogorov–Sinai entropy* of f with respect to μ is given by

$$h_\mu(f) = \sup\big\{h_\mu(f,\xi) : H_\mu(\xi) < \infty\big\},$$

where

$$h_\mu(f,\xi) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n),$$

for the partition ξ_n of Λ into the sets $\bigcap_{k=0}^{n-1} f^{-k}C_{k+1}$ with $C_1, \ldots, C_n \in \xi$. In the case of invariant measures in repellers, the entropy can be obtained as follows. Given a Markov partition of the repeller Λ, we consider the partition

$$\xi_n = \left\{\Delta_{i_1\cdots i_n} : (i_1 \cdots i_n) \in \Sigma_{A,n}\right\}$$

of Λ. Its entropy is given by

$$H_\mu(\xi_n) = -\sum_{i_1\cdots i_n} \mu(\Delta_{i_1\cdots i_n}) \log \mu(\Delta_{i_1\cdots i_n}),$$

and

$$h_\mu(f) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\xi_n) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n).$$

The following is a variational principle for the topological pressure.

Theorem 4.1 ([3]). *Let Λ be a repeller of a C^1 map f, and let $\Phi = (\varphi_n)_n$ be an almost additive sequence of continuous functions on Λ with tempered variation. Then*

$$\begin{aligned} P(\Phi) &= \max_{\mu\in\mathcal{M}} \left(h_\mu(f) + \int_\Lambda \lim_{n\to\infty} \frac{\varphi_n(x)}{n}\, d\mu(x) \right) \\ &= \max_{\mu\in\mathcal{M}} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \varphi_n \, d\mu \right), \end{aligned} \tag{23}$$

including the existence in $L^1(\Lambda, \mu)$ of the first limit, and the existence of the second limit.

In a similar manner to that in the classical theory, it is easier to show that

$$P(\Phi) \ge \max_{\mu\in\mathcal{M}} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \varphi_n \, d\mu \right)$$

when compared to the reverse inequality. The argument uses the subadditivity of the sequence $\psi_n = \varphi_n + C$ (see (18)), that is, the property

$$\psi_{n+m} \le \psi_n + \psi_m \circ f^n, \quad n, m \in \mathbb{N},$$

together with Kingman's subadditive ergodic theorem. The proof of the reverse inequality uses analogous arguments to those in the proof of Theorem 1.7 in [1], which in their turn were inspired in arguments of Bowen in [14]. The fact that the supremum can be replaced by a maximum in (23) follows from the upper semi-continuity of the map

$$\mathcal{M} \ni \mu \mapsto h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \varphi_n \, d\mu, \tag{24}$$

since $\mu \mapsto h_\mu(f)$ is upper semi-continuous in this setting, and since the limit in (24) is continuous in μ.

4.2. **Equilibrium and Gibbs measures.** We continue to consider a repeller Λ of a C^1 map f. In an analogous manner to that in the classical additive theory, we say that a measure $\mu \in \mathcal{M}$ is an *equilibrium measure* for the almost additive sequence Φ (with respect to f on Λ) if it attains any of the maxima in (23) (and thus both maxima), that is, if

$$P(\Phi) = h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \varphi_n \, d\mu.$$

The existence of equilibrium measures is thus an immediate consequence of Theorem 4.1.

Theorem 4.2 ([3])**.** *Let Λ be a repeller of a C^1 map. Then any almost additive sequence of continuous functions on Λ with tempered variation has at least one equilibrium measure.*

We also say that a probability measure μ in Λ (which need not be f-invariant) is a *Gibbs measure* for the sequence Φ (with respect to f on Λ, and to a given Markov partition of Λ) if there exists a constant $K > 0$ such that

$$K^{-1} \le \frac{\mu(\Delta_{i_1\cdots i_n})}{\exp[-nP(\Phi) + \varphi_n(x)]} \le K$$

for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in \Sigma_{A,n}$, and $x \in \Delta_{i_1\cdots i_n}$. It turns out, as in the classical additive theory, that invariant Gibbs measures are always equilibrium measures. The argument is simple. We first note that if μ is an f-invariant Gibbs measure, then the limit

$$h_\mu(x) := \lim_{n\to\infty} -\frac{1}{n}\log\mu(\Delta_{i_1\cdots i_n}) = P(\Phi) - \lim_{n\to\infty}\frac{\varphi_n(x)}{n} \tag{25}$$

exists for μ-almost every $x \in \Lambda$ (by Theorem 4.1 the second limit in (25) exists in $L^1(\Lambda,\mu)$, and thus it also exists for μ-almost every $x \in \Lambda$). By Shannon–McMillan–Breiman's theorem we obtain

$$h_\mu(f) = \int_\Lambda h_\mu(x)\,d\mu(x) = P(\Phi) - \int_\Lambda \lim_{n\to\infty}\frac{\varphi_n(x)}{n}\,d\mu(x),$$

and hence μ is an equilibrium measure.

To formulate the following result we need to consider the stronger notion of bounded variation. We say that the sequence of functions $\Phi = (\varphi_n)_n$ has *bounded variation* if $\sup_{n\in\mathbb{N}}\gamma_n(\Phi) < \infty$ (see (19) for the definition of $\gamma_n(\Phi)$). For example, one can easily verify that if Φ is the additive sequence $\varphi_n = \sum_{k=0}^{n-1}\varphi\circ f^k$ for some Hölder continuous φ in a repeller, then Φ has bounded variation. Clearly, if Φ has bounded variation, then it has tempered variation.

The following statement says in particular that for each almost additive sequence with bounded variation there exists a unique equilibrium measure.

Theorem 4.3 ([3])**.** *Let Λ be a repeller of a C^1 map, and let Φ be an almost additive sequence of continuous functions on Λ with bounded variation. Then:*

1. *there is a unique equilibrium measure for Φ;*
2. *there is a unique invariant Gibbs measure for Φ;*
3. *the two measures coincide and are mixing.*

In particular, the unique equilibrium measure for an almost additive sequence with bounded variation is an invariant Gibbs measure.

We refer to [33] for some results related to those in this section, although using a different notion of equilibrium measure.

4.3. **Characterizations of unique equilibrium measures.** The unique equilibrium measure in Theorem 4.3 can be characterized as follows. We denote by δ_x the probability measure with $\delta_x(\{x\}) = 1$.

