Inverse scattering for elastic plane cracks

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Abstract

It is known that hard planar acoustic screens generate null far-field amplitude on the directions defined by the plane that encloses them. We proved in an earlier paper that this property actually characterizes plane screens among scatterers of any shape, and we generalized it to other boundary conditions. Furthermore it was proved that a single incident wave is sufficient to distinguish screens that lie in the same plane. Generalizations of these results are carried out in this paper to the elastic case.

1 Introduction

Identifying the shape of an object by means of the far field pattern of an acoustic wave, scattered by the object, is a mathematical and numerical problem with many applications. For an overview on the subject we refer to the book by Colton and Kress ([6]). However, until the works of Kress ([12]) and Alves ([2]) scatterers of *crack shapes*, i.e. with neglectible thickness, were not considered. In ([4]), we proved the uniqueness theorem considering admissible scatterers including boundaries of open sets as well as cracks, and Robin boundary conditions with unknown impedance. Furthermore, we derived a simple characterization of plane cracks, which also enabled us to determine the plane of the screen. When the plane direction is known, we also obtained an uniqueness result for plane cracks, considering only one incident plane wave.

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These results are generalized here to elastic scattering problems with Neumann boundary condition.

2 The Uniqueness theorem

Since this paper is a continuation of [4], we shall use the same notations and assumptions and omit details of proofs that were already presented in that paper.

Briefly, we call *admissible scatterer* a surface Γ union of members Γ_1 of $\mathcal{S}_1$ and Γ_2 of $\mathcal{S}_2$, where $\mathcal{S}_1$ designates the set of closed C^2 surfaces, boundaries of bounded domains D with *connected* exterior $\bar{D}^c$ and $\mathcal{S}_2$ the set of open parts of a surface in $\mathcal{S}_1$, called here *cracks*. The crack part Γ_2 is supposed to be exterior to the bounded domain D of Γ_1. The scatterer is a (pure) crack if Γ_1 is void, and a *round object* if Γ_2 is void. We shall set $\Omega = \mathbb{R}^3 \backslash (\bar{D} \cup \Gamma_2)$, the connected exterior domain where waves are propagating.

We consider direct and inverse scattering problems of these *admissible scatterers* by elastic waves in an isotropic, homogeneous medium characterized by its Lamé coefficients λ, μ and by its density ρ.

The direct problem is governed by the usual Navier's equations :

$$(\boldsymbol{PV}) \begin{cases} \mu \Delta \boldsymbol{u} + (\lambda + \mu) \nabla \text{ div } \boldsymbol{u} + \rho \omega^2 \boldsymbol{u} = 0 & \text{in } \Omega & (1) \\ \mathcal{B}\boldsymbol{u} = -\mathcal{B}\boldsymbol{u}^{inc} = \boldsymbol{g} & \text{in } \Gamma & (2) \\ |\boldsymbol{u}| = O(\frac{1}{r});\ |\boldsymbol{\sigma}(\boldsymbol{u}).\hat{x} - i\boldsymbol{W}\boldsymbol{u}| = o(\frac{1}{r}) & \text{when } r = |x| \to \infty & (3) \end{cases}$$

where

$$\boldsymbol{W}\boldsymbol{u} = (\lambda + 2\mu)\kappa_p(\boldsymbol{u}.\hat{x})\hat{x} + \mu\kappa_s(\boldsymbol{u} - (\boldsymbol{u}.\hat{x})\hat{x}).$$

As for the boundary condition, we'll consider the classical Neumann problem

$$\mathcal{B}\boldsymbol{u} = \boldsymbol{\sigma}(\boldsymbol{u}).\boldsymbol{n}_{|\Gamma} \tag{4}$$

