NUMERICAL SCHEMES FOR MULTI PHASE QUADRATURE DOMAINS

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ABSTRACT. In this work, numerical schemes to approximate the solution of one and multi phase quadrature domains are presented. We shall construct a monotone, stable and consistent finite difference method for both one and two phase cases, which converges to the viscosity solution of the partial differential equation arising from the corresponding quadrature domain theory. Moreover, we will discuss the numerical implementation of the resulting approach and present computational tests.

CONTENTS

1. Preliminaries

The subject of the quadrature domains, QDs, has been extensively studied over the last half-century and most of the papers deal with the one phase case, e.g., see [10], [12] and [17]. There is a wide range of applications of quadrature domains

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in physical problems. For instance, Richardson in [19] has studied the Hele-Shaw problem involving a moving boundary problem by driving a flow between two parallel planes without considering surface tension. He opened a crucial and new theory which now is a well developed subject. The solution of the Hele-Shaw problem can be figure out as a one phase quadrature domain.

To the best of our knowledge, most of the authors have studied theoretical aspects of this field and there is a few literature on numerical approach to the quadrature domains. The authors have presented some numerical schemes to approach the one phase quadrature domain in [6]. The main contribution of this paper is to investigate different numerical approximations for the one, two and multi phase quadrature domains.

The outline of this paper is as follows

- We will state the problem in Section two and provide the explanation of the one and the two phase cases and the corresponding partial differential equations, PDEs.
- Section three consists of an introduction to the degenerate elliptic equation and the viscosity solutions.
- Section four is devoted to reformulate the problem for the one and the two phase case. We provide two degenerate elliptic equations and investigate the relation between their viscosity solutions and the weak solutions of the PDEs.
- In Section five we discretize the reformulated problems and introduce our numerical algorithms based on finite difference method. Through this section we concentrate on a special measure, the Dirac measure, and explain the schemes for this case.
- In the last section we shall examine the algorithms by studying some numerical examples.

2. Problem Setting

Let $\mu_i, i=1,\cdots,m$ be given finite measures with compact supports and $\lambda_i(x)$ be non-negative Lipschitz continuous functions. In this article, we investigate the following problem.

Problem: Find functions u_i and domains $\Omega_i := \{x \in \mathbb{R}^N | u_i(x) > 0\}$ for $i = 1,\cdots,m$ such that $\operatorname{supp}(\mu_i) \subset \Omega_i$ and

$$
\begin{cases}
\Delta u_i = \lambda_i \chi_{\Omega_i} - \mu_i & \text{in } \mathbb{R}^N,\\
u_i = 0 & \text{in } \Omega_j,\, j \neq i,\\
|\nabla u_i| = |\nabla u_j| & \text{on } \Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j,\\
|\nabla u_i| = 0 & \text{on } \partial\Omega_i \setminus \bigcup \Gamma_{ij},
\end{cases}
\tag{2.1}
$$

which is understood in the distribution sense. For an illustration of the problem see Figure 1. This problem is related to geometric flows and integral identity in potential theory.

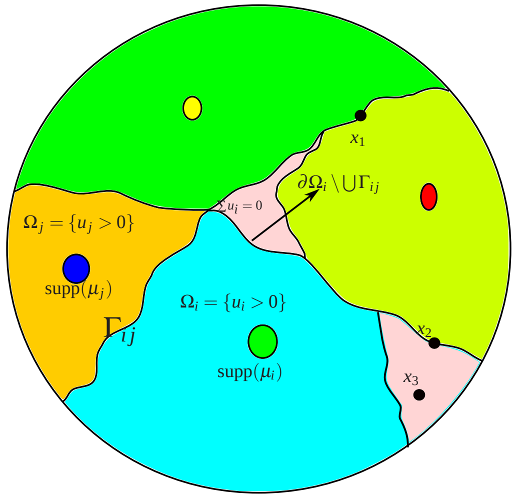


FIGURE 1. This figure shows the supports of the measures and supports of the solution of (2.1) and the corresponding free boundary. The points x_1, x_2 and x_3 are examples of points with different multiplicity, see definition 2.2.

The main contribution of this paper is to construct a finite difference method to approximate the solution of (2.1). We also prove that the numerical approximation converges to the viscosity solution of problem (2.1) for the cases $m = 1$ and $m = 2$. These cases arise from the quadrature domains theory which is quite well studied for the one phase case, see for instance [12] and [17] and the references therein.

2.1. Case $m = 1$: One phase quadrature domain

Let μ be a Radon measure with compact support in $\mathbb{R}^N$. An open connected domain $\Omega \subset \mathbb{R}^N$, is called *quadrature domain* with respect to μ if

$$\int_\Omega h\,dx \geq \int h\,d\mu, \qquad \forall h \in SL^1(\Omega), \quad supp(\mu) \subset \Omega, \tag{2.2}$$

where $SL^1(\Omega)$ is the space of all subharmonic functions contained in $L^1(\Omega)$. There is a strong relation between the concept of quadrature domains, potential theory and the theory of partial differential equations (PDEs). Sakai in [17] has shown that if Ω is a quadrature domain with respect to μ then the pair (u, Ω) with $\Omega := \{x \in \mathbb{R}^N \,|\, u(x) > 0\}$ is the unique solution of the following one phase free boundary problem

$$\begin{cases} \Delta u = \chi_\Omega - \mu, & \text{in} \quad \mathbb{R}^N, \\ u \geq 0, & \text{in} \quad \mathbb{R}^N, \\ u = |\nabla u| = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.3}$$

To ensure this work self contained, we will explain the theory of the two phase case.

2.2. Case $m = 2$: Two phase quadrature domain

Emamizadeh, Prajapat and Shahgholian introduced the two phase quadrature domain in [9]. They proved the existence of the solution of (2.1) by minimization techniques in the case of $m = 2$. The uniqueness of quadrature domain is a challenging problem even in the one phase case, but if one consider sign assumption, see (2.4) then the problem has a unique solution, see [3].

Here we briefly review the definition of the two phase quadrature domain. Before that we need to introduce some notations.

Suppose that $\Omega \subset \mathbb{R}^N$. Let

- $S^+(\Omega)$ and $S^-(\Omega)$ be the set of all subharmonic and superharmonic functions in Ω respectively.
- $SL^\pm(\Omega)$ be the set of all functions in L^1 which are in $S^\pm(\Omega)$.

We note that $SL^+(\Omega)$ is exactly the same as $SL^1(\Omega)$ in the one phase case.

Definition 2.1. Let $\Omega^\pm$ be two disjoint subsets of $\mathbb{R}^N$ and $\mu^\pm$ be two positive Radon measures with compact supports in $\Omega^\pm$. Moreover, suppose that $\lambda^\pm$ are two non-negative Lipschitz continuous functions. If $\mu = \mu^+ - \mu^-$ and

$$\int_{\Omega^+} \lambda^+ h\,dx - \int_{\Omega^-} \lambda^- h\,dx \geq \int h\,d\mu, \qquad \forall h \in SL^+(\Omega^+) \cap SL^-(\Omega^-),$$

then we say that $\Omega := \Omega^+ \cup \Omega^-$ is a *two phase quadrature domain* w.r.t μ for the class

$$S(\Omega) := SL^+(\Omega^+) \cap SL^-(\Omega^-),$$

and we write $\Omega \in Q(\mu, S)$.