Theorem 4.4 ([3])**.** *Let Λ be a repeller of a C^1 map, and let $\Phi = (\varphi_n)_n$ be an almost additive sequence of continuous functions on Λ with bounded variation. Then the unique equilibrium measure for Φ is the weak limit of the sequence of invariant probability measures*

$$\mu_n = \sum_{x\in\mathrm{Fix}(f^n)} e^{\varphi_n(x)}\delta_x \Big/ \sum_{x\in\mathrm{Fix}(f^n)} e^{\varphi_n(x)}. \tag{26}$$

Now we present another characterization of the unique equilibrium measures. Given a sequence of continuous functions $\Phi = (\varphi_n)_n$ with bounded variation, we set

$$a_{i_1\cdots i_n} = \max\big\{ \exp\varphi_n(y) : y \in \Delta_{i_1\cdots i_n}\big\},$$

with the convention that $a_{i_1\cdots i_n} = 0$ if $\Delta_{i_1\cdots i_n} = \varnothing$. We also set

$$\alpha_n = \sum_{i_1\cdots i_n} a_{i_1\cdots i_n}.$$

We define a probability measure ν_n in the algebra generated by the sets $\Delta_{i_1\cdots i_n}$ by

$$\nu_n(\Delta_{i_1\cdots i_n}) = a_{i_1\cdots i_n}/\alpha_n$$

for each $(i_1\cdots i_n) \in \Sigma_{A,n}$, and we extend it arbitrarily to the Borel σ-algebra of Λ. Since Λ is compact, the family of probability measures in Λ is compact in the weak* topology, and hence, there exists a subsequence $(\nu_{n_k})_k$ converging to some probability measure ν in the weak* topology. A priori the accumulation point ν need not be unique. We denote the set of all accumulation points of the sequence $(\nu_n)_n$ by $\mathcal{M}(\Phi)$. As explained above, $\mathcal{M}(\Phi) \neq \varnothing$. The following statement shows that all accumulation points are Gibbs measures.

Theorem 4.5 ([3])**.** *Let Λ be a repeller of a C^1 map, and let Φ be an almost additive sequence of continuous functions on Λ with bounded variation. Then each measure in $\mathcal{M}(\Phi)$ is an ergodic Gibbs measure for Φ.*

Moreover, the following is a characterization of the unique invariant Gibbs measure.

Theorem 4.6 ([3])**.** *Let Λ be a repeller of a C^1 map, and let Φ be an almost additive sequence of continuous functions on Λ with bounded variation. Then the unique invariant Gibbs measure for Φ is the unique invariant measure in $\mathcal{M}(\Phi)$.*

When Φ is an almost additive sequence of continuous functions in Λ with *tempered* variation (but not necessarily with bounded variation), we can still show that there exist an ergodic probability measure ν in Λ, a constant $K > 0$, and a positive sequence $(\rho_n)_n$ decreasing to 0, such that

$$K^{-1}e^{-n\rho_n} \le \frac{\nu(\Delta_{i_1\cdots i_n})}{\exp[-nP(\Phi) + \varphi_n(x)]} \le Ke^{n\rho_n} \tag{27}$$

for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in \Sigma_{A,n}$, and $x \in \Delta_{i_1 \cdots i_n}$. We emphasize that the measure ν need not be invariant. Furthermore, in general it may not be possible to obtain an invariant measure through an averaging procedure, due to the extra small exponentials in (27). On the other hand, it is still reasonable to call the measure ν in (27) a *weak Gibbs measure* for Φ, as proposed by Yuri in [47].

5. Results for hyperbolic sets

We consider in this section the case of functions defined in a hyperbolic set, and we formulate corresponding results to those in Section 4 for functions defined in a repeller.

5.1. **Hyperbolic sets and Markov partitions.** Let $f \colon M \to M$ be a diffeomorphism of a smooth manifold M, and let $\Lambda \subset M$ be a compact f-invariant set. We say that Λ is a *hyperbolic set* for f if for every point $x \in \Lambda$ there exists a decomposition of the tangent space

$$T_x M = E^s(x) \oplus E^u(x)$$

such that

$$d_x f E^s(x) = E^s(f(x)) \quad \text{and} \quad d_x f E^u(x) = E^u(f(x)),$$

and there exist constants $\lambda \in (0,1)$ and $c > 0$ such that

$$\|d_x f^n | E^s(x)\| \le c\lambda^n \quad \text{and} \quad \|d_x f^{-n} | E^u(x)\| \le c\lambda^n$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$. In addition, we always assume in this presentation that there is an open set $U \supset \Lambda$ such that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n U, \tag{28}$$

and that f is topologically mixing on Λ. Given $\varepsilon > 0$ sufficiently small, for each $x \in \Lambda$ the *local stable* and *unstable manifolds* (of size ε) are given by

$$V^s(x) = \big\{ y \in M : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \ge 0 \big\}$$

and

$$V^u(x) = \big\{ y \in M : d(f^n(y), f^n(x)) < \varepsilon \text{ for every } n \le 0 \big\},$$

where d is the distance on M.

Now we briefly recall the notion of Markov partition for a hyperbolic set. A collection of closed sets $R_1, \ldots, R_p \subset \Lambda$ with sufficiently small diameter (given by (14)) is called a *Markov partition* of Λ if:

1. $\Lambda = \bigcup_{i=1}^p R_i$, and $\overline{\operatorname{int} R_i} = R_i$ for $i = 1, \ldots, p$;
2. $V^s(x) \cap V^u(x) \in R_i$ and $\operatorname{card}(V^s(x) \cap V^u(x)) = 1$ for $x, y \in R_i$;
3. $\operatorname{int} R_i \cap \operatorname{int} R_j = \varnothing$ whenever $i \ne j$;
4. if $x \in f(\operatorname{int} R_i) \cap \operatorname{int} R_j$, then

$$f^{-1}(V^u(f(x)) \cap R_j) \subset V^u(x) \cap R_i$$

and

$$f(V^s(x) \cap R_i) \subset V^s(f(x)) \cap R_j.$$

The interior of each set R_i is computed with respect to the induced topology on Λ. Any hyperbolic set satisfying (28) has Markov partitions with arbitrarily small diameter (see for example [14]).