When Γ is a closed surface, that is, an element of $\mathcal{S}_1$, the well-posedness of problem $(\boldsymbol{PV})$ can be found in many classical treaties on elastodynamics. When Γ is a crack, one can easily adapt the proof of well-posedness in [15] for the static case and in [5] for the case of a complex frequency ω. See also the proof of Kress in [13] for the 2D problem.
We shall concentrate ourselves in studying the inverse problem of identifying the scatterer Γ from the scattering amplitude or far-field pattern of $\boldsymbol{u}$. This

scattering amplitude $\boldsymbol{F} = \boldsymbol{F}^p + \boldsymbol{F}^s$ represents the asymptotic behavior of the scattered wave $\boldsymbol{u}$ as indicated by the following well known formula (cf.[1]) :

$$\boldsymbol{u}(x) = \left\{ \frac{e^{i\kappa_p|x|}}{|x|}\boldsymbol{F}^p(\hat{x}) + \frac{e^{i\kappa_s|x|}}{|x|}\boldsymbol{F}^s(\hat{x}) \right\} + \boldsymbol{o}(\frac{1}{|x|}). \tag{5}$$

More precisely, as in the acoustic case, one can reconstruct $\boldsymbol{u}$ in Ω from its scattering amplitude. One possible method for that is by the elastic Atkinson expansion (see e.g. [3]). Another important feature of this elastic scattering amplitude, proved in ([3]), is that its pressure and shear parts $\boldsymbol{F}^p(\theta)$ and $\boldsymbol{F}^s(\theta)$ can be expressed with the Fourier transform of *grad div* and *curl curl* of a same vector distribution supported by Γ. From that, it follows easily that they are respectively parallel and orthogonal to the observation direction θ. Their expressions depend on different layer potentials one can use to represent $\boldsymbol{u}$. In terms of the traces of $\boldsymbol{u}$ on Γ, one gets the following formula (see [1]):

$$\boldsymbol{F}^p(\theta) = \frac{\theta}{4\pi(\lambda+2\mu)}\left(i\kappa_p \int_\Gamma \left[\lambda(\boldsymbol{n}.\boldsymbol{u}) + 2\mu(\theta.\boldsymbol{u})(\theta.\boldsymbol{n})\right] e^{-i\kappa_p\theta.x}\, ds \right.$$
$$\left. + \int_\Gamma \sigma(\boldsymbol{u})\boldsymbol{n}.\theta\, e^{-i\kappa_p\theta.x} ds \right) \tag{6}$$

and

$$\boldsymbol{F}^s(\theta) = \frac{-\theta}{4\pi\mu} \wedge \left(\theta \wedge \int_\Gamma i\kappa_s\mu \left[(\theta.\boldsymbol{u})\boldsymbol{n} + (\theta.\boldsymbol{n})\boldsymbol{u}\right] e^{-i\kappa_s\theta.x} ds + \right.$$
$$\left. + \theta \wedge \int_\Gamma \sigma(\boldsymbol{u})\boldsymbol{n}\, e^{-i\kappa_s\theta.x}\, ds \right). \tag{7}$$

In the next section, we will have to represent $\boldsymbol{u}$ as a double layer potential with density $\boldsymbol{\varphi}$, which is its *crack opening displacement* (COD). In this case, one has :

$$\boldsymbol{F}^p(\theta) = \frac{\theta}{4\pi(\lambda+2\mu)} \int_\Gamma i\kappa_p \left[\lambda \boldsymbol{n}.\boldsymbol{\varphi} + 2\mu(\theta.\boldsymbol{\varphi})(\theta.\boldsymbol{n})\right]\ e^{-i\kappa_p\theta.x} ds \tag{8}$$

and

$$\boldsymbol{F}^s(\theta) = \frac{-\theta}{4\pi\mu} \wedge \left(\theta \wedge \int_\Gamma i\kappa_s\mu \left[(\theta.\boldsymbol{\varphi})\boldsymbol{n} + (\theta.\boldsymbol{n})\boldsymbol{\varphi}\right]\ e^{-i\kappa_s\theta.x}\, ds \right). \tag{9}$$