Similar to the one phase case we can provide a PDE formulation for the two phase case. Consider the following free boundary problem in the distribution sense

$$\begin{cases} \Delta u = \lambda^+ \chi_{\Omega^+} - \mu^+ - (\lambda^- \chi_{\Omega^-} - \mu^-) & \text{in } \mathbb{R}^N, \\ \Omega^\pm = \{\pm u \geq 0\}, \end{cases} \tag{2.4}$$

where $\operatorname{supp}(\mu^\pm) \subset \Omega^\pm$ and $u, \Omega^\pm$ are unknown. If $\Omega = \Omega^+ \cup \Omega^-$ then one can show that $\Omega \in Q(\mu, S)$ if and only if Ω is the unique solution of (2.4), see [3] and [9].

Remark 1. If we set $u = u_1 - u_2$, then the problem (2.4) is a special case of (2.1) where

$$u_1 = u^+ = \max(u, 0), \qquad u_2 = u^- = \max(-u, 0).$$

Consider problem (2.1) and let $\Omega = \bigcup_i \Omega_i$. We define now multiplicity of a point and discuss on the multiplicity of one and two phase points.

Definition 2.2. The multiplicity of a point $x \in \overline{\Omega}$, denoted by $m(x)$, is defined by

$$m(x) = \operatorname{card}\{i : meas(\Omega_i \cap B(x,r)) > 0 \text{ for all } r > 0\}.$$

The interface between two densities is defined as

$$\partial\Omega_i \cap \partial\Omega_j \cap \{x \in \Omega : m(x) = 2\}.$$

Our numerical scheme is based on the following properties, which are straightforward to verify.

Lemma 2.3. *Let $x_0 \in \Omega$. Then the following holds:*
1) If $m(x_0) = 0$, then there is a $r > 0$ such that for every $i = 1, \cdots, m$; $u_i \equiv 0$ in $B(x_0, r)$.
2) If $m(x_0) = 1$, then there are i and $r > 0$ such that

$$\Delta u_i = \lambda_i \chi_{\Omega_i} - \mu_i, \qquad u_j \equiv 0 \quad \text{for } j \neq i, \text{ in } B(x_0, r).$$

3) If $m(x_0) = 2$, then there are i, j and $r > 0$ such that for every $k \neq i, j$ we have $u_k \equiv 0$ and

$$\Delta(u_i - u_j) = \lambda_i \chi_{\Omega_i} - \lambda_j \chi_{\Omega_j} - \mu_i + \mu_j, \text{ in } B(x_0, r).$$

The last part of Lemma 2.3 states that for points with multiplicity two, the problem locally turns to the two phase case of quadrature domain problem. For example in Figure (1), the points x_1, x_2 and x_3 have multiplicity $2, 1, 0$, respectively. Thus we can find a small ball $B(x_1, \varepsilon)$ such that problem (2.1) turns to (2.4) in $B(x_1, \varepsilon)$.

3. Degenerate elliptic equations and Viscosity Solutions

In this section we recall the definition of a degenerate elliptic equation and a viscosity solution.

Let Ω be a bounded open subset in $\mathbb{R}^N$ and $L(x, r, p, M)$ be a continuous real valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathscr{M}^N$ where $\mathscr{M}^N$ is the space of all symmetric $N \times N$ matrices. Moreover, suppose that Du and D^2u denote the gradient and Hessian matrix of function u, respectively.

Definition 3.1. The fully non-linear second order partial differential equation

$$Lu = L(x, u, Du, D^2u) = 0, \tag{3.1}$$

is called a *degenerate elliptic* equation if for $r_1 \leq r_2$ and $M_1, M_2 \in \mathscr{M}^N$ with $M_1 \leq M_2$

$$L(x, r_1, p, M_2) \leq L(x, r_2, p, M_1),$$

where $M_1 \leq M_2$, means $M_2 - M_1$ is a nonnegative definite symmetric matrix.

For the reader's convenience we recall the viscosity solution whose importance and its merits can be seen in the convergence analysis of the numerical schemes, for instance see [15]. To have a complete review of this topic we refer to [8] where Crandall and Lions introduced the viscosity solution for the first order Hamilton-Jacobi equation. We also refer the reader to [4] which is a great reference for viscosity solutions.

Definition 3.2. A continuous function u is called a

- *viscosity sub-solution* for the equation (3.1) if for every $\psi \in C^2(\Omega)$ and local maximum point $x_0 \in \Omega$ of $u - \psi$,

$$L\left(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)\right) \leq 0,$$

- *viscosity super-solution* for the equation (3.1) if for every $\psi \in C^2(\Omega)$ and local minimum point $x_0 \in \Omega$ of $u - \psi$,
$$L\Big(x_0, u(x_0), D\psi(x_0), D^2\psi(x_0)\Big) \geq 0,$$
- *viscosity solution* of (3.1) if and only if it is both viscosity sub and super-solution.

4. Reformulation of the problem for $m = 1, 2$.

In this section we reformulate the one phase and the two phase QD problems. This reformulation enable us to introduce viscosity solutions for these problems. We show that the viscosity solutions for these degenerate elliptic equations are equivalent to weak solutions of our original QD problems.

4.1. Min-formula for the one phase case

Consider the following one phase QD problem, which was introduced in subsection 2.1. Find (u, Ω) such that

$$\begin{cases} \Delta u = \lambda \chi_\Omega - \mu & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N, \\ u = |\nabla u| = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ supp(\mu) \subset \Omega, \end{cases} \tag{4.1}$$

where $\Omega = \{u > 0\}$. From now in this paper, for a given measure μ we convolve μ with a mollifier and we work with the regularized measure. Now one can easily see that the equation

$$L(x, u, Du, D^2u) := \min(-\Delta u + \lambda - \mu, u) = 0, \quad \text{in } \mathbb{R}^N \tag{4.2}$$

is degenerate elliptic. We refer to (4.2) as *Min-formula*.

Now we consider the following problem

$$\begin{cases} \min(-\Delta u + \lambda - \mu, u) = 0 & \text{in } \mathbb{R}^N, \\ u = |\nabla u| = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ supp(\mu) \subset \Omega. \end{cases} \tag{4.3}$$

By considering maximum principle and Perron's method we can prove that (4.3) has a unique viscosity solution, see [8]. Next lemma shows the relation between (4.1) and (4.3).