Given a Markov partition $R_1, \ldots, R_p$ of a hyperbolic set Λ, we define as in the case of repellers a $p \times p$ matrix $A = (a_{ij})$ with entries given by (15), and we consider the corresponding two-sided topological Markov chain defined by the shift map on the set

$$\Sigma_A = \Big\{(i_1 i_2 \cdots) \in \{1, \ldots, p\}^{\mathbb{Z}} : a_{i_k i_{k+1}} = 1 \text{ for every } k \in \mathbb{Z}\Big\}. \tag{29}$$

We continue to denote by $\Sigma_{A,n}$ the set of n-tuples $(i_1 \cdots i_n)$ for which there is a sequence $(\cdots j_0 j_1 j_2 \cdots) \in \Sigma_A$ such that $i_\ell = j_\ell$ for $\ell = 1, \ldots, n$. For each $(i_1 \cdots i_n) \in \Sigma_{A,n}$ we consider again the sets $\Delta_{i_1 \cdots i_n}$ defined by (17).

5.2. **Formulation of the results.** Repeating arguments in the proofs of Theorems 3.1 and 3.2 we obtain the following statement, thus providing formulas for the topological pressure of an almost additive sequence.

Theorem 5.1 ([3]). *Let Λ be a hyperbolic set of a C^1 map, and let Φ be an almost additive sequence of continuous functions on Λ with tempered variation. Then identities* (20) *and* (22) *hold for any points $x_{i_1 \cdots i_n} \in \Delta_{i_1 \cdots i_n}$, for each $(i_1 \cdots i_n) \in \Sigma_{A,n}$ and $n \in \mathbb{N}$.*

We also formulate corresponding versions of Theorems 4.1 and 4.3.

Theorem 5.2 ([3]). *Let Λ be a hyperbolic set of a C^1 map, and let Φ be an almost additive sequence of continuous functions on Λ with tempered variation. Then* (23) *holds, including the existence in $L^1(\Lambda, \mu)$ of the first limit, and the existence of the second limit.*

In particular, this shows that the sequence Φ has at least one equilibrium measure.

Theorem 5.3 ([3]). *Let Λ be a hyperbolic set of a C^1 map, and let Φ be an almost additive sequence of continuous functions on Λ with bounded variation. Then:*

1. *there is a unique equilibrium measure for Φ;*
2. *there is a unique invariant Gibbs measure for Φ;*
3. *the two measures are equal, are mixing, and coincide with the weak limit of the sequence of invariant probability measures μ_n in* (26).

6. Further generalizations

Some of the former results for repellers and hyperbolic sets can be generalized to more general classes of dynamics. We first present a variational principle for the topological pressure.

Theorem 6.1 ([3]). *Let f be a continuous map in a compact metric space Λ, and let Φ be an almost additive sequence of continuous functions in Λ satisfying* (6). *Then*

$$\begin{aligned} P(\Phi) &= \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \int_\Lambda \lim_{n\to\infty} \frac{\varphi_n(x)}{n} \, d\mu(x) \right) \\ &= \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \varphi_n \, d\mu \right), \end{aligned}$$

including the existence in $L^1(\Lambda, \mu)$ *of the first limit, and the existence of the second limit.*

We also formulate a criterion for the existence of equilibrium measures.

Theorem 6.2 ([3])**.** *Let* f *be a continuous map in a compact metric space* Λ *such that* $\mathcal{M} \in \mu \mapsto h_\mu(f)$ *is upper semi-continuous, and let* Φ *be an almost additive sequence of continuous functions on* Λ *satisfying* (6)*. Then there exists an equilibrium measure for* Φ*.*

For example, if f is an expansive continuous map in Λ, then the entropy is upper semi-continuous, and hence each almost additive sequence has an equilibrium measure. We recall that f is said to be *expansive* if there exists $\delta > 0$ such that if

$$d(f^n(x), f^n(y)) < \delta \quad \text{for every} \quad n \in \mathbb{N},$$

then $x = y$ (when f is invertible we replace $\mathbb{N}$ by $\mathbb{Z}$). For example, when f is a one-sided or two-sided topologically mixing topological Markov chain, the entropy is upper semi-continuous. Incidentally, all these transformations satisfy specification. On the other hand, there are plenty transformations not satisfying specification for which the entropy is still upper semi-continuous. For example, all β-shifts are expansive, and thus the entropy is upper semi-continuous (see [30] for details), but for β in a residual set of full Lebesgue measure (although the complement has full Hausdorff dimension) the corresponding β-shift does not satisfy specification (see [42]).

Finally, we describe some regularity properties of the topological pressure. We denote by $A(\Lambda)$ the family of almost additive sequences of continuous functions satisfying (6). Let also $E(\Lambda) \subset A(\Lambda)$ be the family of sequences with a unique equilibrium measure.

Theorem 6.3 ([5])**.** *Let* f *be a continuous map in a compact metric space* Λ *such that* $\mathcal{M} \ni \mu \mapsto h_\mu(f)$ *is upper semi-continuous. Then:*

1. *given* $\Phi \in A(\Lambda)$*, the function* $t \mapsto P(\Phi + t\Psi)$ *is differentiable at* $t = 0$ *for every* $\Psi \in A(\Lambda)$ *if and only if* $\Phi \in E(\Lambda)$*; in this case the unique equilibrium measure* μ *of* Φ *is ergodic, and*
$$\frac{d}{dt} P(\Phi + t\Psi)|_{t=0} = \lim_{n\to\infty} \int_\Lambda \frac{\psi_n}{n}\, d\mu; \tag{30}$$
2. *for each open set* $U \subset \mathbb{R}$*, if* $\Phi + t\Psi \in E(\Lambda)$ *for every* $t \in U$*, then the function* $t \mapsto P(\Phi + t\Psi)$ *is of class* C^1 *in* U*.*

The proof of Theorem 6.3 follows partially arguments in [30].

7. Application I: Nonconformal repellers

We describe in this section a class of nonconformal repellers considered by Barreira and Gelfert in [7] to which one can apply the results in Section 4, in connection with the study of Lyapunov exponents of nonconformal transformations.

7.1. **Cone condition and bounded distortion.** To describe the class of repellers under consideration, we first introduce what we call a cone condition.

