Riemann-Hilbert problems, Toeplitz operators and $\mathfrak{Q}$-classes

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Abstract. We generalize the notion of $\mathfrak{Q}$-classes C_{Q_1,Q_2}, which was introduced in the context of Wiener-Hopf factorization, by considering very general 2×2 matrix functions Q_1, Q_2. This allows us to use a mainly algebraic approach to obtain several equivalent representations for each class, to study the intersections of $\mathfrak{Q}$-classes and to explore their close connection with certain non-linear scalar equations. The results are applied to various factorization problems and to the study of Toeplitz operators with symbol in a $\mathfrak{Q}$-class. We conclude with a group theoretic interpretation of some of the main results.

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1. Introduction

1.1 We start by introducing some notation.

Let $H_p^{\pm} := H^p(\mathbb{C}^{\pm})$, $p \in]0, +\infty]$, denote the Hardy spaces over the half-planes $\mathbb{C}^{\pm}$ ([19]), identified as usual with subspaces of the Lebesgue spaces $L_p(\mathbb{R})$. For $p \in]1, \infty[$, we have $L_p(\mathbb{R}) = H_p^+ \oplus H_p^-$ and we denote by P^+ the projection of $L_p(\mathbb{R})$ onto H_p^+ parallel to H_p^-.

Let $C_\mu(\dot{\mathbb{R}})$ denote the Banach algebra of all functions that are continuous and satisfy a Hölder condition with exponent $\mu \in]0, 1[$ on $\dot{\mathbb{R}}$. Let moreover $C_\mu^{\pm}(\dot{\mathbb{R}}) := C_\mu(\dot{\mathbb{R}}) \cap H_\infty^{\pm}$.

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Denoting by $\mathcal{R}$ the set of all rational functions with poles off $\dot{\mathbb{R}}$, let $\mathcal{M}^{\pm}_{\infty} = H^{\pm}_{\infty} + \mathcal{R}$.

For any set X, we denote by X^n (resp., $X^{n\times m}$) the set of all n-vectors (resp., $n \times m$ matrices) with entries in X, and for any unital algebra $\mathcal{A}$, let $\mathcal{GA}$ denote the group of invertible elements in $\mathcal{A}$.

We say that $G \in \mathcal{G}(L_\infty(\mathbb{R}))^{2\times 2}$ admits a *bounded factorization* (resp., *$\mathcal{M}$-bounded factorization*) if and only if G admits a representation

$$G = G_- D G_+^{-1} \tag{1.1}$$

where

$$G_\pm \in \mathcal{G}(H^{\pm}_{\infty})^{2\times 2} \qquad (\text{resp.,} G_\pm \in \mathcal{G}(\mathcal{M}^{\pm}_{\infty})^{2\times 2}) \tag{1.2}$$

and $D = \text{diag}(d_1, d_2) \in \mathcal{G}(L_\infty(\mathbb{R}))^{2\times 2}$ ([5]). When $D = I$ and $G_\pm$ satisfy the first condition in (1.2), the representation (1.1) is called a *canonical bounded factorization.* This is a particular form of the Douglas-Rudin factorization, which is known to exist for every log-integrable G in $(L_\infty(\mathbb{R}))^{2\times 2}$([1, 4]). If $D = I$ and $G_\pm \in \mathcal{G}(\mathcal{M}^{\pm}_{\infty})^{2\times 2}$, we say that G admits a *meromorphic factorization* ([6, 28]).This type of factorization appears naturally, for instance, in the study of certain elastodynamic diffraction problems ([7, 29]).

Representations of the form (1.1) are closely associated with the study of *Riemann-Hilbert problems*, which can be formulated as follows: for a given matrix function G and a given vector function g defined (a.e.) on $\mathbb{R}$, find two vector functions $\phi_\pm$, analytic in the upper and lower half-planes $\mathbb{C}^\pm$, respectively, satisfying the boundary condition

$$G\phi_+ = \phi_- + g \tag{1.3}$$

on $\mathbb{R}$. The existence of a bounded factorization for G means that the matrix Riemann-Hilbert (RH) problem (1.3) can be decoupled into scalar RH problems, and several meaningful conclusions regarding the solvability of (1.3) can be drawn from the factorization (1.1), if it exists, even without additional information about the diagonal elements of D ([5]). More can be said, of course, if some particular form is imposed for the elements of D, as it happens in the cases of Wiener-Hopf or almost periodic factorization.