Lemma 4.1. *The viscosity solution of (4.3) is a solution of (4.1) and vice versa.*

Proof. Suppose that u is a weak solution of (4.1) in $\Omega = \{x | u(x) > 0\}$. To prove that u is a viscosity solution of (4.2) it is sufficient to show that it is both viscosity sub and super-solution. We argue by contradiction and suppose that u is not a viscosity super-solution, then there exists a point $x_0 \in \Omega$ such that $u-\psi$ has local minimum at x_0 and

$$L(x,u(x_0),D\psi(x_0),D^2\psi(x_0)) = \min(-\Delta\psi(x_0)+\lambda-\mu, u(x_0)) < 0.$$

By positivity assumption of u we obtain that

$$F(\psi)(x_0) := -\Delta\psi(x_0)+\lambda-\mu < 0,$$

and by continuity of F, we can find a $r>0$ such that $F(\psi)(x)<0$ for all $x \in B(x_0,r)$. Let

$$s = \inf_{x\in\partial B_r}(u-\psi)(x) > 0,$$

and set $\tilde{\psi} = \psi + s$, then $\tilde{\psi}(x) \le u(x)$ for all $x \in \partial B_r$. Moreover, $F(\tilde{\psi}) < 0$ in B_r and comparison principle gives $\tilde{\psi} \le u$ in B_r. On the other hand we know that $u(x_0) = \psi(x_0)$ then

$$\tilde{\psi}(x_0) = \psi(x_0)+s = u(x_0)+s > u(x_0),$$

which is a contradiction and consequently u is a viscosity super-solution. Similarly, u is also a viscosity sub-solution, which proves the first part of the lemma.

For the converse part, consider the unique viscosity solution u of (4.2). If $u>0$ then $\Delta u = \lambda - \mu$ in Ω in the viscosity sense. Then by the uniqueness of the solution of (4.1) one gets u as a weak solution for (4.1). □

4.2. Min-Max formula for the two phase case

Suppose that u is the solution of (2.4) and let

$$\Omega^+ = \{x : u(x) > 0\}, \quad \Omega^- = \{x : u(x) < 0\} \quad \text{and } \Omega = \Omega^+ \cup \Omega^-.$$

Our objective is to prove that u satisfies the following non-linear problem

$$\begin{cases} \min\Big(-\Delta u+\lambda^+-\mu^+, \max(-\Delta u-\lambda^-+\mu^-, u)\Big) = 0 & \text{in } \mathbb{R}^N, \\ u = 0 & \text{in } \mathbb{R}^N\setminus\Omega. \end{cases} \tag{4.4}$$

The equation

$$Lu := \min\Big(-\Delta u+\lambda^+-\mu^+, \max(-\Delta u-\lambda^-+\mu^-, u)\Big) = 0, \tag{4.5}$$

is called *Min-Max* formula and introduced in [2] for the two phase obstacle problem.

Remark 2. One can rewrite the equation of the two phase quadrature domain (2.4) in another form of Min-Max form as,

$$\min(-\Delta u+\lambda^+-\mu^+, u) + \max(-\Delta u-\lambda^-+\mu^-, u) - u = 0.$$

First, we show that (4.5) is a degenerate elliptic equation and then we prove that its viscosity solution is a weak solution of the corresponding PDE. A similar proof of the next proposition could also be found in [1].

Proposition 4.2. *Equation (4.5) is a degenerate elliptic equation and problem (4.4) has a unique viscosity solution.*

Proof. Suppose that $r_1 \leq r_2$ and $M_1, M_2 \in \mathscr{M}^N$ with $M_1 \leq M_2$, then $trace(M_1) \leq trace(M_2)$. It is clear that

$$-\lambda^- + \mu^- - trace(M_2) \leq -\lambda^- + \mu^- - trace(M_1)$$

and consequently

$$\max\big(-\lambda^- + \mu^- - trace(M_2), r_1\big) \leq \max\big(-\lambda^- + \mu^- - trace(M_1), r_2\big).$$

Similarly

$$\lambda^+ - \mu^+ - trace(M_2) \leq \lambda^+ - \mu^+ - trace(M_1)$$

and these inequalities sum up to

$$L(x, r_1, p, M_2) \leq L(x, r_2, p, M_1),$$

which shows that (4.5) is a degenerate elliptic equation.

For the second part, suppose that u and v are two different viscosity solutions of (4.5). We consider the following different cases.

First assume that $u > v \geq 0$. According to what we assumed, u is both viscosity super and sub-solution for the problem. Let $\psi_1, \psi_2 \in C^2$ such that $u - \psi_1$ has local maximum at x_1 and $u - \psi_2$ has local minimum in x_2. Therefore by definition

$$L(\psi_1)(x_1) = L(x_1, u(x_1), D\psi_1(x_1), D^2\psi_1(x_1)) \leq 0, \tag{4.6}$$

and

$$L(\psi_2)(x_2) = L(x_2, u(x_2), D\psi_2(x_2), D^2\psi_2(x_2)) \geq 0. \tag{4.7}$$

Therefore

$$-\Delta\psi_2 + \lambda^+ - \mu^+ \geq 0,$$

which is equivalent to claim that u is a viscosity super-solution of the

$$\Delta u = \lambda^+ - \mu^+. \tag{4.8}$$

By the positivity of u we obtain

$$\max(-\Delta\psi_1 - \lambda^- + \mu^-, u(x_1)) > 0,$$

and inequality (4.6) implies

$$-\Delta\psi_1 + \lambda^+ - \mu^+ \leq 0,$$

which is equivalent to say that u is also a viscosity sub-solution of

$$\Delta u = \lambda^+ - \mu^+. \tag{4.9}$$

Consequently (4.8) and (4.9) yield u is a viscosity solution of $\Delta u = \lambda^+ - \mu^+$, where $u > 0$.

On the other hand all these results are valid for $v > 0$, i.e., v is a viscosity solution of $\Delta v = \lambda^+ - \mu^+$, where $v > 0$. Finally we have

$$\begin{cases} \Delta(u - v) = 0 & \text{in } \{u > 0\}, \\ u = v = 0 & \text{on } \partial\{u > 0\}. \end{cases} \tag{4.10}$$

By applying maximum principle we get $u = v = 0$ which contradicts our assumption.

For other cases, $u < v \leq 0$ or $u > 0 \geq v$ one can apply the same techniques and prove the lemma. □

Lemma 4.3. *If $\mu^{\pm}$ are Dirac measures then any weak solution of (2.4) is a viscosity solution of (4.5) and vice versa.*

Proof. Suppose that u solves (2.4) in the weak sense. We treat the problem in two cases.