Given a number $\gamma \le 1$ and a 1-dimensional subspace $E(x) \subset \mathbb{R}^2$, we consider the *cone*

$$C_\gamma(x) = \big\{(u,v) \in E(x) \oplus E(x)^\perp : \|v\| \le \gamma \|u\|\big\}.$$

We say that a differentiable map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies a *cone condition* on a set $\Lambda \subset \mathbb{R}^2$ if there exist $\gamma \le 1$ and for each $x \in \Lambda$ a 1-dimensional subspace $E(x) \subset \mathbb{R}^2$ varying continuously with x such that

$$(d_x f) C_\gamma(x) \subset \{0\} \cup \operatorname{int} C_\gamma(fx). \tag{31}$$

Following [7], we present several examples of maps satisfying a cone condition.

Example 7.1. *Assume that for each $x \in \Lambda$ the derivative $d_x f$ is represented by a positive 2×2 matrix. Then the first quadrant Q is invariant under these linear transformations, that is, $(d_x f)Q \subset Q$ for each $x \in \Lambda$. Therefore, the map f satisfies the cone condition in* (31) *with $\gamma = 1$, taking for $E(x)$ the* 1*-dimensional subspace making an angle of $\pi/4$ with the horizontal direction.*

This example is related to work in [25] (see also [22]).

Another class of examples corresponds to the existence of a strongly unstable foliation.

Example 7.2. *Let Λ be a locally maximal repeller in the sense that in some open neighborhood U the repeller Λ is the only invariant set. In this case $f^{-1}\Lambda \cap U = \Lambda$. Assume that there exists a* strongly unstable foliation *of the set U, that is, a foliation by* 1*-dimensional C^2 leaves $V(x)$ such that:*

1. *$f(V(x)) \supset V(fx)$ for every $x \in U \cap f^{-1}U$;*
2. *there exist constants $c > 0$ and $\lambda \in (0,1)$ such that*

$$\frac{|\det d_x f^n|}{\|d_x f^n | T_x V(x)\|^2} \le c\lambda^n \text{ for all } x \in \bigcap_{i=0}^{n} f^{-i}U \text{ and } n \in \mathbb{N}.$$

It is shown by Hu in [28] *that this assumption is equivalent to:*

1. *for some choice of subspaces $E(x)$ varying continuously with x, the cone condition in* (31) *holds for every $x \in U \cap f^{-1}U$;*
2. *there exist* 1*-dimensional subspaces $F(x) \subset \{0\} \cup \operatorname{int} C_\gamma(x)$ for each $x \in U \cap f^{-1}U$ such that $d_x f F(x) = F(fx)$.*

Thus, repellers with a strongly unstable foliation satisfy a cone condition.

Notice that the cone condition in (31) is weaker then assuming the existence of a strongly unstable foliation. In particular, (31) does not ensure the existence of an invariant distribution $F(x)$ as in Example 7.2. On the other hand, when there exists a strongly unstable foliation, the invariant distribution $F(x)$ is given by (see [28])

$$F(x) = \bigcap_{n\in\mathbb{N}} \bigcup_{y \in f^{-n}x} d_y f^n C_\gamma(y).$$

It is thus independent of the particular preimages $x_n \in f^{-n}x$, that is,

$$F(x) = \bigcap_{n\in\mathbb{N}} d_{x_n} f^n C_\gamma(x_n).$$

We can also consider repellers with a dominated splitting.

Example 7.3. *We say that the repeller* Λ *possesses a* dominated splitting *if there exists a decomposition* $T_\Lambda\mathbb{R}^2 = E \oplus F$ *such that:*

1. $d_x f E(x) = E(fx)$ *and* $d_x f F(x) = F(fx)$ *for every* $x \in \Lambda$*;*
2. *there exist constants* $c > 0$ *and* $\lambda \in (0,1)$ *such that*

$$\|d_x f^n|E\| \cdot \|(d_x f)^{-n}|F\| \le c\lambda^n \text{ for all } x \in \Lambda \text{ and } n \in \mathbb{N}.$$

It follows easily from the definition that the subspaces $E(x)$ *and* $F(x)$ *vary continuously with* x*. Furthermore, one can verify that when there exists a dominated splitting of* Λ*, the map* f *satisfies a cone condition on* Λ*.*

We note that the existence of a strongly unstable foliation does not ensure the existence of a dominated splitting, due to the requirement of a df-invariant decomposition $E \oplus F$ (more precisely, the existence of a strongly unstable foliation only ensures the existence of the invariant distribution F in Example 7.2).

Now we consider certain almost additive sequences of functions obtained from the singular values of a 2×2 matrix A, namely

$$\sigma_1(A) = \|A\| \quad \text{and} \quad \sigma_2(A) = \|A^{-1}\|^{-1}$$

(with respect to the 2-norm in $\mathbb{R}^2$). Given a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, we define sequences of functions $\Phi_i = (\varphi_{i,n})_n$ for $i = 1, 2$ by

$$\varphi_{i,n}(x) = \log \sigma_i(d_x f^n) \tag{32}$$

for each $n \in \mathbb{N}$ and $i = 1, 2$. Clearly, the functions $\varphi_{i,n}$ are continuous. These sequences are related to the Lyapunov exponents of the map f (see Section 7.2). We first present a criterium for almost additivity.

Proposition 7.4 ([7])**.** *Let* Λ *be a repeller of a* C^1 *map* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$*. If* f *satisfies a cone condition on* Λ*, then* Φ_i *is almost additive for* $i = 1, 2$*.*

For a map f as in Proposition 7.4, we consider a number $\delta > 0$ such that for every $x \in \Lambda$ the map is invertible on the ball $B(x, \delta)$ (simply take a Lebesgue number of a cover by balls with the property that f is invertible on each of them). For each $x \in \Lambda$ and $n \in \mathbb{N}$ we define

$$B_n(x, \delta) = \bigcap_{\ell=0}^{n-1} f^{-\ell} B(f^\ell x, \delta).$$

We always assume that the diameter of the Markov partition used to define the sets $\Delta_{i_1\cdots i_n}$ in (17) is at most $\delta/2$ (we recall that any repeller has Markov partitions of arbitrarily small diameter). This ensures that

$$\Delta_{i_1\cdots i_n} \subset B_n(x, \delta)$$

for every $x = \chi(i_1 i_2 \cdots) \in \Lambda$ and $n \in \mathbb{N}$. We say that f has *bounded distortion* on Λ if there exists $\delta > 0$ such that

$$\sup\bigl\{\|d_y f^n (d_z f^n)^{-1}\| : x \in \Lambda \text{ and } y, z \in B_n(x, \delta)\bigr\} < \infty.$$

Now we give a condition for bounded distortion in the case of $C^{1+\alpha}$ transformations. Given $\alpha > 0$, we say that f is *α-bunched* on Λ if

$$\|(d_xf)^{-1}\|^{1+\alpha}\|d_xf\| < 1$$

for every $x \in \Lambda$ (this notion was introduced in [1] in the context of dimension theory of nonconformal transformations). The following statement is an immediate consequence of the proof of Theorem 4 in [2].

