On inverse scattering by screens

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Abstract

A generalization of Isakov's theorem ([13]) to admissible shapes that include screens with a Robin boundary condition (in which the impedance is also an unknown) is presented. We also establish a characterization that enables us to distinguish the far-field of plane screens from others. Numerical results using a quasi-Newton method are presented.

1 Introduction

Identifying the object shape by means of the far field pattern of some acoustic wave, scattered by the object, is a mathematical and numerical problem with large applications in industry. For an overview on the subject we cite, for instances, the book of Colton and Kress ([7]).

The first uniqueness result known on the subject, was presented by Schiffer in 1967 (cf. [18]) for the case of the Dirichlet problem, based on the eigenvalues of the interior problem. The same technique was not possible neither for screens shapes nor for the Neumann case. And it was only in 1990 that Isakov ([13]) gave a new proof that enabled Kirsch and Kress ([14]) to establish an uniqueness result for the Neumann problem.

However, in this uniqueness results, screen shapes were not considered. Here, we deduce the uniqueness theorem also considering screens as admissible shapes and Robin boundary conditions with unknown impedance. Further, we derive a simple characterization of plane screens, which also enables us to determine the plane of the screen. When the plane direction is known, we also obtain an uniqueness result for plane screens, considering only one incident plane wave.

On the numerical point of view, in the 80's, Kirsch and Kress or Colton and Monk developed original numerical techniques that enabled a good recovering of the shape, in the case of round objects (cf. [16]). But this techniques are no longer available if we work with screens, so, a different technique, by using a quasi-Newton type method and a Levenberg-Marquardt minimization algorithm, was tested in the recovering of plane screen shapes. The numerical results obtained were good, even considering non-regular screen shapes, such as a cardioid.

It is worth noticing that a large part of this work was already presented in the PhD thesis of Alves [1] (1995), and that there is a very recent work by Eller [10] (1996) on the uniqueness for screens (considering the near field). Also, a recently published work by Kress in [15] (1995) shows uniqueness results for 2-D screens and a numerical scheme using a Newton method. However, both these results prove the uniqueness assuming a screen shape.

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2 Isakov's Theorem and Consequences

The uniqueness theorem for the inverse Dirichlet problem, given by Schiffer ([18]), considers the far-field knowledge for an infinity of incident frequencies. Several improvements were made since, namely a result by Colton and Sleeman [8], which proves the uniqueness result using only one incident wave, with frequency κ, by supposing the object in a sphere with radius $R < \pi/\kappa$.

Since these proofs were based on the eigenvalues of the interior problems, they were not extendible neither to the Neumann case, because of the possible irregularity of the intersection of two regular domains, nor to screen shapes. The recent proof from Isakov ([13]), presented for the Dirichlet problem, was adapted for the Neumann problem by Kirsch and Kress [14], and, as we'll see, it can be generalized for screen shapes.

To present the uniqueness result including screen shapes, we begin by defining which kind of shapes we'll consider *admissible*.

– We call $\mathcal{S}_1$ the set made by all closed C^1 surfaces, boundaries of bounded domains D with *connected* exterior $\Omega = \bar{D}^c$. This last hypothesis is necessary in total wave reflection problems, and means that we can't "*see*" the interior of D.

– We call $\mathcal{S}_2$ the *set of all screens*. By *screen* we mean a C^1-surface Γ with boundary $\gamma = \partial\Gamma$, such that:

i) $\overset{o}{\Gamma} = \Gamma\backslash\gamma$ is homeomorphic to an open bounded set of $I\!R^2$

ii) exists a $\tilde{\Gamma} \in \mathcal{S}_1 : \tilde{\Gamma} \supset \Gamma$.

In particular this means that we only consider orientable screens (non-orientable shapes, such as the Möbius strip, are not considered), and we'll take a continuous vector normal field by using one of the two possible orientations.

The *admissible surfaces* considered here after will be taken in $\mathcal{S}_1$, in $\mathcal{S}_2$, or will be a disjoint union of members from $\mathcal{S}_1$ or $\mathcal{S}_2$, which means the scatterer can have two disjoint parts of different geometrical nature.

$$\left\{ \begin{array}{c} \Gamma = \Gamma_1 \cup \Gamma_2, \\ \Gamma_1 \in \mathcal{S}_1, \ \ \Gamma_1 = \partial D_1 = \partial\Omega_1 \ \ (\Omega_1 = \bar{D}_1^c) \\ \Gamma_2 \in \mathcal{S}_2, \ \ \Gamma_2 \cap \bar{D}_1 = \emptyset \end{array} \right.$$

(Γ_1 or Γ_2 may be empty sets) and we'll write $\Omega = I\!R^3\backslash(\bar{D}_1 \cup \Gamma_2)$.

The cases where we have two screens with non-void intersection, or a screen and an object with non-void intersection, are too complicated to be considered on this framework.

If $\Gamma \in \mathcal{S}_2$, we'll write $\Omega = I\!R^3\backslash\bar{\Gamma}$, while if $\Gamma \in \mathcal{S}_1$ we consider $\Omega = I\!R^3\backslash\bar{D}$, where D is the interior of Γ (that is the bounded components of $I\!R^3\backslash\bar{\Gamma}$). Consider then the following scattering problem by a screen:

$$(P) \left\{ \begin{array}{lll} (\Delta + \kappa^2)u = 0 & \text{in } \Omega & (1) \\ \mathcal{B}u = -\mathcal{B}(u^{inc}) = g & \text{on } \Gamma & (2) \\ \dfrac{\partial u}{\partial r} - i\kappa u = o(\dfrac{1}{r}) & \text{when } r \to \infty & (3) \end{array} \right.$$

where

$$u^{inc}(x) = e^{i\kappa d.x},$$

is an incident plane wave in Γ, with $d \in S^2$, and $\mathcal{B}u$ defines a boundary condition – we'll be mainly interested in Robin boundary conditions of the form:

$$\mathcal{B}u = \frac{\partial u}{\partial n} + iZ(x)\kappa u \tag{4}$$

where $Z(x)$ is the acoustic impedance ($\Re e\, Z \geq 0$).