- If $x \in \Omega^+$ then $u(x) > 0$ and Lemma 2.3 verifies that $\Delta u = \lambda^+ - \mu^+$ holds in a ball $B := B(x, r_x)$ for some $r_x > 0$. This means that

$$\max(-\Delta u - \lambda^- + \mu^-, u) > 0 \quad \text{and} \quad -\Delta u + \lambda^+ - \mu^+ = 0,$$

  hold in the viscosity sense in B, for details see [11] Section 4. Consequently (4.5) is obtained by similar discussion in the proof of Lemma 4.1.
- If $x \in \Omega^-$, according to Lemma 2.3 we get $\Delta u = -\lambda^- + \mu^-$ in a ball $B := B(x, r_x)$ for some $r_x > 0$ and therefore

$$\max(-\Delta u - \lambda^- + \mu^-, u) = 0.$$

  On the other hand by the assumptions for the measures one gets

$$-\Delta u + \lambda^+ - \mu^+ = \lambda^- - \mu^- + \lambda^+ - \mu^+ \geq 0 \quad a.e. \text{ in } B,$$

  in the viscosity sense. Then again according to the proof of Lemma 4.1 it yields

$$\min\left(-\Delta u + \lambda^+ - \mu^+, \max(-\Delta u - \lambda^- + \mu^-, u)\right) = 0.$$

  It turns out that u is a viscosity super-solution of (4.5).

Similarly we can prove that u is also a viscosity sub-solution. For the other side we also consider two cases.

- Suppose that $u > 0$ solves the Min-Max formula in the viscosity sense. We argue by contradiction and assume that $-\Delta u + \lambda^+ - \mu^+ > 0$ then $\max(-\Delta u - \lambda^- + \mu^-, u) = 0$ which violates the positivity of u.
- If $u < 0$ solves the Min-Max formula in the viscosity sense and if $\max(-\Delta u - \lambda^- + \mu^-, u) = -\Delta u - \lambda^- + \mu^- > 0$, then $-\Delta u + \lambda^+ - \mu^+ = 0$. It turns out that $\mu^+ - \lambda^+ - \lambda^- + \mu^- > 0$. But it is clear that $\lambda^+ + \lambda^- - \mu^+ - \mu^- \geq 0$ a.e. in $\mathbb{R}^N$.

Hence u solves (2.4) in viscosity sense which is also a weak solution according to the uniqueness of the solution, see [3].

□

5. Numerical approximation

In this section we will discretize Min and Min-Max formulas and will build numerical algorithms based on the finite difference method. Then we shall apply the schemes for Dirac measure.

We define a structured grid $\mathcal{N}$ with mesh size h on a domain D, consisting of a set of grid points $x_i \in \mathcal{N}$, $i = 1, \cdots, N$. Each grid point x_i is endowed with a list of neighbors $N(i)$. A grid function is a real valued function which is defined on the grid, with values $u_i := u(x_i)$. Usually a finite difference scheme at each grid point can be written as an equation of the form

$$L_h^i[u] = L_h[u_i, u_i - u_j\big|_{j=N(i)}], \quad i = 1, \ldots, N.$$

In other words, we regard a scheme as an equation that holds at each $x_i \in \mathcal{N}$. For having a simple notation from now on, we drop h and write

$$L^i[u] := L_h[u_i, u_i - u_j],$$

where u_j is the shorthand for the list of neighbors $u_j\big|_{j=N(i)}$. Also by $\overline{u_i}$ we mean the average of $u_j\big|_{j=N(i)}$. Thus a solution for a scheme L, with components L^i, is a grid function which satisfies $L^i[u] = 0$ for all $i = 1, \ldots, N$.

5.1. Discretization of the Min-formula

Consider the Min formula in Ω and suppose that D is a big enough domain where $\Omega \subset D$, for existence of such a D, see [16]. Let $\mathcal{N}$ be a uniform mesh on D with the mesh size h. We discretize the the equation in (4.3) (Min formula) as follows

$$\min(-\Delta_h u + \lambda_h - \mu_h, u) = 0,$$

where μ_h is an appropriate discretization of μ and $-\Delta_h u$ is a discretization of Laplacian operator. We remind that μ is the regularized measure with compact support. For instance, in dimension two using standard finite difference with five points, we discretize the Laplacian operator at the point $(x_i, y_i) \in \mathcal{N}$ as

$$-\Delta_h u(x_i, y_j) = \frac{4u_(x_i, y_j) - \overline{u}(x_i, y_j)}{h^2},$$

where $\overline{u}(x_i, y_j) = \frac{1}{4}(u(x_{i-1}, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j-1}) + u(x_i, y_{j+1}))$. By simple calculation we get

$$u(x_i, y_j) = \max(\overline{u}(x_i, y_j) + \frac{(\mu_h - \lambda_h)(x_i, y_j)}{4} h^2, 0). \tag{5.1}$$

Algorithm I: For the one phase case the numerical algorithm to approximate the corresponding quadrature domain is as follows:

(1) Choose a domain D with $\Omega \subset D$, initial guess u^0 and a tolerance $TOL <<$ 1.
(2) Find a discretization μ_h for μ.

(3) For $k \geq 1$ update the values at each grid points by (5.1). More precisely

$$u^{k+1}(x_i,y_j) = \max(\overline{u}^k(x_i,y_j) + \frac{(\mu_h - \lambda_h)(x_i,y_j)}{4}h^2, 0).$$

(4) If $\sup_{(x_i,y_j)\in\mathcal{N}} |u^{k+1}(x_i,y_j) - u^k(x_i,y_j)| < TOL$ then stop otherwise set $k = k+1$ and iterate the previous step.

In the next subsection we mainly work with Dirac measure.

5.1.1. Discretization of Min-formula for Dirac measure

As mentioned before we mainly work with Dirac measure. There are several papers dealing with differential equations with jump as Dirac function, e.g., see [18].

In [14] Mayo has considered the discrete version of delta function. Following his work, consider the delta function δ_a and let x_i and h be the grid points and the mesh size. We consider two types of the grid points. If $x_{i-1} \leq a \leq x_i$ then the points x_i, x_{i-1} are called *irregular points*, otherwise they are *regular points.*

In one dimension, by applying Taylor expansion, one can easily obtain the discretization form of (4.2) as follows

$$u''(x_i) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - \widetilde{\delta}_i + O_I(h) + O(h^2),$$

where O_I denotes the error which occurs at the irregular points. Here $\widetilde{\delta}_i$ is given by $\widetilde{\delta}_i = \widetilde{\delta_i^+} + \widetilde{\delta_i^-}$ where

$$\widetilde{\delta_i^+} = \begin{cases} \frac{(x_{i+1}-a)}{h^2} & \text{if } x_i \leq a < x_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\widetilde{\delta_i^-} = \begin{cases} \frac{(a-x_{i-1})}{h^2} & \text{if } x_{i-1} < a < x_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that if the source point a is not a grid point, i.e., $x_i < a < x_{i+1}$, then the function $\widetilde{\delta}_i$ is nonzero only at the two grid points x_i, x_{i+1}. If the the source point a is a grid point, then $\widetilde{\delta}_i$ is nonzero only at $x_i = a$ and we have $\widetilde{\delta}_i = \frac{1}{h}$. Moreover, if $a = \frac{x_{i+1}+x_i}{2}$ then $\widetilde{\delta}_i = \frac{1}{2h}$.