Proposition 7.5. *Let Λ be a repeller of a $C^{1+\alpha}$ map $f\colon M \to M$. If f is α-bunched on Λ, then f has bounded distortion on Λ.*

Now we consider the sequences Φ_i for $i = 1, 2$ introduced in (32) and we present a criterium for bounded variation.

Proposition 7.6 (see [7, Proposition 1])**.** *Let Λ be a repeller of a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. If f has bounded distortion on Λ, then Φ_i has bounded variation for $i = 1, 2$.*

7.2. **Variational principle and Gibbs measures.** It follows from Propositions 7.4 and 7.6 that if a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies a cone condition on Λ and has bounded distortion on Λ, then Φ_i is an almost additive sequence with bounded variation for $i = 1, 2$. This allows us to apply the results in Section 4 to recover the corresponding statements of Barreira and Gelfert in [7].

To explain the relation between the sequences Φ_i and the theory of Lyapunov exponents, we first recall some basic notions. Given a differentiable transformation $f\colon M \to M$ (which is not necessarily invertible), for each $x \in M$ and $v \in T_xM$ we define the *Lyapunov exponent* of (x, v) by

$$\chi(x,v) = \limsup_{n\to+\infty} \frac{1}{n} \log\|d_xf^nv\|, \tag{33}$$

with the convention that $\log 0 = -\infty$. It follows from the abstract theory of Lyapunov exponents (see [8] for full details) that for each $x \in M$ there exist a positive integer $s(x) \le \dim M$, numbers $\chi_1(x) < \cdots < \chi_{s(x)}(x)$, and linear subspaces

$$\{0\} = E_0(x) \subset E_1(x) \subset \cdots \subset E_{s(x)}(x) = T_xM$$

such that for $i = 1, \ldots, s(x)$ we have

$$E_i(x) = \bigl\{v \in T_xM\colon \chi(x,v) \le \chi_i(x)\bigr\},$$

and $\chi(x, v) = \chi_i(x)$ whenever $v \in E_i(x) \setminus E_{i-1}(x)$. It follows from Oseledets' multiplicative ergodic theorem (see for example [8]), or more precisely from its version for noninvertible transformations, that for each finite f-invariant measure in M there is a set $X \subset M$ of full measure such that if $x \in X$, then

$$\lim_{n\to+\infty} \frac{1}{n} \log\|d_xf^nv\| = \chi_i(x)$$

for every $v \in E_i(x) \setminus E_{i-1}(x)$ and $i = 1, \ldots, s(x)$, with uniform convergence in v on each subspace $F \subset E_i(x)$ such that $F \cap E_{i-1}(x) = \{0\}$ (in particular, the lim sup in (33) is now a limit).

For $M = \mathbb{R}^2$ and each $x \in \mathbb{R}^2$, when $s(x) = 1$ we set

$$\lambda_1(x) = \chi_1(x) \quad\text{and}\quad \lambda_2(x) = \chi_1(x),$$

and when $s(x) = 2$ we set

$$\lambda_1(x) = \chi_1(x) \quad \text{and} \quad \lambda_2(x) = \chi_2(x).$$

The numbers $\lambda_1(x)$ and $\lambda_2(x)$ are the values of the Lyapunov exponent $v \mapsto \chi(x, v)$ counted with multiplicities. It follows again from Oseledets' multiplicative ergodic theorem that for each finite f-invariant measure in $\mathbb{R}^2$ there is a set $X \subset \mathbb{R}^2$ of full measure such that

$$\lim_{n\to+\infty} \frac{\varphi_{i,n}(x)}{n} = \lim_{n\to+\infty} \frac{1}{n} \log \sigma_i(d_x f^n) = \lambda_i(x)$$

for each $x \in X$ and $i = 1, 2$ (see (32)). Combining these observations with the criteria in Propositions 7.4 and 7.6, we readily obtain the following statement of Barreira and Gelfert by applying the results in Section 4.

Theorem 7.7 ([7])**.** *Let Λ be a repeller of a C^1 map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. If f satisfies a cone condition on Λ, and f has bounded distortion on Λ, then for $i = 1$, 2 the following properties hold:*

1. *the topological pressure satisfies the variational principle*

$$\begin{aligned} P(\Phi_i) &= \max_{\mu\in\mathcal{M}} \left(h_\mu(f) + \int_\Lambda \lambda_i(x)\, d\mu(x) \right) \\ &= \max_{\mu\in\mathcal{M}} \left(h_\mu(f) + \lim_{n\to\infty} \frac{1}{n} \int_\Lambda \log \sigma_i(d_x f^n)\, d\mu(x) \right); \end{aligned}$$

2. *there is a unique equilibrium measure μ_i for Φ_i, and this is the unique invariant Gibbs measure for Φ_i;*
3. *there is a constant $K > 0$ such that*

$$K^{-1} \le \frac{\mu_i(\Delta_{i_1\cdots i_n})}{\exp[-nP(\Phi_i)]\sigma_i(d_x f^n)} \le K$$

for every $n \in \mathbb{N}$, $(i_1 \cdots i_n) \in \Sigma_{A,n}$, and $x \in \Delta_{i_1\cdots i_n}$;

4. *the measure μ_i is mixing, and*

$$\sum_{x\in\mathrm{Fix}(f^n)} \sigma_i(d_x f^n)\delta_x \Big/ \sum_{x\in\mathrm{Fix}(f^n)} \sigma_i(d_x f^n) \rightharpoonup \mu_i \quad \text{as} \quad n \to \infty.$$

8. Application II: Multifractal analysis

We describe in this section a conditional variational principle for the u-dimension spectrum established by Barreira and Doutor in [5]. This contains as a particular case a conditional variational principle for the entropy spectrum (see Theorem 8.3 below). For simplicity of the exposition we do not consider the multidimensional case in [5] but only the case of a single ratio of almost additive functions. We emphasize that this is already a nontrivial result when compared to the existing results in the classical case of additive sequences.

