CHAPTER 1: DELZANT'S CONJECTURE

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1. Symplectic quotients

Theorem 1. If the action of a Lie group G on a manifold M is free and proper[1], then the orbit space M/G has a unique manifold structure such that $\pi : M \twoheadrightarrow M/G$ is a submersion.

Remark. If the action is not free (but proper), then M/G is still a smooth stratified space [DK, Theorem 2.7.4].

Example 1. Let G be a compact Lie group and consider its (proper, but not free) coadjoint action on $\mathfrak{g}^*$. Let $T \subset G$ be a maximal torus. We can view $\mathfrak{t}^*$ as a subset of $\mathfrak{g}^*$:

$$\mathfrak{t}^* = \{\xi \in \mathfrak{g}^* : \mathrm{Ad}_g^* \xi = \xi, \forall g \in T\}.$$

Let $\overline{C} \subset \mathfrak{t}^*$ be a closed Weyl chamber. The composition

$$\overline{C} \hookrightarrow \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}^*/G.$$

is a homeomorphism. The strata of $\overline{C}$ are its faces.

When (M, ω) is a symplectic manifold and G acts by symplectomorphisms, the quotient is in general no longer a symplectic manifold. Since $C^\infty(M/G) = C^\infty(M)^G$, we do retain the Poisson structure.

Proposition 2. If the symplectic action of a Lie group G on (M, ω) is proper and free, then M/G has a Poisson bracket such that $\pi : M \twoheadrightarrow M/G$ is a Poisson map.

To obtain a quotient within the symplectic category, we define Hamiltonian G-spaces.

Definition 3. Let (M, ω) be a symplectic manifold on which G acts by symplectomorphisms. Let $\mu : M \to \mathfrak{g}^*$ be a momentum map, that is a map such that for every $X \in \mathfrak{g}$

(1) $\iota_{X_M}\omega = \mathrm{d}\langle \mu, X\rangle$;
(2) μ is equivariant,

where X_M is the is the vector field on M generated by X.[2] We then call the triple (M, ω, μ) a *Hamiltonian G-space.*

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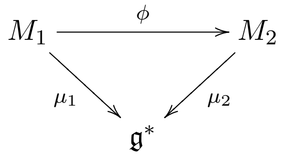
[1]An action $G \times M \to M$ is called *proper* if the map $M \times G \to M, (p, g) \mapsto (p, gp)$ is proper. This is true of any G-action when G is compact.

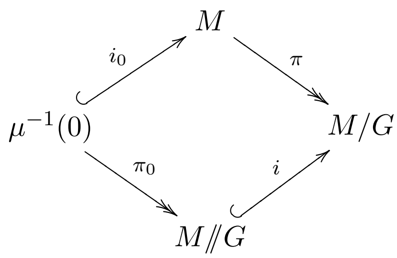
[2]For $X \in \mathfrak{g}$, the vector field X_M on M is defined as the unique vector field with flow $p \mapsto \exp(tX).p$.

A *morphism* $\phi : (M_1, \omega_1, \mu_1) \to (M_2, \omega_2, \mu_2)$ between Hamiltonian G-spaces is a G-equivariant Poisson map which makes the following diagram commutative:



Exercise. Define Hamiltonian $\mathfrak{g}$-spaces and check that they are just Poisson maps $\mu : (M, \omega) \to \mathfrak{g}^*$.

Theorem 4 (Meyer, Marsden–Weinstein)**.** If the symplectic action of G on (M, ω) is proper and free, then $M/\!\!/G := \mu^{-1}(0)/G$ has a symplectic form ω_{red} which is determined by $i_0^*\omega = \pi_0^*\omega_{\text{red}}$.



Remark. π and i are Poisson maps.

Example 2. Consider the proper and free S^1-action

$$S^1 \times \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\}, \theta.(z_0, \dots, z_n) = (e^{i\theta} z_0, \dots, e^{i\theta} z_n).$$

It is a symplectic action for the standard symplectic form:

$$\omega = \frac{i}{2} \sum_{k=0}^{n} dz_k \wedge d\bar{z}_k = \sum_{k=0}^{n} dx_k \wedge dy_k.$$

Moreover, it is Hamiltonian with moment map:

$$\mu : \mathbb{C}^{n+1} - \{0\} \to \mathfrak{t}^* \simeq \mathbb{R}, \mu(z_0, \dots, z_n) = \frac{1}{2} - \frac{1}{2} \sum_k |z_k|^2.$$

The symplectic quotient $\mu^{-1}(0)/S^1$ is $\mathbb{P}^n$ and the form ω_{red} is also known as the Fubini-Study symplectic from ω_{FS}.

Example 3 (Cotangent bundle of a Lie group)**.** [AM, Example 4.3.4.v] Let G act on itself by left multiplication. Like the cotangent bundle of any G-manifold, the cotangent bundle T^*G of G itself is a Hamiltonian G-manifold.