By a *(bounded) Wiener-Hopf factorization* we mean a bounded factorization (1.1) with $D = \text{diag}(r^{k_1}, r^{k_2})$, where $k_1, k_2 \in \mathbb{Z}$ are called the *partial indices* and

$$r(\xi) = \frac{\xi - i}{\xi + i}, \quad \xi \in \mathbb{R} \tag{1.4}$$

([16, 28, 31]). If G admits a meromorphic factorization $G = M_- M_+$ with $M_\pm \in \mathcal{G}(\mathcal{M}^{\pm}_{\infty})^{2\times 2}$, then it also has a Wiener-Hopf factorization that can be obtained from the former by a finite number of elementary algebraic operations and rational factorization ([7, 8, 28]).

By an *almost periodic factorization* we mean a bounded factorization(1.1) where $D = \text{diag}(e_{\mu_1}, e_{\mu_2})$, with $\mu_1, \mu_2 \in \mathbb{R}$ and $e_\mu(\xi) := e^{i\mu\xi}$, and $G_\pm \in AP^\pm$ with $AP^\pm := AP \cap H^{\pm}_{\infty}$, where AP denotes the closure of the set of all almost periodic polynomials $\sum_j c_j e_{\lambda_j}$ with $\lambda_j \in \mathbb{R}, c_j \in \mathbb{C}$, with respect to the uniform norm ([3]).

We use the abbreviations *WH*-factorization for the former and *AP*-factorization for the latter.

From an operator theoretic perspective, factorization of matrix functions and RH problems are also closely connected with the study of Toeplitz operators. Namely, for $G \in (L_\infty(\mathbb{R}))^{2\times 2}$, the Toeplitz operator T_G defined by

$$T_G : (H_p^+)^2 \to (H_p^+)^2, \quad T_G(\phi_+) = P^+(G\phi_+) \tag{1.5}$$

$(1 < p < \infty)$ is Fredholm if its symbol G admits a *WH*-factorization; T_G is invertible if this factorization is canonical, i.e, the partial indices k_j are equal to zero, and in this case $(T_G)^{-1} = G_+P^+(G_-)^{-1}I_+$, where I_+ denotes the identity operator in $(H_p^+)^2$. On the other hand, the kernel of T_G consists of all the functions $\phi_+ \in (H_p^+)^2$ satisfying the RH equation

$$G\phi_+ = \phi_- \tag{1.6}$$

for some $\phi_- \in (H_p^-)^2$.

We say that T_G is *nearly Fredholm equivalent* to $T_{\widetilde{G}}$ if and only if

$$T_G \text{ is Fredholm } \Leftrightarrow T_{\widetilde{G}} \text{ is Fredholm.} \tag{1.7}$$

If moreover T_G and $T_{\widetilde{G}}$ have the same Fredholm index, we say that they are *Fredholm equivalent* (and *strictly Fredholm equivalent* when $\dim\ker T_G = \dim\ker T_{\widetilde{G}}$, $\dim\operatorname{coker} T_G = \dim\operatorname{coker} T_{\widetilde{G}}$) ([10]). With this notation, if G admits an $\mathcal{M}$-bounded factorization (1.1), then T_G is nearly Fredholm equivalent to T_D, and if this factorization is a bounded one then T_G is strictly Fredholm equivalent to T_D ([28, 31]).

1.2 It was pointed out in [15] that every 2×2 matrix function admitting a *WH*-factorization of the form (1.1) satisfies a relation

$$G^T Q_1 G = \det G \cdot Q_2 \tag{1.8}$$

where Q_1 and Q_2 are symmetric matrices such that

$$Q_1 \in \mathcal{G}(\mathcal{M}_\infty^-)^{2\times 2}, \quad Q_2 \in \mathcal{G}(\mathcal{M}_\infty^+)^{2\times 2} \tag{1.9}$$

$$\det Q_1 = \det Q_2 = q \in \mathcal{GR}. \tag{1.10}$$

The class of all 2×2 invertible matrix functions G satisfying (1.8) was denoted by $C(Q_1, Q_2)$, following the notation of [13] where these classes were defined and studied for the first time.