8.1. Notion of u-dimension.

We recall in this section the notion of u-dimension introduced by Barreira and Schmeling in [10]. Let $f\colon X \to X$ be a continuous transformation of a compact metric space, and let $\mathcal{U}$ be a finite open cover of X. Let also $u\colon X \to \mathbb{R}^+$ be a continuous function. Given a set $Z \subset X$ and a number $\alpha \in \mathbb{R}$, we define the function

$$N(Z,\alpha,u,\mathcal{U}) = \lim_{n\to\infty} \inf_{\Gamma} \sum_{U\in\Gamma} \exp(-\alpha u(U)),$$

where $u(U)$ is defined as in (8), and where the infimum is taken over all finite or countable collections $\Gamma \subset \bigcup_{k\ge n} \mathcal{W}_k(\mathcal{U})$ such that $\bigcup_{u\in\Gamma} X(U) \supset Z$. Setting

$$\dim_{u,\mathcal{U}} Z = \inf\bigl\{\alpha \in \mathbb{R}\colon N(Z,\alpha,u,\mathcal{U}) = 0\bigr\},$$

one can show that the limit

$$\dim_u Z = \lim_{\operatorname{diam}\mathcal{U}\to 0} \dim_{u,\mathcal{U}} Z$$

exists. The number $\dim_u Z$ is called the *u-dimension* of the set Z (with respect to f). For example, if $u = 1$, then $\dim_u Z$ is equal to the topological entropy $h(f|Z)$ of f on Z (see Section 2).

The following result is an easy consequence of the definitions.

Proposition 8.1. *The number* $\dim_u Z = \alpha$ *is the unique root* α *of the equation* $P_Z(-\alpha U) = 0$*, where* $U = (u_n)_n$ *with* $u_n = \sum_{k=0}^{n-1} u\circ f^k$ *for each* $n \in \mathbb{N}$.

Furthermore, given a probability measure μ in X, we set

$$\dim_{u,\mathcal{U}} \mu = \inf\bigl\{\dim_{u,\mathcal{U}} Z : \mu(Z) = 1\bigr\}.$$

One can show that the limit

$$\dim_u \mu = \lim_{\operatorname{diam}\mathcal{U}\to 0} \dim_{u,\mathcal{U}} \mu$$

exists, and we call it the *u-dimension* of μ. Moreover, the *lower* and *upper u-pointwise dimensions* of μ at the point $x \in X$ are defined by

$$\underline{d}_{\mu,u}(x) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \liminf_{n\to\infty} \inf_{U} -\frac{\log \mu(X(U))}{u(U)}$$

and

$$\overline{d}_{\mu,u}(x) = \lim_{\operatorname{diam}\mathcal{U}\to 0} \limsup_{n\to\infty} \sup_{U} -\frac{\log \mu(X(U))}{u(U)},$$

where the infimum and supremum are taken over all vectors $U \in \mathcal{W}_n(\mathcal{U})$ such that $x \in X(U)$. If $\mu \in \mathcal{M}$ is ergodic, then

$$\dim_u \mu = \underline{d}_{\mu,u}(x) = \overline{d}_{\mu,u}(x) = \frac{h_\mu(f)}{\int_X u\,d\mu}$$

for μ-almost every $x \in X$ (see [10]).

8.2. **Conditional variational principle.** We formulate in this section a conditional variational principle for the u-dimension of sets defined in terms of ratios of almost additive sequences. This corresponds to a multifractal analysis of the level sets of limits of ratios of almost additive sequences.

We continue to consider a continuous map $f\colon X \to X$ of a compact metric space. Let $\Phi = (\varphi_n)_n$ and $\Psi = (\psi_n)_n$ be almost additive sequences of functions in X. We assume that

$$\liminf_{m\to\infty} \frac{\psi_m(x)}{m} > 0 \quad \text{and} \quad \psi_n(x) > 0$$

for every $x \in X$ and $n \in \mathbb{N}$. Given $\alpha \in \mathbb{R}$ we define

$$K_\alpha = \left\{ x \in X : \lim_{n\to\infty} \frac{\varphi_n(x)}{\psi_n(x)} = \alpha \right\}. \tag{34}$$

The function $\mathcal{F}_u \colon \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{F}_u(\alpha) = \dim_u K_\alpha$$

is called the *u-dimension spectrum of the pair* (Φ, Ψ) (with respect to f). We also consider the function $\mathcal{P} \colon \mathcal{M} \to \mathbb{R}$ defined by

$$\mathcal{P}(\mu) = \lim_{n\to\infty} \frac{\int_X \varphi_n \, d\mu}{\int_X \psi_n \, d\mu}.$$

The following is a conditional variational principle for the spectrum $\mathcal{F}_u$. We consider the (additive) sequence of functions $U = (u_n)_n$ with $u_n = \sum_{k=0}^{n-1} u \circ f^k$ for each $n \in \mathbb{N}$. We recall that $E(X)$ denotes the family of almost additive sequences satisfying (6) with a unique equilibrium measure.

Theorem 8.2 ([5]). *Let f be a continuous map of a compact metric space X such that $\mu \mapsto h_\mu(f)$ is upper semi-continuous, and assume that*

$$\operatorname{span}\{\Phi, \Psi, U\} \subset E(X).$$

If $\alpha \notin \mathcal{P}(\mathcal{M})$, then $K_\alpha = \varnothing$. Otherwise, if $\alpha \in \operatorname{int} \mathcal{P}(\mathcal{M})$, then $K_\alpha \neq \varnothing$, and the following properties hold:

1. *$\mathcal{F}_u$ satisfies the variational principle*
$$\mathcal{F}_u(\alpha) = \max \left\{ \frac{h_\mu(f)}{\int_X u \, d\mu} : \mu \in \mathcal{M} \text{ and } \mathcal{P}(\mu) = \alpha \right\};$$
2. *we have*
$$\mathcal{F}_u(\alpha) = \min \big\{ T_u(\alpha, q) : q \in \mathbb{R} \big\},$$
where $T_u(\alpha, q)$ is the unique real number satisfying
$$P\big(q(\Phi - \alpha\Psi) - T_u(\alpha, q) U\big) = 0; \tag{35}$$
3. *there is an ergodic measure $\mu_\alpha \in \mathcal{M}$ such that $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and*
$$\dim_u \mu_\alpha = \frac{h_{\mu_\alpha}(f)}{\int_X u \, d\mu_\alpha} = \mathcal{F}_u(\alpha).$$

In addition, the spectrum $\mathcal{F}_u$ is continuous in $\operatorname{int} \mathcal{P}(\mathcal{M})$.