It's well known (e.g. [4]) that the problem (P) is well posed when $\Gamma = \Gamma_1$ and when $\Gamma = \Gamma_2$ (cf.[20]). The well-posedness proved in [20] can be extended easily to the general case considered here, ie: $\Gamma_1 \cup \Gamma_2$ with $\Gamma_1 \in \mathcal{S}_1, \Gamma_1 = \partial D_1$ and $\Gamma_2 \in \mathcal{S}_2$, $\Gamma_2 \cap \bar{S}_1 = \emptyset$.

We'll note a solution of (P) by $u(x;\kappa,d)$ and its far-field by $F(\hat{x};\kappa,d)$, and begin by recalling some density results (e.g. [7]) :

Lemma 1 *:*
a) *If κ^2 is not an eigenvalue of the Laplacian $-\Delta$ in D_1, with Dirichlet boundary conditions, the vectorial space*

$$V_0(\kappa,\Gamma) = Vect\ \left(u^{inc}(.;\kappa,d)_{|\Gamma}; d \in S^2\right)$$

is dense in $L^2(\Gamma)$.

b) *If κ^2 is not an eigenvalue of the Laplacian $-\Delta$ in D_1, with Neumann boundary conditions, the vectorial space*

$$V_1(\kappa,\Gamma) = Vect\ \left(\frac{\partial}{\partial n}u^{inc}(.;\kappa,d)_{|\Gamma}; d \in S^2\right)$$

is dense in $L^2(\Gamma)$.

Lemma 2 *Let D be an open set in $I\!R^3$ and $u \in C^2(D)\bigcap C^1(\bar{D})$ a solution of the Helmholtz equation in D, so*

$$\exists v_n \in V(\kappa;D) = Vect\ \left(u^{inc}(.;\kappa,d)_{|D}; d \in S^2\right)$$

such that $v_n \to u$ et $\nabla v_n \to \nabla u$ uniformly on the compact subsets of D.

Noticing that the eigenvalues of the Dirichlet and Neumann interior problems are not the same (for the same domain), one can use Lemma 1 to show Lemma 2 without the use of a mixed potential (c.f.[1]).

We are now in condition to establish an uniqueness result for the inverse problem, here considering also screen shapes and unknown Robin boundary conditions. We associate to a boundary Γ_k the impedance Z_k, supposed to be a regular and bounded function.

Theorem 1 *For a fixed κ, let Γ_1 and Γ_2 be two admissible surfaces with impedance functions Z_1 and Z_2 (respectively) defined on those boundaries and bounded. Suppose that u_1 and u_2 are solutions of (P) for Γ_1 and Γ_2 (respectively) and that their far-fields verify:*

$$F_1(.;\kappa,d) = F_2(.;\kappa,d),\ \forall d \in S^2, \tag{5}$$

then $\Gamma_1 = \Gamma_2$ and $Z_1 = Z_2$.

Proof:
a) Consider X the connected component of the infinity neighborhood in $\Omega_1 \cap \Omega_2$.
Let $x_0 \in X$ and w_j $(j = 1,2)$ be solutions of the auxiliary problem

$$(P_m)\begin{cases} (\Delta + \kappa^2)w_j = 0 & \text{in } \Omega_j \\ \mathcal{B}_j w_j = -\mathcal{B}_j G(x_0 - x) & \text{on } \Gamma_j \\ \dfrac{\partial w_j}{\partial r} - i\kappa w_j = o(\dfrac{1}{r}) & \text{when } r \to \infty \end{cases}$$

where G is the Green function, and $\mathcal{B}_j$ the Robin boundary condition (4), with impedance function Z_j. We'll see that $w_1 = w_2$ in X.

Consider D an open bounded set with connected exterior having Γ_1 and Γ_2 and with $x_0 \in \bar{D}^c$. $G(x_0 - x)$ is a solution of the Helmholtz equation in D and Lemma 2 allows to approach it by $v_n^{inc} \in V$.

Let $v_{n,j}^{sc}$ be solutions of (P) for Γ_j and boundary data induced by v_n^{inc}. As v_n^{inc} is a superposition of plane waves the far-fields of $v_{n,1}^{sc}$ and $v_{n,2}^{sc}$ are equal, using the hypothesis (5). We then have $v_{n,1}^{sc} = v_{n,2}^{sc}$ in X. But on Γ_j, $\frac{\partial v_n^{inc}}{\partial n}$ converges uniformly to $\frac{\partial}{\partial n}G(x_0 - x)$ and v_n^{inc} converges uniformly to $G(x_0 - x)$. Thus, convergence also occurs for $\mathcal{B}v_n^{inc}$ and the well-posedness of problem (P_m) allows us to conclude that $v_{n,j}^{sc}$ converges to w_j in Ω_j (in H^1_{loc}, for instances, by analyticity). In particular $w_1 = w_2$ in X.

b) Consider now the case when $\Gamma_1 = \Gamma_2 = \Gamma$
Since $w_1 = w_2$ in X, this means that the traces are equal on Γ and from the boundary condition we now get

$$(Z_1 - Z_2)(w(x) - e^{i\kappa\theta.x}) = 0 \quad (x \in \Gamma)$$

calling $w = w_1 = w_2$. To conclude that $Z_1 = Z_2$ we now follow an idea in [12].
Consider the set

$$\Gamma' = \{x \in \Gamma : w(x; d) = e^{-i\kappa d.x}\}$$

with $d \in S^2$. We then have $\frac{\partial w}{\partial n}(x; d) = -\frac{\partial}{\partial n}(e^{i\kappa d.x})$. Applying Holmgren's unique continuation principle for real analytic elliptic equations, if the interior of Γ' is non empty, then

$$\tilde{w}(x) = w(x; d) + e^{i\kappa d.x}$$

is null on an open set in Ω, by analyticity we should also have $\tilde{w} = 0$ in Ω, but this is a contradiction with the behavior of $\tilde{w}$ at infinity.

So the interior of Γ' must be void which means that $Z_1 = Z_2$ almost everywhere in Γ.

c) Suppose now $\Gamma_1 \neq \Gamma_2$.
Without any loss of generality we consider $\partial X \backslash \partial\Omega_2 \neq \emptyset$.
If fact, if we suppose $\partial X \subset \partial\Omega_2 = \bar{\Omega}_2 \bigcap \Omega_2^c$ then $\partial X \bigcap \Omega_2 = \emptyset$, so $\partial\Omega_2 \subset \partial X^c = \bar{X}^c \bigcup X$. This way $\Omega_2 = \Omega_2 \bigcap (\bar{X}^c \bigcup X) = (\Omega_2 \bigcap X) \bigcup (\Omega_2 \bigcap \bar{X}^c)$.
As $\Omega_2 \bigcap X \neq \emptyset$ (infinity neighborhood), we have $\Omega_2 \bigcap \bar{X}^c = \emptyset$, or else Ω_2 is not connected. Thus $\Omega_2 \subset X \subset \Omega_1 \bigcap \Omega_2 \subset \Omega_2$, so $\Omega_2 = X \subset \Omega_1$. So, we have $\partial X \backslash \partial\Omega_1 \neq \emptyset$.