As for the plane incident wave, one has to distinguish the cases of P-wave and of S-wave, respectively, with the polarization direction parallel and orthogonal to the propagation direction. In particular, we will consider

$$\boldsymbol{u}^{inc,p}(x;d) = i\kappa_p d e^{i\kappa_p d.x} = grad\ e^{i\kappa_p d.x} \tag{10}$$

and

$$\boldsymbol{u}^{inc,s}(x;q,d) = i\kappa_s q \wedge d e^{i\kappa_s d.x} = curl\ q\ e^{i\kappa_s d.x} \tag{11}$$

where q is a given non null vector and $d \in S^2$. Then, we have the following uniqueness theorem as in the acoustic inverse problem.

Theorem 1 *For a fixed frequency ω, let Γ_1 and Γ_2 be two admissible surfaces such that the associated solutions of problem* $(\boldsymbol{PV})$ *have identical far-field patterns :*

$$\boldsymbol{F}_1^p = \boldsymbol{F}_2^p \textit{ and } \boldsymbol{F}_1^s = \boldsymbol{F}_2^s,$$

for all incident waves satisfying either *(10)* or *(11) then* $\Gamma_1 = \Gamma_2$.

The proof of Theorem 1 is the same as in Theorem 1 of [4]. One has only to verify that the density results proved by Kirsch and Kress in ([10]) for acoustic plane waves, and used in the proof of the cited theorem, remain valid in the two situations considered here. Concretely, one needs the approximation of $grad(G_p(x-x_0))$ in a bounded domain excluding the point x_0 by the functions (10) when d varies in S^2, and the approximation of $curl(qG_s(x-x_0))$ by the functions (11), where G_α is the outgoing Green function of the scalar Helmholtz equation with wavenumber κ_α. The result was already obtained by Hähner and Hsiao ([8]) treating the inverse problem with only *round obstacles* as admissible scatterers. Note that we use only S-incident waves with polarization orthogonal to the fixed direction q. See also, for two dimensional problems, the cited paper of Kress ([13]). $\square$

The before-mentioned density result in a domain is proved in ([8]) in a different manner than the analogous result for scalar waves by Kirsch and Kress ([10]). However, we will need the following density result on the surface Γ, like the one in ([10]) :

Lemma 1 *If Γ is a crack or a closed surface in $\mathcal{S}_1$, boundary of a domain D and if ω^2 is not an eigen-value of the elastic Neumann problem in D, then the space*

$$V = Span\Big\{\boldsymbol{\sigma}(\boldsymbol{u}^{inc}).\boldsymbol{n}_{|\Gamma}; \boldsymbol{u}^{inc} \textit{satisfying (10) or (11)}\Big\}$$

is dense in $(L^2(\Gamma))^3$.

Proof
Let $\boldsymbol{\varphi}$ be a function in $(L^2(\Gamma))^3$ such that

$$\int_\Gamma \boldsymbol{\varphi\sigma}(\boldsymbol{u}^{inc}).\boldsymbol{n} ds = 0 \tag{12}$$

for all $\boldsymbol{u}^{inc}$ satisfying (10) or (11). Consider $\boldsymbol{v}$ the elastic double layer potential with density $\boldsymbol{\varphi}$. A simple calculation using (12) with $\boldsymbol{u}^{inc}$ satisfying (10) and formula (8) shows that the P-far field pattern of $\boldsymbol{v}$ is null. The same for its S-far field pattern, using (12) with $\boldsymbol{u^{inc}}$ satisfying formula (11) and (9). Thus, $\boldsymbol{v}$ is null in the exterior domain, and by the hypothesis on Γ, its COD also. □

3 Characterizations of plane cracks

Now, let us consider the case of plane cracks. We can first observe the following properties on the far field pattern of incident waves propagating parallel to the crack.