The proof of Theorem 8.2 builds on earlier work of Barreira, Saussol and Schmeling in [9]. We note that the number $T_u(\alpha, q)$ is defined implicitly by (35). By Theorem 6.3, the function

$$(p, \alpha, q) \mapsto P\big(q(\Phi - \alpha\Psi) - pU\big)$$

is of class C^1. By the Implicit function theorem, we conclude that $(\alpha, q) \mapsto T_u(\alpha, q)$ is also of class C^1 in $\mathbb{R}^2$, since by (30),

$$\frac{\partial}{\partial p} P\big(q(\Phi - \alpha\Psi) - pU\big)\bigg|_{(p,q)=(T_u(\alpha,q),q)} = -\int_X u \, d\mu_q < 0,$$

where μ_q is the unique equilibrium measure of $q(\Phi - \alpha\Psi) - T_u(\alpha, q)U$.

Now we formulate explicitly a particular case of Theorem 8.2. Let $\Phi = (\varphi_n)_n$ be an almost additive sequence of functions $\varphi_n\colon X \to \mathbb{R}$. Given $\alpha \in \mathbb{R}$, we consider the level set

$$K_\alpha = \left\{x \in X : \lim_{n\to\infty} \varphi_n(x) = \alpha\right\}.$$

The *entropy spectrum* $\mathcal{E}\colon \mathbb{R} \to \mathbb{R}$ (of the sequence Φ) is defined by

$$\mathcal{E}(\alpha) = h(f|K_\alpha),$$

where $h(f|K_\alpha)$ denotes the topological entropy of f on K_α (see Sections 2 and 8.1). We also consider the function $\mathcal{P}\colon \mathcal{M} \to \mathbb{R}$ defined by

$$\mathcal{P}(\mu) = \lim_{n\to\infty} \frac{1}{n} \int_X \varphi_n \, d\mu.$$

The following statement is a conditional variational principle for the entropy spectrum $\mathcal{E}$. It is an immediate consequence of Theorem 8.2 below.

Theorem 8.3. *Let f be a continuous map of a compact metric space X such that $\mu \mapsto h_\mu(f)$ is upper semi-continuous, and assume that the almost additive sequence Φ has a unique equilibrium measure. If $\alpha \notin \mathcal{P}(\mathcal{M})$, then $K_\alpha = \varnothing$. Otherwise, if $\alpha \in$ int $\mathcal{P}(\mathcal{M})$, then $K_\alpha \neq \varnothing$, and the following properties hold:*

1. *$\mathcal{E}$ satisfies the variational principle*
$$\mathcal{E}(\alpha) = \max\left\{h_\mu(f) : \mu \in \mathcal{M} \text{ and } \mathcal{P}(\mu) = \alpha\right\};$$
2. $$\mathcal{E}(\alpha) = \min\left\{P(q\Phi) - q\alpha : q \in \mathbb{R}\right\};$$
3. *there is an ergodic measure $\mu_\alpha \in \mathcal{M}$ such that $\mathcal{P}(\mu_\alpha) = \alpha$, $\mu_\alpha(K_\alpha) = 1$, and $h_{\mu_\alpha}(f) = \mathcal{E}(\alpha)$.*

In addition, the spectrum $\mathcal{E}$ is continuous in int $\mathcal{P}(\mathcal{M})$.

Now we consider the associated irregular sets, on which the limits in (34) do not exist. We consider only the particular case of topological Markov chains. Namely, let Φ and Ψ be almost additive sequences in Σ_A, either as in (16) or as in (29). The *irregular set* of the pair (Φ, Ψ) is defined by

$$I = \left\{x \in \Sigma_A : \liminf_{n\to\infty} \frac{\varphi_n(x)}{\psi_n(x)} < \limsup_{n\to\infty} \frac{\varphi_n(x)}{\psi_n(x)}\right\},$$

and we denote by m_u the equilibrium measure of u, when it is unique.

Theorem 8.4 ([5])**.** *Let $\sigma|\Sigma_A$ be a topologically mixing topological Markov chain. If*

$$\text{span}\{\Phi, \Psi, U\} \subset E(\Sigma_A),$$

and $\mathcal{P}(m_u) \in \operatorname{int} \mathcal{P}(\mathcal{M}_\sigma)$, then

$$\dim_u I = \dim_u \Sigma_A.$$

Theorem 8.4 follows from the application of results in [10] combined with Theorem 8.2.

9. Application III: Dimension spectra

Our last application of the almost additive thermodynamic formalism considers dimension spectra of level sets associated to the limits of ratios of almost additive sequences. Moreover, we take into account simultaneously limits of ratios of sequences into the future and into the past.