Let $x^* \in \partial X \backslash \partial\Omega_2 = \partial X \backslash \Gamma_2$. We have $x^* \in \Gamma_1$, since as $\Omega_1 \bigcap \Omega_2 = X \bigcup T$ with T a bounded open set with empty intersection with X, $\partial(\Omega_1 \bigcap \Omega_2) = \partial X \bigcup \partial T$ thus $\partial X = \partial(\Omega_1 \bigcap \Omega_2)$ and it is clear that $\partial(\Omega_1 \bigcap \Omega_2) \backslash \partial\Omega_2 \subset \partial\Omega_1$.

d) It follows that for $m \in \mathbb{N}$ sufficiently large, $x_m = x^* + \frac{\boldsymbol{n}_1(x^*)}{m} \in X$ ($\boldsymbol{n}_1(x^*)$ being the normal vector to Γ_1 on x^*, oriented to Ω_1).
Considering the problems (P_m) with x_m instead of x_0, we have solutions w_1^m and w_2^m identical in X.
When $m \to \infty$,

$$G_m(x) = \frac{e^{i\kappa|x_m - x|}}{4\pi|x_m - x|}$$

converges uniformly to $G(x^*-x)$ when x is on $\partial\Omega_2=\Gamma_2$, and also ∇G_m converges to $\nabla G(x^*-x)$. We deduce then that $w_2^m(x)$ converges to the finite limit $w_2^m(x^*)$ and $\boldsymbol{n}_1(x^*).\nabla w_2^m(x)$ converges to the finite limit $\boldsymbol{n}_1(x^*).\nabla w_2^m(x^*)$ when $x\in X$ tends to x^*, and since Z_1 is bounded, the sequence

$$\boldsymbol{n}_1(x^*).\nabla w_2^m(x^*)+i\kappa Z_1(x^*)w_2^m(x^*) \tag{6}$$

is bounded when $m\to\infty$.
Since $w_2^m(x)=w_1^m(x)$ and $\boldsymbol{n}_1(x^*).\nabla w_2^m(x)=\boldsymbol{n}_1(x^*).\nabla w_1^m(x)$ for all $x\in X$, it follows that

$$\begin{cases} i\kappa Z_1(x^*)w_2^m(x^*)=i\kappa Z_1(x^*)w_1^m(x^*)\\ \boldsymbol{n}_1(x^*).\nabla w_2^m(x^*)=\boldsymbol{n}_1(x^*).\nabla w_1^m(x^*)\end{cases}$$

This means that sequence (6) converges to

$$-\frac{\partial}{\partial\boldsymbol{n}_1}G_m(x^*)-i\kappa Z_1(x^*)G_m(x^*)$$

which is clearly not bounded, and so we have a contradiction.□

Remark **1:**
Each admissible surface Γ_i may be a reunion of two surfaces, for instances $\Gamma_1=\Gamma_{11}\cup\Gamma_{12}$ with $\Gamma_{11}\in\mathcal{S}_1$, $\Gamma_{12}\in\mathcal{S}_2$. Then the far-fields allow not only to distinguish a screen Γ among other screens, but also from other closed surfaces, boundaries of open bounded sets.

It is also worth noting that the information on the far field pattern contains the information on the impedance function used in the Robin boundary condition.

Remark **2:**
Except for part b), the above proof follow essentially an idea of Isakov [13] reviewed by Kirsch and Kress [14]. On the other hand, part c) is added in the proof to ensure that this doesn't demand that Γ_1 and Γ_2 should be boundaries of open sets with interior D_1 and D_2.

Remark **3:**
As we mentioned in the Introduction, a similar uniqueness proof appeared recently in [15] for the 2-D case, and in [10] for the near-field, but they assume (as admissible surfaces) only screen shapes, and known boundary conditions.

• We may now question ourselves on the existence of some criteria on $F(.;\kappa,d)$ to say if it is the far-field of a scattered wave by a screen. We have not achieved yet to answer this question, so we'll concentrate ourselves on the case of a plane screen.

2.1 Characterization of plane screens

2.1.1 Neumann boundary conditions

In the case of the Neumann problem, the scattered wave can be represented, for all screens, by a double layer potential (e.g. [11]), and its far-field is given by:

$$F(\kappa,d;\theta)=\frac{i\kappa}{4\pi}\int_\Gamma(\theta.\boldsymbol{n}_x)\varphi(x)e^{-i\kappa\theta.x}\,ds(x) \tag{7}$$

so, if $\boldsymbol{n}_x$ is constant (the case when Γ is a plane screen),

$$F(\kappa,d;\theta)=\frac{i\kappa}{4\pi}(\theta.\boldsymbol{n})\int_\Gamma\varphi(x)e^{-i\kappa\theta.x}\,ds(x), \tag{8}$$

φ being given as a solution of the associated variational problem (see [11]), with $g = -\dfrac{\partial}{\partial n}(e^{i\kappa d.x})$.

Proposition 1 *Let* $\boldsymbol{n}$ *be an unitary fixed vector, and* F *the far-field associated to an admissible surface* Γ*, with a Neumann boundary condition.*

$$F(\kappa, d; \theta) = 0, \ \forall\theta, \theta.\boldsymbol{n} = 0, \ \forall d \in S^2, \tag{9}$$

if and only if Γ *is a plane screen.*

Moreover, the screen plane is well determined by

$$\mathcal{P} = \{\alpha\boldsymbol{n} + y : y.\boldsymbol{n} = 0\}$$

where α *is given by (we suppose* Γ *in some ball* $B(0,R)$*)*

$$\alpha = \frac{-1}{2\kappa\beta} arcos \ (-\Re e \left(\frac{F(\kappa, d, \theta)}{F(\kappa, d, \theta')}\right)) \tag{10}$$

where θ, θ' *are distinct observation directions, and* d *is one fixed incident direction such that* $d.n \neq 0$*.*

Proof:

a) After (8), if Γ is a plane screen, we have (9).

b) Suppose that Γ is a screen, or a closed surface – boundary of an open set with interior D such that κ^2 is not an eigenvalue of the interior Neumann problem – in a way that the operator $\mathcal{G}(\kappa) : g \to \varphi$ is invertible (for instances, from $H^1(\Gamma)$ to $L^2(\Gamma)$).