Hence the discrete form of Min-formula is

$$\min(\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} - \lambda_h + \widetilde{\delta}_i, u_i) = 0,$$

and by simple calculations we get

$$u_i = \max(\overline{u_i} + \frac{1}{2}(\lambda_h - \widetilde{\delta}_i)h^2, 0). \tag{5.2}$$

In this case the third step of **Algorithm I** for $\mu = \delta$ is as follows:

- For $k \geq 1$ update the values at each grid points by (5.2). More precisely

$$u_i^{k+1} = \max(\overline{u_i^k} + \frac{1}{2}(\lambda_h h^2 - \widetilde{\delta}_i h^2), 0).$$

Remark 3. One can extend the previous results to find a similar discretization of Dirac measure in two dimensions. For more details see [18].

5.2. Finite difference discretization for the two phase case

We will present two methods to simulate the solution of the problem (2.4).

5.2.1. First method

Consider the two phases free boundary problem (2.4) and set $u_1 = \max\{u, 0\}$, $u_2 = \max\{-u, 0\}$. Clearly u_1 and u_2 are the solutions of the following one phase free boundary problems

$$\begin{cases} \Delta u_1 = \lambda^+ \chi_{\Omega_1} - \mu^+, & \text{in } \Omega_1 = \{u_1 > 0\}, \\ u_1 = 0 & \text{on } \partial\Omega_1, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \Delta u_2 = \lambda^- \chi_{\Omega_2} - \mu^-, & \text{in } \Omega_2 = \{u_2 > 0\}, \\ u_2 = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{5.4}$$

respectively, see [7]. Note that u_1 and u_2 have disjoint supports, i.e, $u_1 \cdot u_2 = 0$. Indeed, the gradient of $u = u_1 - u_2$ vanishes on $\partial\Omega \setminus (\partial\Omega_1 \cap \partial\Omega_2)$ where $\Omega = \Omega_1 \cup \Omega_2$. The solutions u_1 and u_2 are coupled with the condition

$$|\nabla u_1| = |\nabla u_2| \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2$$

Now consider a domain D such that $\Omega \subset D$ and let $\mathscr{N}$ be a uniform mesh on D with the mesh size h and $(x_i, y_i) \in \mathscr{N}$. We use the five stencil points finite difference for Laplace operator to get

$$\begin{aligned} \frac{4}{h^2}\left(\overline{u_1}(x_i,y_j) - u_1(x_i,y_j) - \overline{u_2}(x_i,y_j) - u_2(x_i,y_j)\right) = \\ = (\lambda_h^+ \chi_{\Omega_1} - \mu_h^+) - (\lambda_h^- \chi_{\Omega_2} - \mu_h^-), \end{aligned} \tag{5.5}$$

We can obtain $u_1(x_i, y_j)$ and $u_2(x_i, y_j)$ from (5.5) and impose the following conditions

$$u_1(x_i,y_j) \cdot u_2(x_i,y_j) = 0 \text{ and } u_1(x_i,y_j) \geq 0,\ u_2(x_i,y_j) \geq 0.$$

The iteration method for the grid points in $\text{supp}(\mu^{\pm})$ is set up as follows,

$$u_1^{k+1}(x_i,y_j) = \max\left(\overline{u_1^k}(x_i,y_j) - \overline{u_2^k}(x_i,y_j) + \frac{(\mu_h^+ - \lambda_h^+)h^2}{4}, 0\right), \tag{5.6}$$

and

$$u_2^{k+1}(x_i,y_j) = \max\left(\overline{u_2^k}(x_i,y_j) - \overline{u_1^k}(x_i,y_j) + \frac{(\mu_h^- - \lambda_h^-)h^2}{4}, 0\right), \tag{5.7}$$

where $\mu_h^{\pm}$ are the discretizations of $\mu^{\pm}$. For other points we have the same iteration formula without $\mu_h^{\pm}$.

Now assume that the measures $\mu^{\pm}$ are Dirac measures located at the source points $X_1 = (x_{i_1}, y_{j_1}) \in \mathscr{N}$ and $X_2 = (x_{i_2}, y_{j_2}) \in \mathscr{N}$. We can discretize the Dirac measures $\mu^+ = c_1\delta_{X_1}$ and $\mu^- = c_2\delta_{X_2}$, by

$$\mu_h^+ = \frac{3c_1}{\pi h^2}, \qquad \mu_h^- = \frac{3c_2}{\pi h^2}. \tag{5.8}$$

Therefore we update the values at each grid point due to (5.6) and (5.7) as follows:

- For source points

$$u_1^{k+1}(x_{i_1}, y_{j_1}) = \max\left(\overline{u_1^k}(x_{i_1}, y_{j_1}) - \overline{u_2^k}(x_{i_1}, y_{j_1}) + \frac{3c_1/(\pi h^2) - \lambda_h^+}{4}h^2, 0\right),$$

$$u_2^{k+1}(x_{i_2}, y_{j_2}) = \max\left(\overline{u_2^k}(x_{i_2}, y_{j_2}) - \overline{u_1^k}(x_{i_2}, y_{j_2}) + \frac{3c_2/(\pi h^2) - \lambda_h^-}{4}h^2, 0\right),$$

- otherwise

$$u_1^{k+1}(x_i, y_j) = \max\left(\overline{u_1^k}(x_i, y_j) - \overline{u_2^k}(x_i, y_j) - \frac{\lambda_h^+ h^2}{4}, 0\right),$$

$$u_2^{k+1}(x_i, y_j) = \max\left(\overline{u_2^k}(x_i, y_j) - \overline{u_1^k}(x_i, y_j) - \frac{\lambda_h^- h^2}{4}, 0\right).$$

Now we are ready to construct the first algorithm for the two phase quadrature domain based on the PDE formulation.

This algorithm is constructed as follows using the discretization formulas (5.6) and (5.7).

(1) Choose a tolerance $TOL << 1$ and a big domain D and consider a finite mesh on it.
(2) Find an appropriate discretization for the measures $\mu^{\pm}$.
(3) By using (5.6) and (5.7) find u_1 and u_2.
(4) For $u = u_1 - u_2$, if $\sup\limits_{(x_i, y_j) \in \mathscr{N}} |u^{k+1}(x_i, y_j) - u^k(x_i, y_j)| < TOL$, then set $k = k+1$ and go to previous step.