Let $f\colon M \to M$ be a $C^{1+\varepsilon}$ surface diffeomorphism with a hyperbolic set Λ satisfying the same hypotheses as in Section 5.1. We always assume that

$$\dim E^s(x) = \dim E^u(x) = 1$$

for every $x \in \Lambda$. Let t_s and t_u be the unique real numbers such that

$$P(t_s \log \|df|E^s\|) = P(t_u \log \|df^{-1}|E^u\|) = 0,$$

where P denotes the (classical) topological pressure with respect to f on Λ. It was shown by McCluskey and Manning in [32] that

$$\dim_H(\Lambda \cap V^s(x)) = t_s \quad \text{and} \quad \dim_H(\Lambda \cap V^u(x)) = t_u$$

for every $x \in \Lambda$, where $\dim_H$ denotes the Hausdorff dimension. Moreover, it was shown by Palis and Viana in [34] that

$$\begin{aligned}\dim_H(\Lambda \cap V^s(x)) &= \overline{\dim}_B(\Lambda \cap V^s(x)),\\ \dim_H(\Lambda \cap V^u(x)) &= \overline{\dim}_B(\Lambda \cap V^u(x))\end{aligned}$$

for every $x \in \Lambda$, where $\overline{\dim}_B$ denotes the upper box dimension. Since the stable and unstable distributions have codimension 1, it follows from results of Hasselblatt in [27] that the maps $x \mapsto E^s(x)$ and $x \mapsto E^u(x)$ are Lipschitz. This implies that

$$\begin{aligned}\dim_H\Lambda &= \dim_H[(\Lambda \cap V^s(x)) \times (\Lambda \cap V^u(x))]\\ &= \dim_H(\Lambda \cap V^s(x)) + \dim_H(\Lambda \cap V^u(x)) = t_s + t_u.\end{aligned} \tag{36}$$

Indeed, if $\dim_H A = \overline{\dim}_B A$, then for any set B we have

$$\dim_H(A \times B) = \dim_H A + \dim_H B.$$

Now we proceed with the description of the dimension spectra. We denote by L^+ (respectively L^-) the family of almost additive sequences of continuous functions with respect to f (respectively f^{-1}) that have bounded variation with respect to f (respectively f^{-1}). We only consider almost additive sequences

$$\Phi^+ = (\varphi_n^+)_n, \quad \Phi^- = (\varphi_n^-)_n, \quad \Psi^+ = (\psi_n^+)_n, \quad \text{and} \quad \Psi^- = (\psi_n^-)_n$$

such that

$$\liminf_{m\to\infty} \frac{\psi_m^\pm(x)}{m} > 0 \quad \text{and} \quad \psi_n^\pm(x) > 0$$

for every $n \in \mathbb{N}$ and $x \in \Lambda$. Given $(\Phi^+, \Psi^+) \in L^+ \times L^+$ and $\alpha \in \mathbb{R}$ we define

$$K_\alpha^+ = \left\{x \in \Lambda : \lim_{n\to\infty} \frac{\varphi_n^+(x)}{\psi_n^+(x)} = \alpha\right\},$$

and given $(\Phi^-, \Psi^-) \in L^- \times L^-$ and $\alpha \in \mathbb{R}$ we define

$$K_\alpha^- = \left\{x \in \Lambda : \lim_{n\to\infty} \frac{\varphi_n^-(x)}{\psi_n^-(x)} = \alpha\right\}.$$

We also consider the *dimension spectrum* $\mathcal{D}\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\mathcal{D}(\alpha,\beta) = \dim_H(K_\alpha^+ \cap K_\beta^-).$$

The following is a conditional variational principle for the spectrum $\mathcal{D}$.

Theorem 9.1 ([6]). *If* $\alpha \in \operatorname{int}\mathcal{P}^+(\mathcal{M})$ *and* $\beta \in \operatorname{int}\mathcal{P}^-(\mathcal{M})$, *then*

$$\begin{aligned}\mathcal{D}(\alpha,\beta) &= \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda\\ &= \max\left\{\frac{h_\mu(f)}{-\int_\Lambda \log\|df|E^s\|\,d\mu} : \mu \in \mathcal{M} \text{ and } \mathcal{P}^+(\mu) = \alpha\right\}\\ &\quad + \max\left\{\frac{h_\mu(f)}{\int_\Lambda \log\|df|E^u\|\,d\mu} : \mu \in \mathcal{M} \text{ and } \mathcal{P}^-(\mu) = \beta\right\}.\end{aligned} \tag{37}$$

Moreover, the spectrum $\mathcal{D}$ *is analytic in* $\operatorname{int}\mathcal{P}^+(\mathcal{M}) \times \operatorname{int}\mathcal{P}^-(\mathcal{M})$.

The proof Theorem 9.1 follows to some extent arguments of Barreira and Valls [11] in the additive case. In particular, it involves constructing a measure $\nu = \nu_{\alpha\beta}$ sitting on the set $K_\alpha^+ \cap K_\beta^-$, that is, such that

$$\nu(K_\alpha^+ \cap K_\beta^-) = 1,$$

having the "right" pointwise dimension. This means that

$$\liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \ge \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda$$

for ν-almost every $x \in \Lambda$, and

$$\limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \le \dim_H K_\alpha^+ + \dim_H K_\beta^- - \dim_H \Lambda$$

for every $x \in K_\alpha^+ \cap K_\beta^-$. These properties, together with general results in dimension theory (see for example [4]) readily yield the first identity in (37). The second identity follows from Theorem 8.2. The measure ν, although never invariant, is constructed essentially as a product of (invariant) equilibrium measures along the stable and unstable directions, for which the results in Section 4 are essential. More precisely, set

$$U = q^+(\Phi - \alpha\Psi) - (\dim_H K_\alpha^+ - t_s)D_u$$

and

$$S = q^-(\Phi - \beta\Psi) - (\dim_H K_\beta^- - t_u)D_s,$$

where D_u and D_s are the additive sequences

$$\sum_{k=0}^{n-1} \log\|df|E^u\| \circ f^k \quad\text{and}\quad \sum_{k=0}^{n-1} \log\|df^{-1}|E^s\| \circ f^k,$$

and where $q^+, q^- \in \mathbb{R}$ are such that

$$P(U) = P(S) = 0.$$

By the almost additive thermodynamic formalism there exist unique equilibrium measures ν^u and ν^s respectively of U and S. Roughly speaking, the measure $\nu_{\alpha\beta}$ is given by the product $\nu^u \times \nu^s$ at the level of symbolic dynamics. It is also shown in [6] that

$$\dim_H K_\alpha^+ = \dim_H(K_\alpha^+ \cap V^u(x)) + t_s$$

and

$$\dim_H K_\beta^- = \dim_H(K_\beta^- \cap V^s(y)) + t_u$$

for every $x \in K_\alpha^+$ and $y \in K_\beta^-$. Together with (36) and (37) this shows that

$$\mathcal{D}(\alpha, \beta) = \dim_H(K_\alpha^+ \cap V^u(x)) + \dim_H(K_\beta^- \cap V^s(y))$$

for every $x \in K_\alpha^+$ and $y \in K_\beta^-$.

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