The hypothesis allow us to show that $F(u^{inc}; \theta)$ verifies still the nullity property if $\theta.\boldsymbol{n} = 0$, for each u^{inc} with $g = \frac{\partial u^{inc}}{\partial n}$ in $L^2(\Gamma)$ (by the density Lemma 1). If Γ is not on the normal plane to $\boldsymbol{n}$ and since $\varphi = \mathcal{G}(\kappa)^{-1}g$, we can choose φ with a small support in a neighborhood $V(y)$ of a point $y \in \Gamma$ such that $\boldsymbol{n}_x.\boldsymbol{n} \neq 1$, and

$$F(\kappa, d, \theta) = \frac{i\kappa}{4\pi}\int_\Gamma (\theta.\boldsymbol{n}_x)\varphi(x)e^{-i\kappa\theta.x} \ ds(x) \neq 0$$

contradicting the hypothesis.

In the case where κ^2 is an eigenvalue of the Neumann interior problem on D, we must take a different representation of u, namely we can consider $u = (\mathcal{D} + i\eta\mathcal{S})\varphi$ for some constant η, where $\mathcal{S}$ is the single layer potential and $\mathcal{D}$ the double layer potential. The far-field becomes then

$$F(\kappa, d, \theta) = \frac{1}{4\pi}\int_\Gamma (i\kappa(\theta.\boldsymbol{n}_x) + i\eta)\varphi(x)e^{-i\kappa\theta.x} \ ds(x),$$

which is not null for an appropriate choice of η.

c) Suppose now that we have determined the normal vector n of the plane screen. We take d fixed, and we'll write $x \in \Gamma$

$$x = (x.n)n + y = \alpha n + y$$

with $y.n = 0$. Thus

$$F(\theta) = \frac{i\kappa}{4\pi}\beta e^{-i\kappa\alpha\beta}\int_\Gamma \varphi(\alpha n + y)e^{-i\kappa y.\tau} \ ds(x) \neq 0$$

where $\theta = \beta n + \tau,\ (\tau \perp n)$.
Taking another observation direction $\theta' = -\beta n + \tau$, we get

$$\frac{F(\theta)}{F(\theta')} = -e^{-2i\kappa\alpha\beta}$$

and we obtain (10), with an *a priori* hypothesis ($\Gamma \subset B(0,R)$ for instances), we then determine α ($-R < \alpha < R$).

Γ plane is then perfectly determined.□

Remark **1:**
It's necessary to locate the screen in some ball, since we can have screens in parallel planes. More, to determine the plane, we suppose (by writing $x = \alpha n + y$) that the screen has not connected components over two distinct planes.

Remark **2:**
Another possible plane screen characterization comes from the well known reciprocity relations, which adds to the previous result that if $F(\kappa, d; .) = 0$ for an incident wave with direction d then the screen plane must be parallel to d. The screen is identified and its plane well determined if, for two independent directions d_1 et d_2, we have $F(\kappa, d_1; .) = 0$ et $F(\kappa, d_2; .) = 0$.

2.1.2 Dirichlet boundary conditions

For the Dirichlet problem, if we want to determine whether the object is a plane screen or not, we don't have *a priori* an evident condition as (7).

In this case the scattered wave can be represented by a single layer potential and its far-field is now given by:

$$F(\kappa, d; \theta) = \frac{1}{4\pi} \int_\Gamma \psi(d; x) e^{-i\kappa\theta.x}\ ds(x). \tag{11}$$

where ψ is the normal displacement of the solution with Dirichlet boundary condition $u(d; x) = -e^{i\kappa d.x} = g(x),\ \ (x \in \Gamma)$.

To overcome this characterization difficulty, we questioned ourselves over the tangential gradient ∇^* on S^2. It's easy to verify that

$$\nabla^*_\theta F(\kappa, d; \theta) = \frac{1}{4\pi} \int_\Gamma \psi(d; x)(x - (x.\theta)\theta) e^{-i\kappa\theta.x}\ ds(x).$$

so, when Γ is a plane screen with normal $\boldsymbol{n}$

$$\boldsymbol{n}.\nabla^*_\theta F(\kappa, d; \theta) = \frac{i\kappa}{4\pi} \int_\Gamma \psi(d; x)((\boldsymbol{n}.x) - (x.\theta)(\boldsymbol{n}.\theta)) e^{-i\kappa\theta.x}\ ds(x). \tag{12}$$

is null when $\theta \perp \boldsymbol{n}$.

Proposition 2 *Let $\boldsymbol{n}$ be an unitary fixed vector, and F the far-field associated to a surface Γ, with a Dirichlet boundary condition.*

$$\boldsymbol{n}.\nabla^*_\theta F(\kappa, d; \theta) = 0,\ \forall\theta, \theta.n = 0,\ \forall d \in S^2, \tag{13}$$

if and only if Γ is a plane screen.

Proof:
The direct statement comes from (12).
Suppose inversely that Γ is a screen, or a closed surface – boundary of an open set with interior D such that κ^2 is not an eigenvalue of the interior Dirichlet problem.
The same reasoning as for the Neumann case can be applied, using now the operator $\mathcal{G}(\kappa) : g \to \psi$ invertible from $H^1(\Gamma)$ to $L^2(\Gamma)$.

Since (13) holds if $\theta.n = 0$, for each u^{inc} with $g = -u^{inc}$ in $L^2(\Gamma)$ (by the density Lemma 1), we take $\psi = \mathcal{G}(\kappa)^{-1}g$. Notice that ψ can be chosen with a small support in a neighborhood $V(x)$ of a point $x \in \Gamma$ such that for $y \in V(x)$ we have $\boldsymbol{n}_x.y \neq 0$, if Γ is not perpendicular to n.
Then
$$\nabla^*_\theta F(\kappa, d, \theta) = \frac{i\kappa}{4\pi} \int_\Gamma (\boldsymbol{n}.x)\varphi(x) e^{-i\kappa\theta.x}\, ds(x) \neq 0$$
contradicting the hypothesis (13).