5.2.2. Algorithm for the multi phase case

Now consider the problem (2.1) for $m \geq 2$. Let us now define, for all i

$$\widehat{u_i} := u_i - \sum_{j \neq i} u_j.$$

From here we get

$$\Delta(u_i - \sum_{j \neq i} u_j) = \lambda_i \chi_{\Omega_i} - \mu_i - \sum_{j \neq i} (\lambda_j \chi_{\Omega_j} - \mu_j).$$

The problems (2.1) is equivalent to the following system

$$
\begin{cases}
\Delta u_i = \lambda_i \chi_{\Omega_i} - \mu_i & \text{in } \mathbb{R}^N, \\
\Delta(u_i - \sum_{j\neq i} u_j) = \lambda_i \chi_{\Omega_i} - \mu_i - \sum_{j\neq i}(\lambda_j \chi_{\Omega_j} - \mu_j) & \text{in } \mathbb{R}^N, \\
u_i \geq 0 & \text{in } \mathbb{R}^N \\
supp(\mu_i) \subset \Omega_i.
\end{cases}
\tag{5.9}
$$

Algorithm II: For an arbitrary m, the third step of the above algorithm is generalized for multi phase case. For $l = 1,\cdots,m$ and any $\mathbf{x} \in \mathcal{N}$ we iterate

$$
u_l^{(k+1)}(\mathbf{x}) = \max\left(\overline{u}_l^{(k)}(\mathbf{x}) - \sum_{p\neq l} \overline{u}_p^{(k)}(\mathbf{x}) + \frac{(\mu_l - \lambda_l)(\mathbf{x})h^2}{4}, 0\right),
\tag{5.10}
$$

when $\mathbf{x} \in \operatorname{supp}(\mu_l)$. Otherwise

$$
u_l^{(k+1)}(\mathbf{x}) = \max\left(\overline{u}_l^{(k)}(\mathbf{x}) - \sum_{p\neq l} \overline{u}_p^{(k)}(\mathbf{x}) - \frac{\lambda_l(\mathbf{x})h^2}{4}, 0\right).
\tag{5.11}
$$

Remark 4. Note that this iterative method is slow since information propagates from the support of the measures.

Lemma 5.1. *Assume that μ_i for $i = 1,\ldots,m$ are Dirac measures. The iterative method* (5.10) *and* (5.11) *for any $\mathbf{x} \in \mathcal{N}$ satisfy*

$$
u_l^{(k)}(\mathbf{x}) \cdot u_q^{(k)}(\mathbf{x}) = 0,
$$

for all $k \in \mathbb{N}$ and $q, l \in \{1,2,\ldots,m\}$, where $q \neq l$.

Proof. We know that the measures μ_i have disjoint supports so for the points in $\operatorname{supp}(\mu_i)$ the proof is obvious. Assume that the point $\mathbf{x}$ does not belong to $\operatorname{supp}(\mu_i)$ for all $1 \leq i \leq m$. Observe that from (5.11) it follows that

$$
u_l^{(k+1)}(\mathbf{x}) \geq 0,
$$

for all $k \in \mathbb{N}$ and $l \in \{1,2,\ldots,m\}$. If $u_l^{(k+1)}(\mathbf{x}) > 0$, then by (5.11) we have

$$
u_l^{(k+1)}(\mathbf{x}) = \overline{u}_l^{(k)}(\mathbf{x}) - \frac{\lambda_l(\mathbf{x})h^2}{4} - \sum_{p\neq l} \overline{u}_p^{(k)}(\mathbf{x}).
$$

This shows that for every $q \neq l$ we obtain

$$
\overline{u}_l^{(k)}(\mathbf{x}) > \sum_{p\neq l} \overline{u}_p^{(k)}(\mathbf{x}) + \frac{\lambda_l(\mathbf{x})h^2}{4} \geq \overline{u}_q^{(k)}(\mathbf{x}).
$$

Thus

$$
\overline{u}_q^{(k)}(\mathbf{x}) < \overline{u}_l^{(k)}(\mathbf{x}) \leq \frac{\lambda_q(\mathbf{x})h^2}{4} + \sum_{p\neq q} \overline{u}_p^{(k)}(\mathbf{x}),
$$

and after rearranging the above inequalities we arrive at

$$
\overline{u}_q^{(k)}(\mathbf{x}) - \frac{\lambda_q(\mathbf{x})h^2}{4} - \sum_{p\neq q} \overline{u}_p^{(k)}(\mathbf{x}) < 0.
\tag{5.12}
$$

In light of (5.12) and (5.10) we derive

$$u_q^{(k+1)}(\mathbf{x}) = \max\left(\overline{u}_q^{(k)}(\mathbf{x}) - \frac{\lambda_q(\mathbf{x})h^2}{4} - \sum_{p\neq q}\overline{u}_q^{(k)}(\mathbf{x}),\, 0\right) = 0,$$

and finally

$$u_l^{(k+1)}(\mathbf{x})\cdot u_q^{(k+1)}(\mathbf{x}) = 0.$$

□

5.2.3. Second method: Discretization of the Min-Max formula

In this part, we construct and implement another numerical scheme for the two phase quadrature domain and prove its convergence. The main tools of this method are monotonicity, consistency and stability to provide the convergence.

The fundamental result for the convergence of numerical schemes for fully non-linear, degenerate equations is Barles-Souganidis Theorem, see [5].

Theorem 5.2. *(Barles-Souganidis, 1991) Consider a degenerate elliptic partial differential equation for which there exists a unique viscosity solution. A consistent, stable approximation scheme converges uniformly on compact subsets to the unique viscosity solution provided it is monotone.*

5.2.4. Monotonicity, Stability and Consistency

In order to construct convergent numerical schemes, we define a class of nonlinear finite difference schemes which are called degenerate elliptic.

Definition 5.3. A scheme L is called *degenerate elliptic* if each component $L^i[u] = L[u_i, u_i - u_j]$ is non-decreasing in each variable.

In other words, all the scheme's components are non-decreasing functions of u_i and $u_i - u_j$.

It is easy to find the following discretization for Min-Max problem (4.5) where $\mu_h^{\pm}$ are appropriate discretizations of $\mu^{\pm}$,

$$\begin{aligned} L^i[u] &= L[u_i, u_i - u_j] \\ &= \min\Bigg(\sum_{j=N(i)}(u_i - u_j) + (\lambda_h^+ - \mu_h^+)h^2, \\ &\qquad \max\Big(\sum_{j=N(i)}(u_i - u_j) - (\lambda_h^- - \mu_h^-)h^2, u_i\Big)\Bigg) = 0. \end{aligned} \tag{5.13}$$

It is easy to see that all components (5.13) are non-decreasing functions of u_i and $u_i - u_j$ and thereby the scheme L is degenerate elliptic.

Lemma 5.4. *The scheme (5.13) is degenerate elliptic.*

Remark 5. If one deals with the uniform grid then it is clear that $|N(i)|$ is a constant. For instance, it arises $|N(i)| = 4$ for the five points discretization of Laplacian in dimension two.

Definition 5.5. For a nonlinear equation $Lu = 0$, the scheme $L^i[u] = L(u_i, u_{j=N(i)})$ is *monotone* if it is non-decreasing in the first variable and non-increasing in the remaining variables, see [13].

For example it is clear that the scheme (5.13) is monotone.

Definition 5.6. By *stability*, we mean that for every $h > 0$, the scheme L has a solution u_h which is uniformly bounded independently of h.

The proof of the next theorem could be found in [15].

Theorem 5.7. *A scheme is monotone and stable if and only if it is degenerate elliptic.*

The next corollary is obtained by Theorem 5.7 and Lemma 5.4.

Corollary 5.8. *The scheme (5.13) is monotone and stable.*

Remark 6. One can easily build monotone numerical schemes which are not stable, see [15]. It means that elliptic degeneracy is stronger than monotonicity.