Proposition 1 *Let $\boldsymbol{n}$ be a fixed unit vector, if Γ is a plane crack orthogonal to $\boldsymbol{n}$, then*
a) if the incident wave is a P-wave propagating parallel to the crack (that is, $\boldsymbol{u}^{inc}$ given by (10) with $d \perp \boldsymbol{n}$), then

$$F^s(\theta) = 0 \; \forall \theta \perp \boldsymbol{n} \tag{13}$$

b) if the incident wave is a S-wave propagating parallel to the crack,

$$F^p(\theta) = 0 \; \forall \theta \perp \boldsymbol{n} \tag{14}$$

Proof
We note that in scattering by cracks, the scattered wave is always representable by a double layer potential which density φ is the crack opening displacement (COD) and solution of the boundary integral equation

$$\boldsymbol{D}\varphi = g := -\boldsymbol{\sigma}(\boldsymbol{u}^{inc}).\boldsymbol{n}_{|\Gamma} \tag{15}$$

where $\boldsymbol{D}\varphi$ is the surface traction of the potential. On the other hand, if Γ is plane with normal vector $\boldsymbol{n}$, using axis such that $\boldsymbol{n} = (0,0,1)$ one knows (cf. [7]) that the operator $\boldsymbol{D}$ can be put in the form

$$\boldsymbol{D} = \boldsymbol{R}_\Gamma \circ \begin{pmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix} \circ \boldsymbol{P}_\Gamma$$

where $\boldsymbol{P}_\Gamma$ is the extension operator by zero out of Γ in its plane, $\boldsymbol{R}_\Gamma$ the restriction operator to Γ, and the matrix is an elliptic pseudo-differential operator in the plane $x_3 = 0$.

From that, and from the explicit expressions of g in (15) for the two incident waves under consideration, one finds easily that in the first case φ is parallel to $\boldsymbol{n}$ and in the second case φ is normal to $\boldsymbol{n}$. Using now (8) and (9), formulae (13) and (14) are obtained. □

However, for the converse of proposition 1, the restriction to waves propagating parallel to the crack raises a major difficulty, since the proof needs the density of the set $\left\{g := -\boldsymbol{\sigma}(\boldsymbol{u}^{inc}).\boldsymbol{n}_{|\Gamma}; d \in S^2\right\}$ in $(L^2(\Gamma))^3$ (cf. Lemma 1). The following property, which is also a simple consequence of formula (9), is then preferable.

Proposition 2 *Let $\boldsymbol{n}$ be a fixed unit vector, if Γ is a plane crack orthogonal to $\boldsymbol{n}$, then for every plane incident wave, one has*

$$\eta.\boldsymbol{F}^s(\theta) = 0 \ \ \forall \theta, \eta \perp \boldsymbol{n} \tag{16}$$

One can say that for a plane crack, the "horizontal part" of the shear far-field patttern of the scattered wave vanishes along any horizontal observation direction. Its converse can now be formulated :

Proposition 3 *Let $\boldsymbol{n}$ be a fixed unit vector. If for any P-incident wave (10)* and *for any S-incident wave (11) - with fixed q -, the property (16) holds, then Γ is a plane crack orthogonal to $\boldsymbol{n}$.*

Proof

Suppose that $\boldsymbol{u}$ can be represented by a double layer potential on Γ, with density φ (this is the case when Γ is a crack non orthogonal to $\boldsymbol{n}$, or a round object which doesn't contain ω as a spurious frequency). Otherwise, one has to modify a little the proof as in proposition 1 of ([4]). Then one has formula (9) for the S-far field pattern of $\boldsymbol{u}$. Thus

$$\boldsymbol{F}^s(\theta) = Const.\, \theta \wedge \left(\theta \wedge \int_\Gamma \left[(\theta.\boldsymbol{\varphi})\boldsymbol{n} + (\theta.\boldsymbol{n})\boldsymbol{\varphi}\right] \, e^{-i\kappa_s \theta.x} \, ds\right)$$