In the case where κ^2 is an eigenvalue of the Dirichlet interior problem on D, we also must take a different representation of u.$\square$

2.1.3 Robin boundary conditions

As for the case of a Robin problem, we are only able to proof a similar result, to the previous ones, if we consider a constant impedance function $Z(x) = Z \geq 0$.
The far-field is now given by:
$$F(\kappa, d; \theta) = \frac{i\kappa}{4\pi} \int_\Gamma ((\theta.\boldsymbol{n}_x) - Z)\varphi(d; x) e^{-i\kappa\theta.x}\, ds(x), \tag{14}$$
where the density φ is obtained by the following representation of u:
$$u = (\mathcal{D} + i\kappa Z\mathcal{S})\varphi.$$
where $\mathcal{S}$ is the single layer potential and $\mathcal{D}$ the double layer potential.

A condition of the type $(\theta.\boldsymbol{n}) = Z,\ \forall\theta$, like we had for the Neumann problem, is only possible for $0 \leq Z < 1$, so we'll use again the tangential gradient, which is now:
$$\nabla^*_\theta F(\kappa, d; \theta) = \frac{i\kappa}{4\pi} \int_\Gamma \left[i\kappa(x - (x.\theta)\theta)((\theta.\boldsymbol{n}_x) - Z) + \boldsymbol{n}_x - (\theta.\boldsymbol{n}_x)\theta\right] e^{-i\kappa\theta.x} \varphi(d; x)\, ds(x).$$
so, when Γ is a plane screen with constant normal $\boldsymbol{n}$ and if we consider observation directions θ normal to $\boldsymbol{n}$ we get
$$\boldsymbol{n}.\nabla^*_\theta F(\kappa, d; \theta) = \frac{i\kappa}{4\pi} \int_\Gamma e^{-i\kappa\theta.x} \varphi(x; d)\, ds(x). \tag{15}$$
it's now easy to see that we obtain the following proposition, in an analogous way to the previous ones:

Proposition 3 *Let $\boldsymbol{n}$ be an unitary fixed vector, and F the far-field associated to a surface Γ with a Robin boundary condition where impedance function $Z \geq 0$ is constant.*
$$Z\boldsymbol{n}.\nabla^*_\theta F(\kappa, d; \theta) + F(\kappa, d; \theta) = 0,\ \forall\theta, \theta.n = 0,\ \forall d \in S^2, \tag{16}$$
if and only if Γ is a plane screen.

Proof:
Similar to the previous, we only remark that if Γ is not a plane screen, we have for $\theta \perp \boldsymbol{n}$ and φ with a small support V that is not a part of a plane normal to $\boldsymbol{n}$, then $\boldsymbol{n}.x \neq 0$ and $\boldsymbol{n}_x.\boldsymbol{n} \neq 1$

$$Z\boldsymbol{n}.\nabla_\theta^* F(\kappa, d; \theta) + F(\kappa, d; \theta) = \frac{i\kappa}{4\pi} \int_\Gamma i\kappa Z(x.\boldsymbol{n}) \{(\theta.\boldsymbol{n}_x) - Z +$$

$$+ Z(\boldsymbol{n}_x.\boldsymbol{n}) + (\theta.\boldsymbol{n}_x) - Z\} e^{-i\kappa\theta.x} \varphi(d; x)\, ds(x) \neq 0,$$

contradicting the hypothesis. This is so, because in that support, the term

$$N(x) = i\kappa Z(x.\boldsymbol{n})((\theta.\boldsymbol{n}_x) - Z) + Z(\boldsymbol{n}_x.\boldsymbol{n}) + (\theta.\boldsymbol{n}_x) - Z$$

for $(\theta.\boldsymbol{n}_x) - Z \neq 0$, has always an imaginary part, which is not zero and we would simply get $N(x) = (\theta.\boldsymbol{n}_x) - Z \neq 0$.
The case $(\theta.\boldsymbol{n}_x) - Z = 0$ implies $N(x) = Z(\boldsymbol{n}_x.\boldsymbol{n})$, so if $N(x) = 0$, and since we consider all $\theta : \boldsymbol{n} \perp \theta$, this would mean $\boldsymbol{n}_x$ orthogonal to all vectors in $\mathbb{R}^3$, thus $N(x) \neq 0$.□

• If we have determined the screen plane, we can show that *only one* incident wave is necessary to completely determine the shape of the screen.

2.1.4 Uniqueness for a single incident wave

Proposition 4 *Consider the screens Γ_1, Γ_2 in the plane $\{x_3 = 0\}$ where we have Robin boundary conditions with constant impedance Z. and* **only one** *incident wave with direction d such that $d_3 \neq Z$. If the associated far-fields F_1, F_2 (of the scattered waves) are equal, i.e:*

$$F_1(\theta) = F_2(\theta), \forall \theta \in S^2,$$

we have $\Gamma_1 = \Gamma_2$.

Proof :
Let $\varphi_1 \in H^{1/2}(\mathbb{R}^2)$ be the extension by 0 of the screen open displacement outside the screen Γ_1 and φ_2 the same for Γ_2. They are not null since $g = i\kappa(d_3 - Z)e^{i\kappa d.x}$

If the far-fields coincide, we have

$$\frac{i\kappa}{4\pi}(\theta_3 - Z) \int_{\mathbb{R}^2} \varphi_1(x) e^{-i\kappa\theta.x}\, ds(x) = \frac{i\kappa}{4\pi}(\theta_3 - Z) \int_{\mathbb{R}^2} \varphi_2(x) e^{-i\kappa\theta.x}\, ds(x)$$

then, since $x_3 = 0$, if we consider θ such that $\theta_3 \neq 0$ we get

$$\mathcal{F}_2(\varphi_1)(\kappa\theta') = \mathcal{F}_2(\varphi_2)(\kappa\theta') \tag{17}$$

where $\mathcal{F}_2$ is the Fourier transform in $\mathbb{R}^2$, and $\theta' = (\theta_1, \theta_2)$. But, since φ_1, φ_2 are functions on $\mathbb{R}^2$ with compact support, their Fourier transforms on $\mathbb{R}^2$ are analytical functions. Since the identity (17) is true for all θ' projection of θ on the plane $\mathbb{R}^2$, we have the identity for θ' in the unitary disk. By consequence we have two analytical functions in $\mathbb{R}^2$ that coincide on a disk with radius $\kappa > 0$, so they are equal. Then φ_1 and φ_2 coincide, and also their support.