Definition 5.9. The scheme L^i is consistent at x_i if for all $\psi \in C^2$ which is defined in a neighborhood of x_i, we have

$$L_h^i[\psi] \to L[\psi](x_i) \text{ when } h \to 0.$$

Lemma 5.10. *The approximation scheme (5.13) is consistent.*

Proof. Suppose that x_i is a grid point and $\psi \in C^2(B)$, where B is a ball centered at x_i. It is clear that

$$\sum_{j=N(i)} \frac{1}{h^2}(\psi_i - \psi_j) - \lambda_h^- + \mu_h^- \to -\lambda^- + \mu^- - \Delta\psi(x_i),$$

when $h \to 0$ and consequently

$$\max\Big(\sum_{j=N(i)} \frac{1}{h^2}(\psi_i - \psi_j) - \lambda_h^- + \mu_h^-, 0\Big) \to \max(-\lambda^- + \mu^- - \Delta\psi(x_i), 0).$$

Similarly

$$\sum_{j=N(i)} \frac{1}{h^2}(\psi_i - \psi_j) + \lambda_h^+ - \mu_h^+ \to \lambda^+ - \mu^+ - \Delta\psi(x_i),$$

and one can obtain

$$L_h^i[\psi](x) \to L[\psi](x), \text{ when } h \to 0.$$

□

The proof of the next corollary is obtained by previous lemma and Barles-Souganidis Theorem.

Corollary 5.11. *The scheme (5.13) converges to the unique viscosity solution of (4.5).*

Remark 7. By simple calculation one can derive the following reformulation of (5.13) in two dimensions,

$$L^i[u] = \min\left(u_i - \overline{u_i} + \frac{\lambda_h^+ - \mu_h^+}{4}h^2, \max(u_i - \overline{u_i} - \frac{\lambda_h^- - \mu_h^-}{4}h^2, u_i)\right)$$
$$= \min\left(u_i - \overline{u_i} + \frac{\lambda^+ - \mu_h^+}{4}h^2, u_i - \min(\overline{u_i} + \frac{\lambda_h^- - \mu_h^-}{4}h^2, 0)\right) = 0,$$

which turns to

$$L^i[u] = u_i - \max\left(\overline{u_i} + \tfrac{\mu_h^+ - \lambda_h^+}{4}h^2, \min(\overline{u_i} + \tfrac{\lambda_h^- - \mu_h^-}{4}h^2, 0)\right) = 0. \tag{5.14}$$

5.2.5. Second algorithm for the two phase QDs

Algorithm III: The second numerical algorithm based on Min-Max formulation is given by,

(1) Choose a tolerance, TOL.
(2) Choose a big domain D and consider a finite mesh on it.
(3) Find an appropriate discretization $\mu_h^\pm$ for the measures $\mu^\pm$.
(4) Apply the finite difference scheme (5.14) and let

$$u_i^{k+1} = \max\left(\overline{u_i^k} + \frac{\mu_h^+ - \lambda_h^+}{4}h^2, \min(\overline{u_i^k} + \frac{\lambda_h^- - \mu_h^-}{4}h^2, 0)\right).$$

(5) If $|u_i^{k+1} - u_i^k| \leq$ TOL then stop.

Remark 8. If we consider two Dirac measures μ_1, μ_2 with two positive densities c_1, c_2 at two points x_1, x_2 then the third step of the algorithm III could be as follows:

$$u_i^{k+1} = u^{k+1}(x_i) = \begin{cases} \max\left(\overline{u_i^k} + \frac{3c_1/\pi h^2 - \lambda_h^+}{4}h^2, \min(\overline{u_i^k} + \frac{\lambda_h^- - 3c_2/\pi h^2}{4}h^2, 0)\right), \\ \qquad\qquad \text{if } x_i = x_1 \text{ or } x_i = x_2, \\ \max\left(\overline{u_i^k} - \frac{\lambda_h^+}{4}h^2, \min(\overline{u_i^k} + \frac{\lambda_h^-}{4}h^2, 0)\right), \\ \qquad\qquad \text{if } x_i \neq x_1 \text{ and } x_i \neq x_2. \end{cases}$$

6. Numerical Simulations

In this section, we examine the numerical algorithms in the case of Dirac measure.

Example 6.1. Suppose that $\mu = \delta_0$ is the Dirac delta function concentrated at the origin and set $\lambda = 1$. It is well known that the corresponding quadrature domain is the ball centered at the origin with radius $r = \frac{1}{\sqrt{\pi}}$, see [6].

To find a numerical approximation, we consider a uniform mesh on

$$D = [-0.7, 0.7] \times [-0.7, 0.7],$$

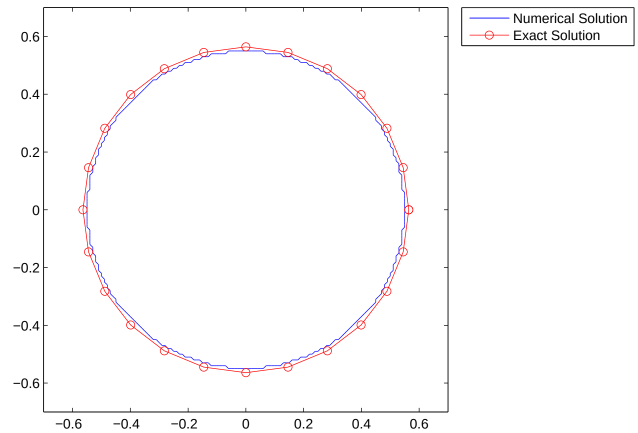


FIGURE 2. The exact and the numerical solution for the one phase quadrature domain by considering Dirac measure.

with mesh size $h = 0.01$. According to the discretization of Dirac measure, Subsection 5.1.1 we set $\mu_h(0) = \frac{3}{\pi h^2}$. We have used **Algorithm I** and Figure 2 shows the result.

Example 6.2. In this example we solve a two phase problem by applying both **Algorithm II** and **Algorithm III**. Consider a mesh $\mathscr{N}$, with size h on

$$D = [-2.5, 2.5] \times [-2.5, 2.5],$$

and assume that $X_1, X_2 \in \mathscr{N}$ are two distinct grid points. Let $\mu^+ = c_1\delta_{x_1}$ and $\mu^- = c_2\delta_{x_2}$, where c_1, c_2 are two positive constants. Figures 3 (a) and 3 (b) are the numerical solution corresponding to **Algorithm II** and **Algorithm III**. It is verified that the second method based on Min-Max is faster than the first one. Figure 4 depicts the surface of the numerical approximation.

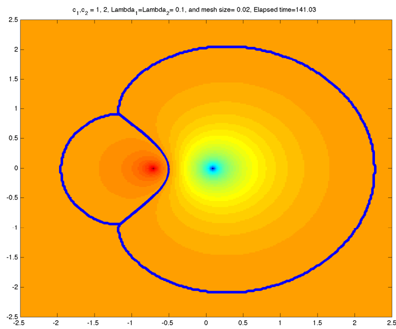
Example 6.3. We can make a three phase non-symmetric quadrature domain by chosing $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{3}, \lambda_3 = \frac{1}{5}, c_i = 1$ in **Algorithm II**. The mesh size is 0.02 and the numerical simulation is illustrated in Figure 5.