Using the classical identity

$$(X \wedge Y) \wedge Z = (X.Z)Y - (Y.Z)X$$

for vector products, and choosing $\eta \perp \theta$, one obtains

$$\eta.\boldsymbol{F}^s(\theta) = Const. \int_\Gamma [(\theta.\varphi)(\eta.\boldsymbol{n}_x) + (\theta.\boldsymbol{n}_x)(\eta.\boldsymbol{\varphi})] \; e^{-i\kappa_s \theta.x} \, ds$$

Now, by the invertibility of operator $\boldsymbol{D}$ (from $(H^1(\Gamma))^3$ to $L^2(\Gamma))^3$), and the density of the functions $g = \boldsymbol{\sigma}(\boldsymbol{u}^{inc}).\boldsymbol{n}_{|\Gamma}$ when d varies in S^2, one can choose φ of the form $\psi \boldsymbol{n}_x$ with the support of ψ contained in a small vicinity V of a point x_0 such that $\boldsymbol{n}_{x_0}.\boldsymbol{n} < 1$, one gets

$$\eta.\boldsymbol{F}^s(\theta) = Const. \int_V [(\theta.\boldsymbol{n}_x)(\eta.\boldsymbol{n}_x)] \; \psi e^{-i\kappa_s \theta.x} \, ds$$

which does not vanish for an appropriate choice of ψ. □

Finally, if Γ is already known to be a plane crack, it can be identified by only one incident wave as indicated in the two following propositions:

Proposition 4 *If $\boldsymbol{n}$ be a fixed unit vector and Γ_1 and Γ_2 are plane cracks orthogonal to $\boldsymbol{n}$. Then, $\Gamma_1 = \Gamma_2$ if one has $\boldsymbol{F}_1^\alpha = \boldsymbol{F}_2^\alpha$ for* either $\alpha = p$ or *$\alpha = S$, and for one P- incident plane (10) wave normal to $\boldsymbol{n}$.*

Proof

We are going to suppose that the cracks are in the plane $\{x_3 = 0\}$. It is easy to see that, for a P- incident plane wave normal to $\boldsymbol{n}$, the surface traction g is given by :

$$g = -\kappa_p^2 \lambda \, e^{i\kappa_p d'.x'} \boldsymbol{n}.$$

We define here $x = (x', x_3)$ for $x \in R^3$. Using the expression of $\boldsymbol{D}$ as recalled in the proof of Proposition 1, one sees that the COD $\boldsymbol{\varphi}_j$ are of the form $\boldsymbol{\varphi}_j = \boldsymbol{\psi}_j \boldsymbol{n}$. Formula (8) then yields

$$\boldsymbol{F}_j^p(\theta) = Const. \, \theta \int_{\{x_3=0\}} (\lambda + 2\mu(\theta_3)^2) \boldsymbol{\psi}_j(x') e^{-i\kappa_p \theta'.x'} ds,$$

for $j = 1, 2$, where $\boldsymbol{\psi}_j$ are null outside of Γ_j.
Thus, the hypothesis implies that

$$\int_{\{x_3=0\}} \boldsymbol{\psi}_1(x') e^{-i\kappa_p \theta'.x'} ds = \int_{\{x_3=0\}} \boldsymbol{\psi}_2(x') e^{-i\kappa_p \theta'.x'} ds \tag{17}$$

that is, the Fourier transforms of $\boldsymbol{\psi}_j$ are equal in the disk $|\theta'| \leq \kappa_p(1 - \theta_3)$. So $\boldsymbol{\psi}_1 = \boldsymbol{\psi}_2$. Now, $\boldsymbol{\psi}_j$ is obtained from the non null analytic function $g_3 = -\kappa_p^2 \lambda \, e^{i\kappa_p d'.x'}$ by inverting an elliptic pseudo-differential operator (the term t_{33}

in $\boldsymbol{D}$), its support is equal to Γ_j, the case $\alpha = p$ is done.
For the case $\alpha = s$, using formula (9), one obtains again equality (17). Thus, the same conclusion follows. $\Box$

For a S-incident wave, the result is a little different.