The support of the screen open displacement coincides with the screen, for a plane wave with $d_3 \neq Z$. Because otherwise, if $[u] = 0$ and $[\frac{\partial u}{\partial n}] = 0$ on a non-empty open subset of Γ, we would have $g = 0$.
Thus, we can conclude that $\Gamma_1 = \Gamma_2$. □

Remark **1 :**
– We can also consider a weaker hypothesis, that is: the far-fields coincide for some observation directions $\theta \in \mathcal{O} \subset S^2$, where $\mathcal{O}$ is a set with non-empty interior for the topology of S^2. The projection of $\mathcal{O}$ on $I\!R^2$ has non-empty interior also, and we have uniqueness.
– By using the reciprocity relations, we can also conclude that for a set $\mathcal{O}$ of incident directions and *only one* observation direction, the shape of the plane screen is completely determined.

Remark **2 :**
It's easy to conclude that this proposition is also true for Dirichlet boundary conditions, by using the same arguments.

However, we were not able to prove the uniqueness for an unknown Z with only one incident wave.

2.2 Numerical Approach for Plane Screens

In the Neumann case, as we saw, we can distinguish a plane screen from other shapes by a simple test on the far-field. Determined the plane of the screen, our aim is to improve a numerical method which enable us to determine the shape of the screen.

Kirsch-Kress or Colton-Monk numerical methods (see [16]) are not possible in this case, since they are based on the fact that the objects have a non-empty interior. Thus, we'll develop a quasi-Newton-type numerical procedure.

First we establish the space that contains the admissible shapes (which we'll suppose to be star-shaped). Let $\partial\Gamma$ be the screen boundary determined by the points (in the plane $x_3 = 0$)

$$x = \rho(\tau)(\cos\tau, \sin\tau), \text{ where } \tau \in [0, 2\pi]$$

fixed some constants a and b, we take a compact subset of

$$\mathcal{S} = \{\rho \in C^{1,\alpha}[0, 2\pi] : ||\rho||_{1,\alpha} \leq b \text{ et } \rho(\tau) \leq a.\}$$

which is the set that contains all admissible border shapes:

$$\mathcal{A}_p = \left\{ r \in \mathcal{S} : r(\theta) = a_0 + \sum_{k=1}^{p} (a_k cos(k\theta) + a'_k sin(k\theta)) \text{ with } a_k, a'_k \in [0, M] \right\} \tag{18}$$

The value p is fixed, and determines the complexity of the screen shape; for instances, if we take $p = 0$, we'll only have disks. The parameters a_k, a'_k determine completely the shape of the screen Γ, and we'll adopt the notation $\vec{a}$ to designate all.

Now, we'll suppose known the values of the far-field $F(d, \theta)$ for some screen Γ unknown. We'll take a finite number of incident directions $d_1, ..., d_m$ and a fixed number of observation directions $\theta_1, ..., \theta_n$.

We must minimize the L^2-difference between $F(d, \theta)$ and $\tilde{F}(d, \theta)$ - the far-field generated by screens with border shapes in $\mathcal{A}_p$.

To find this minimum, we follow the Levenberg-Marquardt algorithm, based on a least-squares method (cf. [9]).

2.2.1 Search Algorithm

(0) We begin by considering one arbitrary initial guess of the border shape defined by $\vec{a}$.
(1) Determine the far-field $F_{\vec{a}}(d_i, \theta_j)$ for all considered incident and observation directions.

(2) Levenberg-Marquardt's Method

$$\mathcal{F}(\vec{a}) = \sum_{i=1}^{m} \sum_{j=1}^{n} |F(d_i, \theta_j) - F_{\vec{a}}(d_i, \theta_j)|^2 = \sum_{k=1}^{mn} f_k(\vec{a})^2$$

$$\vec{b} = \vec{a} + \vec{h}$$

where $\vec{h}$ is the solution of

$$(J^T J + \lambda I)\vec{h} = -J^T \vec{f}.$$

a $p \times p$ system, where J is the $m \times p$ Jacobian matrix of $\vec{f}$.
(3) We consider a new guess defined by the parameter-vector $\vec{b}$ and return to step 1).

3 Numerical Tests

We'll concentrate ourselves in the Neumann case.
Consider the admissible shapes chosen in (18), we notice that $\mathcal{A}_0 \subset \ldots \subset \mathcal{A}_p \subset \ldots$

The tests will always suppose a given far-field. That far-field may be calculated with the same method used in retrieving the shape or with another one. However, to have a real test, we must assure that the given far-field is not a far-field generated by the objects of $\mathcal{A}_p$ (see [7] on "inverse crimes"). To avoid this danger, we can use a different direct method, some random perturbations, or, what becomes the same thing, a different precision on the direct calculations.

If the shape doesn't belong to $\mathcal{A}_p$, we can consider direct calculations of the far-field without any restriction.

To solve the direct problem we'll use a boundary element method (cf.[2]).

As observation directions, we take (in polar coordinates) the directions :

$$\theta_{ij} = (1, 2\pi/i, \pi/j) \quad (i = 1, \ldots, M_1 \text{ and } j = 1, \ldots, M_2)$$

that is $M_1 \times M_2$ observation directions.

The first test presented (figure 1) is very simple, we'll try to retrieve the shape of the acoustic disk (in $\mathcal{A}_0$) , among the shapes of $\mathcal{A}_1$.

We'll consider only one incident wave (with $\kappa = 4$, normal incidence) in a disk, and the associated far-field in 10×10 directions of observation, measured with 10% random noise. We apply our method by using 19 interior nodes, and we start with a shape on $\mathcal{A}_1$ not too close to the disk. We see, after five iterations that results are quite close, and we arrive to a circular shape which is not far from the searched object – the unitary disk.

In the figures 2, 3 and 4, we consider the incidence of a normal incident wave (with $\kappa = 6$) in a square shape.

First we try to approach it by considering only disk shapes (that is in $\mathcal{A}_0$), this way we obtain a *minimal disk* (figure 2).