Given $\lambda \in \mathfrak{g}^*$, let λ^R be the right-invariant 1-form on G such that $\lambda^R(e) = \lambda$. Explicitly, $\lambda^R(g) := T_g^* R_{g^{-1}}(\lambda)$, where R_g is right multiplication by $g \in G$. In the right trivialisation of T^*G,

$$G \times \mathfrak{g}^* \xrightarrow{\sim} T^*G, (g, \lambda) \mapsto \lambda^R(g),$$

the moment map is given by projection onto $\mathfrak{g}^*$.

2. Convexity Theorem and Delzant's Conjecture

Theorem 5 (Atiyah, Guillemin–Sternberg, Kirwan)**.** Let G be a compact Lie group and (M,ω,μ) a Hamiltonian G-space. Let $\overline{C} \subset \mathfrak{t}^* \hookrightarrow \mathfrak{g}^*$ be a closed Weyl chamber. Then the set $\Delta(M) := \mu(M) \cap \overline{C}$ is a convex polytope.

We will sometimes use Δ for $\Delta(M)$ when there is only one manifold under consideration.

Example 4. Consider the proper, but non-free torus action

$$\mathbb{T}^n \times \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}^{n+1} - \{0\}, (\theta_1,\ldots,\theta_n).(z_0,\ldots,z_n) = (z_0, e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n).$$

It is Hamiltonian with moment map

$$\mu : \mathbb{C}^{n+1} - \{0\} \to (\mathbb{R}^n)^* \simeq \mathbb{R}^n, (z_0,\ldots,z_n) \mapsto \left(\frac{|z_1|^2}{\sum_k |z_k|^2},\ldots,\frac{|z_n|^2}{\sum_k |z_k|^2}\right).$$

Since this action commutes with the S^1-action of example 2 and the moment map of each action is invariant under the other action, we obtain a Hamiltonian $\mathbb{T}^n$-action on $(\mathbb{P}^n, \omega_{\text{FS}})$ with moment map $\overline{\mu}$ induced by μ. Since $\mathbb{T}^n$ is abelian, we have that $\overline{C} = \mathfrak{t}^* \simeq \mathbb{R}^n$ and one verifies that the moment image is the standard convex n-polytope:

$$\overline{\mu}(M) = \Big\{(\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n : \lambda_k \geq 0\, \forall k, \sum_k \lambda_k \leq 1\Big\}.$$

From now on, we **assume that both M and G are compact and connected.** In order to state Delzant's conjecture, we require two more concepts.

Definition 6 (Completely integrable systems)**.** A Hamiltonian G-space (M,ω,μ) is called *completely integrable* if

(1) the action of G is locally free[3] in at least one point of M;
(2) $\dim G + \operatorname{rank} G = \dim M$.

Delzant [D2] showed that for a completely integrable Hamiltonian G-space the inverse image under μ of a coadjoint orbit is a G-orbit in M. In other words, μ induces a homeomorphism between M/G and the polytope Δ in theorem 5.

$$\begin{array}{ccccc} M & \xrightarrow{\mu} & \mathfrak{g}^* & & \\ \downarrow & & & \searrow & \\ M/G & \xrightarrow{\text{homeo}} & \Delta & \hookrightarrow & \mathfrak{g}^*/G = \overline{C} \end{array}$$

Proposition 7. Let M be a G-space. There exists an dense open subset U of M such that all its points have the same orbit type: for all x and y in U, the isotropy groups G_x and G_y are conjugate. The conjugacy class of these isotropy groups is called *the principal isotropy type* of the action.

A proof for this proposition can be found in [DK, Proposition 2.7.1]. In place of principal isotropy type, [D2] speaks of *principal isotropy group*. Others still, use *generic isotropy group*.

[3]The G-action on M is called *locally free at $x \in M$* if the isotropy group G_x is discrete.

Remark. If the action is locally free in at least one point, then the generic isotropy group is discrete; that is, the action is locally free on a dense open set.

Conjecture (Delzant)**.** *Given two completely integrable Hamiltonian G-spaces (M_1, ω_1, μ_1) and (M_2, ω_2, μ_2) such that $\Delta(M_1) = \Delta(M_2)$ and such that their principal isotropy types are the same, there is a G-equivariant symplectic diffeomorphism $\phi : M_1 \to M_2$ such that $\mu_2 \circ \phi = \mu_1$.*

References

[AM] R. Abraham and J.E. Marsden, *Foundations of Mechanics*, Benjamin/Cummings, 2nd ed (1978).

[D2] T. Delzant, Classification des actions hamiltoniennes complètement intégrables de rang deux, *Ann. Global Anal. Geom.* **8** (1990), 87–112.

[DK] J.J. Duistermaat and J.A.C. Kolk, *Lie Groups*, Springer–Verlag Berlin Heidelberg (1990).