Example 6.4. Consider four source points with the different intensities

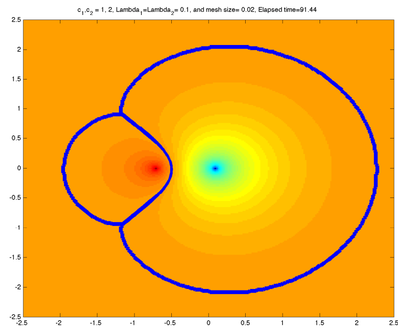
$$c_1 = 1, c_2 = 2, c_3 = 4, c_4 = 6, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1/3.$$

The source points are located on a circle. We use **Algorithm II** and find the numerical solution which is illustrated in Figure 6.

Example 6.5. In this simulation we have considered five source points which one of them, μ_5, is located at the origin and the others are on a ball centered at the



(A) Numerical solution by applying **Algorithm II**.



(B) Numerical solution by applying **Algorithm III**.

FIGURE 3. The numerical approximation of the two phase quadrature domain w.r.t two Dirac measures where we have applied **Algorithm II** and **Algorithm III.** Here $\lambda_1 = 0.1$, $\lambda_2 = 0.1, c_1 = 1$ and $c_2 = 2$. The elapsed time, e, by **Algorithm II** is $e = 141.3s$ and $e = 91.44s$ for **Algorithm III**. It is verified that the second method for the two phase case is faster than the first one.

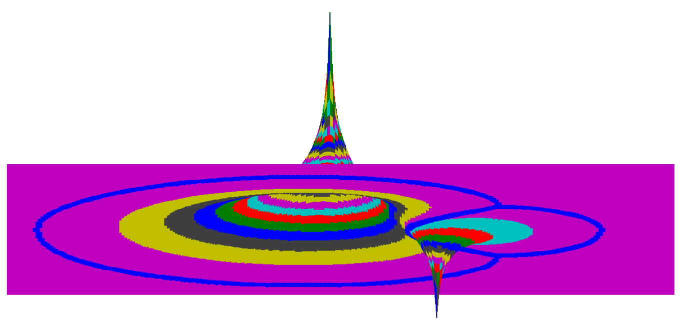


FIGURE 4. The surface of the two phase quadrature domain with two Dirac measures in Example 6.2.

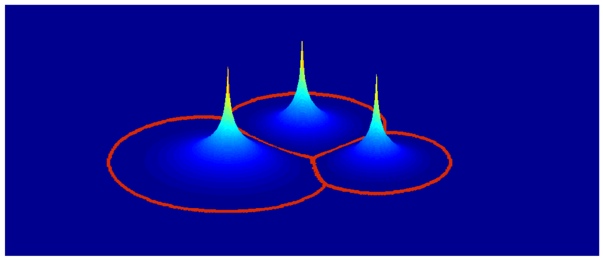


FIGURE 5. A three phase quadrature domain where the Dirac mass are located on a circle. The solution is derived by using **Algorithm II.**

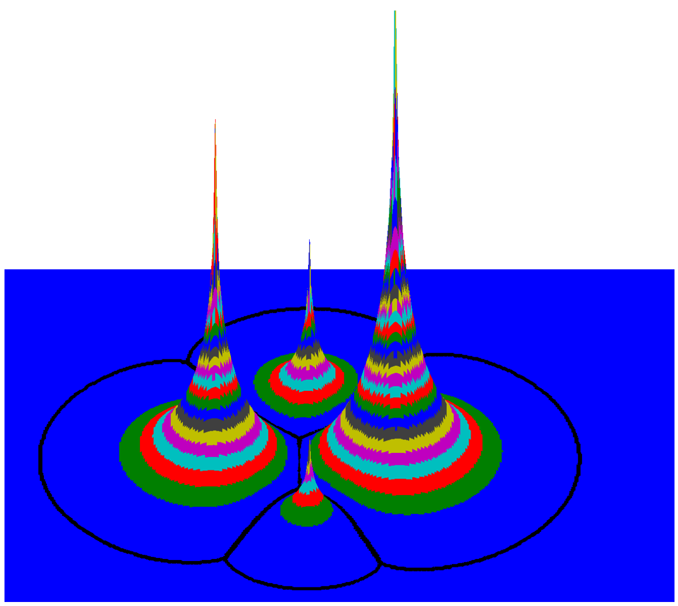


FIGURE 6. A four phase quadrature domain, where the Dirac masses are located on a circle. The solution is derived by using **Algorithm II.**

origin. We use **Algorithm II** and Figure 7 shows the approximation. In this example $\lambda_i = 1/3$ for $1 \le i \le 5$ and $c_1 = c_2 = c_3 = c_4 = 6$ and $c_5 = 5$.

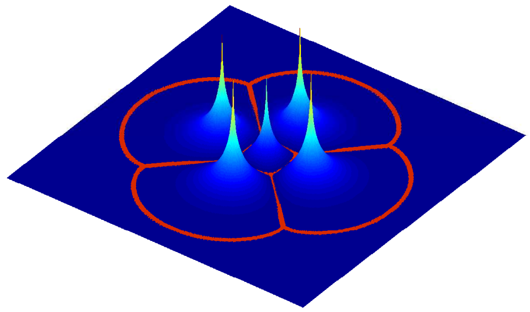


FIGURE 7. A five phase quadrature domain where x_1 is the origin and x_i for $2 \leq i \leq 5$ are on a circle with the same intensity. Here $c_1 = c_2 = c_3 = c_4 = 6$ and $c_5 = 5$.

Example 6.6. This example depicts a five phase quadrature domain for which the source points are located on a circle symmetrically. We use **Algorithm II** and $\lambda_i = .01, c_i = 2$. Figure 8 shows the numerical approximation.

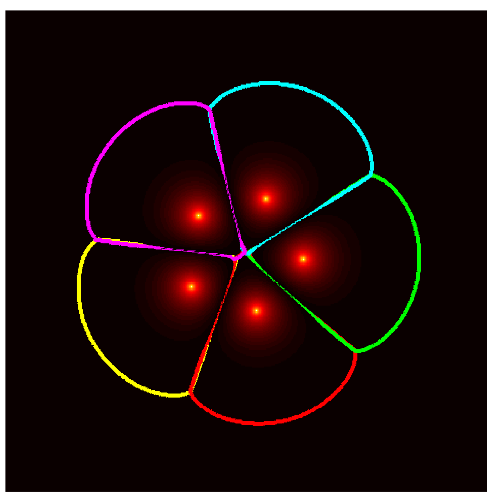


FIGURE 8. The contour of five phase quadrature domain where all source points are located on a circle with the same intensity.

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