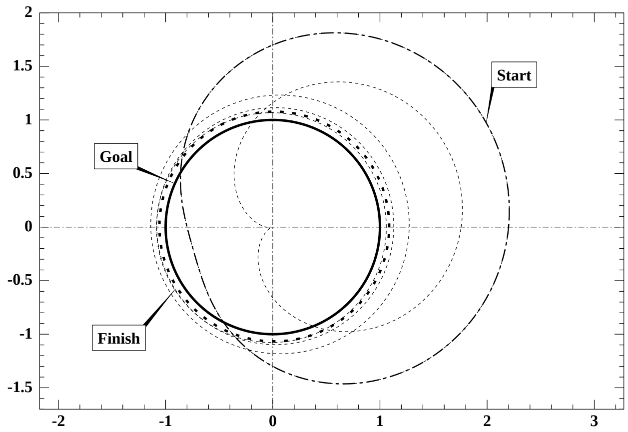

Figure 1: *Convergence to a simple object*

Then, we start with the minimal disk, which has almost the same area as the square, and we search the minimal shape in $\mathcal{A}_4$ (see figure 3). The same procedure is taken to search the minimal shape in $\mathcal{A}_4$ (figure 4).

We can see that the results are quite good, and the strategy (to start with the minimal shape obtained with less parameters) allows us to avoid other local minima.

It seems convenient to find first a shape which has almost the same dimension as the desired shape. This was obtained, in our case, considering first disk shapes, which depend on one parameter only.

We have also tested some results with a *kite*. Again, we have obtained the minimal disk and took it as a start shape, while considering screens on $\mathcal{A}_4$ (figure 5).

We have then used the minimal shape on $\mathcal{A}_4$ as the start shape for screens in $\mathcal{A}_{10}$, but we haven't obtained convergence, so we had to consider a "middle-step" with screens in $\mathcal{A}_6$ (figure 6). The minimal shape obtained there was then used as a start, to find the minimal shape in $\mathcal{A}_{10}$. Convergence was achieved, but no major differences were seen with this increase of parameters .