Proposition 5 *If $\boldsymbol{n}$ be a fixed unit vector and Γ_1 and Γ_2 are plane cracks orthogonal to $\boldsymbol{n}$. Then, $\Gamma_1 = \Gamma_2$ if one has $\boldsymbol{F}_1^s = \boldsymbol{F}_2^s$ for one SH- incident plane wave normal to $\boldsymbol{n}$ (*that is, $\boldsymbol{u}^{inc}$ satisfying (11) with the orthogonal vectors $q, d, \boldsymbol{n}$*).*

Proof
The surface traction g is now given by :

$$g = -\kappa_s^2 \mu\, (q \wedge d).\boldsymbol{n} d e^{i\kappa_s d'.x'} = Const.\, d\; e^{i\kappa_s d'.x'}.$$

so that, because d is orthogonal to $\boldsymbol{n}$, so are $\boldsymbol{\varphi}_j = \boldsymbol{\varphi'}_j$. If one looks at the P-far field pattern, one gets

$$\boldsymbol{F}_j^p(\theta) = Cte\, \theta \int_{\{x_3=0\}} (\theta.\boldsymbol{\varphi}_j)\theta_3 e^{-i\kappa_s \theta'.x'} ds$$

and the equality of $\boldsymbol{F}_j^p$ implies

$$\int_{\{x_3=0\}} \theta'.\boldsymbol{\varphi'}_1\, e^{-i\kappa_s\theta'.x'} ds = \int_{\{x_3=0\}} \theta'.\boldsymbol{\varphi'}_2\, e^{-i\kappa_s\theta'.x'} ds.$$

That means $div(\boldsymbol{\varphi'}_1) = div(\boldsymbol{\varphi'}_2)$ and we cannot conclude here. But, if one considers the S-far field pattern, one gets from formula (9)

$$\boldsymbol{F}_j^s(\theta) = Const.\; \theta \wedge \left(\theta \wedge \int_{\{x_3=0\}} \left[(\theta.\boldsymbol{\varphi'}_j)\boldsymbol{n} + (\theta.\boldsymbol{n})\boldsymbol{\varphi'}_j \right] \; e^{-i\kappa_s \theta.x}\; ds \right).$$

And from the equality of $\boldsymbol{F}_j^s$

$$(2\theta_3\theta - \boldsymbol{n})(\theta'.\boldsymbol{\psi'}_1) - \theta_3 \boldsymbol{\psi'}_1 = (2\theta_3\theta - \boldsymbol{n})(\theta'.\boldsymbol{\psi'}_2) - \theta_3 \boldsymbol{\psi'}_2 \;\; \forall \theta, \qquad (18)$$

where $\boldsymbol{\psi'}_j$ stands for $\int_{\{x_3=0\}} \boldsymbol{\varphi'}_j e^{-i\kappa_s\theta.x}\, ds$.
Taking $\theta = \theta'$ (i.e. $\theta_3 = 0$), one obtains again the equality of $div(\boldsymbol{\varphi'}_j)$. But then, (18) is reduced to $\boldsymbol{\psi'}_1 = \boldsymbol{\psi'}_2$, and the conclusion follows as before, with the property of support of $\boldsymbol{\varphi'}_j$. $\Box$

N.B. We would like to take the opportunity of this paper to rectify an assertion following the Lemma 2 of our precedent paper [4]. It was affirmed in that paper that *the eigenvalues of the Dirichlet and Neumann interior problems are not the same*. It is false, for example for squares or cubes. However, since we raised only a possibility of a different proof of this lemma 2, which was taken from [10], there is no consequence of this error in the paper. On the other hand, it is worth noticing the difference between the end of the proof of propositions 4 and 5 in this paper (from the equality of the COD to the equality of the cracks by a property of support of these COD), compared with the unclear end of the proposition 4 in [4] (similar result for the scalar case).

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