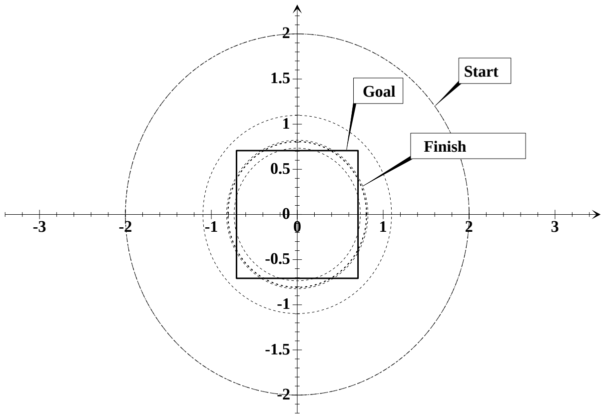

Figure 2: *Approximation of a square by disks.*

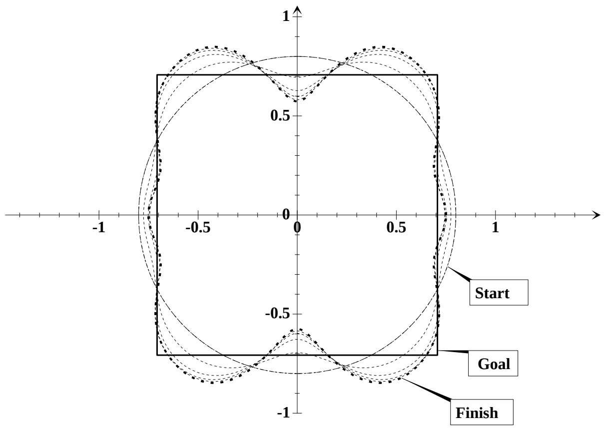

Figure 3: *Approximation of a square by shapes in* $\mathcal{A}_4$

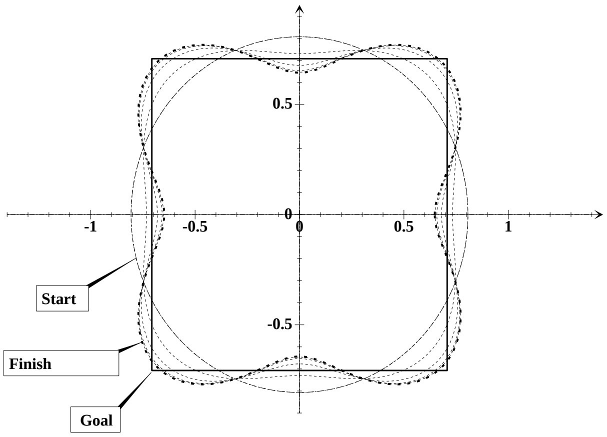

Figure 4: *Approximation of a square by shapes in* $\mathcal{A}_{10}$

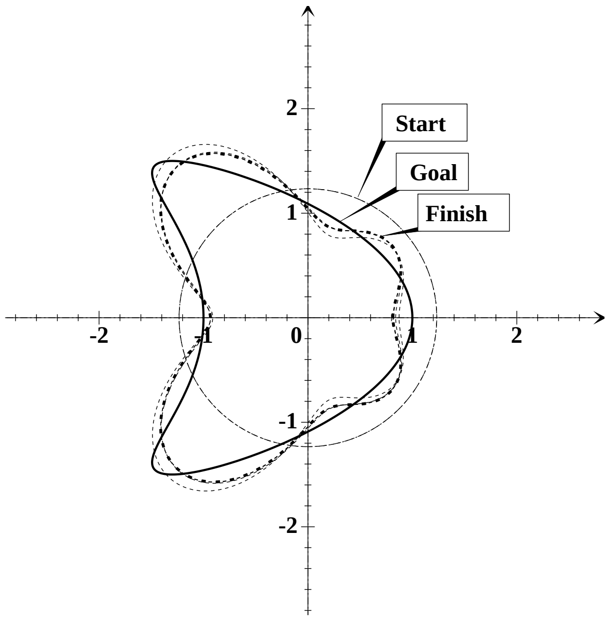

Figure 5: *Approximation of a* kite *by shapes in* $\mathcal{A}_4$

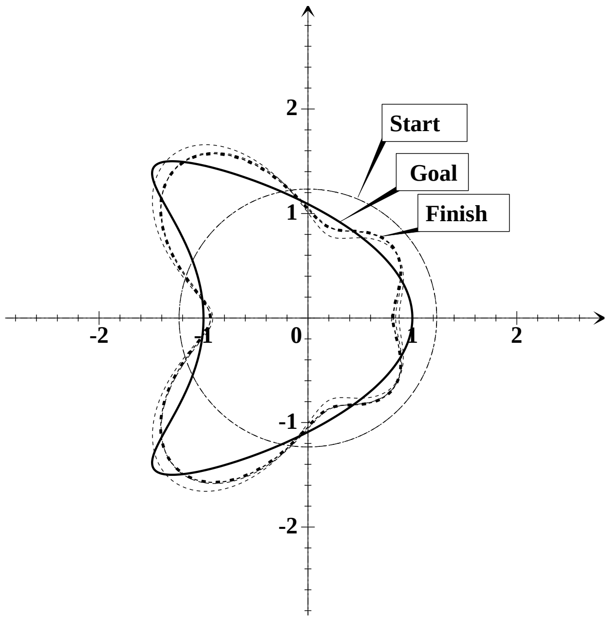

Figure 6: *Approximation of a* kite *by shapes in* $\mathcal{A}_6$

Even if we have shown that the screens can be well determined by the use of only one incident wave, we'll compare the results with two incident waves.

In figure 7 we show the previous test for two incident waves. One of this incident waves will be normal to the plane, and the other has a direction 45^o to the normal vector, and it will be

i) either in the $\{x_2 = 0\}$ plane,

ii) either in the $\{x_1 = x_2\}$ plane.

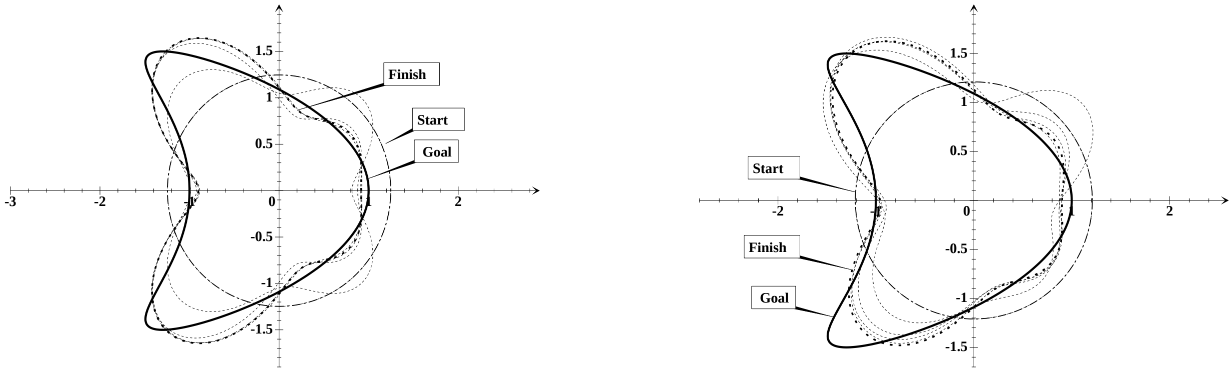

Figure 7: *Approximation of a* kite *by shapes in $\mathcal{A}_4$ using two incident directions: a normal one, and other on the plane defined by i) and ii)*

It can be seen, as already noticed in [16], that the result of the approach depends on the directions of the incident waves – in a clear way – the reconstruction is better on the directions close to the optical reflection.

Finally, we present some results for an irregular shape, such as the cardioid (figures 8 and 9). We can see that the approach is globally very good when we use shapes in $\mathcal{A}_4$, but it almost ignores the cardioid cusp.

When we increase the number of parameters, considering screen shapes on $\mathcal{A}_7$, the minimal shape obtained now, is not globally so good as the previous, but it approaches quite well the geometrical singularity point (the cusp).

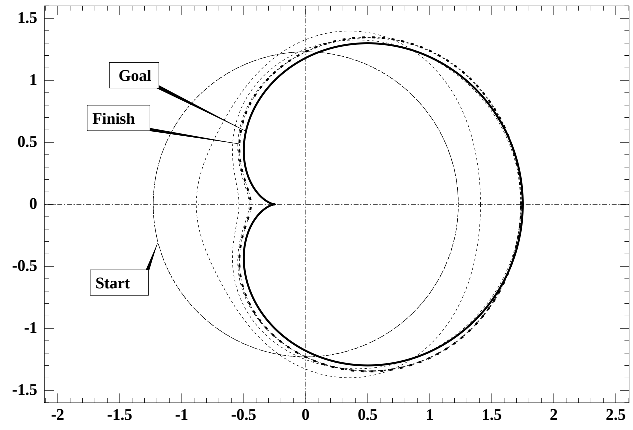

Figure 8: *Approximation of a cardioid by shapes in $\mathcal{A}_4$ by using a normal incident wave.*

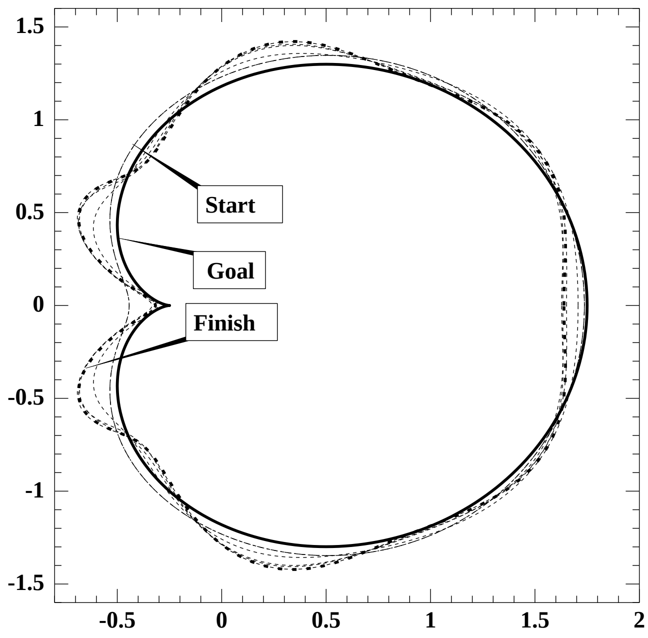

Figure 9: *Approximation of a cardioid by shapes in $\mathcal{A}_7$ by using a normal incident wave.*

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