The importance of the relation (1.8) in solving RH problems of the form (1.3) and in the study of a Toeplitz operator with symbol G was put in evidence in [13, 14, 15]. In fact, in the case where $G \in \mathcal{G}(C_\mu(\dot{\mathbb{R}}))$, it provides certain non-linear equations allowing to solve those problems, as shown in [13], as well as an equivalence between the matrix RH problem (1.3) in the complex plane and a scalar RH problem in a Riemann surface Σ uniquely associated to the class $C(Q_1, Q_2)$, as shown in [14, 15]. In the latter case, an appropriate factorization for scalar functions defined on a contour in Σ can be used to solve (1.3) and, consequently, to study some properties of the Toeplitz operator T_G ([9, 14, 15]).

Although the results of [15] represent a significant step forward in the study of RH problems and some properties of Toepliz operators, a number of questions were left open. Namely, given a 2×2 matrix G with entries in $C_\mu(\dot{\mathbb{R}})$, how to determine a pair (Q_1, Q_2) such that (1.8) holds? And, since G may belong to two different classes $C(Q_1, Q_2)$ and $C(\widetilde{Q}_1, \widetilde{Q}_2)$, can we have two associated Riemann surfaces, with different genuses? Furthermore, is it possible to extend the results to factorization problems that are not of Wiener-Hopf type? We address these and other questions in the present paper.

In the past, the classes $C(Q_1, Q_2)$ have been studied only for matrix functions G admitting a WH-factorization. We will be interested in studying them in a more general context, and hence we will give up the restrictions (1.9) and (1.10). We define here C_{Q_1,Q_2} as consisting of all $G \in \mathcal{G}(L_\infty(\mathbb{R}))^{2\times 2}$ satisfying (1.8), where we assume only that Q_1 and Q_2 are symmetric and invertible in $(L_\infty(\mathbb{R}))^{2\times 2}$, with $\det Q_1 = \det Q_2$.

Several reasons make it natural, and convenient, to study these classes of matrix functions in a more abstract context. On the one hand, by taking advantage of the mainly algebraic nature of their definition and properties, this approach provides a unified treatment of various problems in different settings, allows us to obtain new results, and yields a better understanding of the existing ones. On the other hand, it enables us to give a group theoretic perspective of the results, which is presented here for the first time to the authors' knowledge.

The paper is organized as follows.

In Section 2 we introduce the $\mathfrak{Q}$-classes C_{Q_1,Q_2} and present some of their properties. The main result in this section is Theorem 2.6, which gives different equivalent representations of the matrix functions belonging to a given class C_{Q_1,Q_2}. In Section 3 we study the intersections of $\mathfrak{Q}$-classes and we show that a matrix function G may belong to two different classes, whose intersection consists of scalar multiples of G. Section 4 deals with the so-called product equation ([13]), a non-linear scalar equation associated with each C_{Q_1,Q_2}. It is shown that it can be used to address a major problem related to the use of the relation (1.8) to solve RH problems, which is how to determine a pair (Q_1, Q_2) such that (1.8) holds for a given G. In Section 5 we study the factorization of matrices in C_{Q_1,Q_2} with factors belonging to certain $\mathfrak{Q}$-classes. It is shown that, for every Q_3, $G \in C_{Q_1,Q_2}$ can be represented as a product of two matrices, in C_{Q_1,Q_3} and C_{Q_3,Q_2} respectively, which can be determined from a solution to the associated equation $G\phi = \psi$. In Section 6 the results of the previous sections are applied to several problems regarding bounded factorization of 2×2 matrix functions and Toeplitz operators. It is shown, in particular, that the results of sections 3 and 4 can be used to describe the kernel of a Toeplitz operator, obtain conditions for its invertibility and determine an explicit Wiener-Hopf factorization for its symbol by simple algebraic methods, instead of the much more complicated approach suggested in [15]. Finally, in Section 7, we show that C_{Q_1,Q_2}, endowed with an operation $*$ which reduces to the usual multiplication of matrices when $Q_1 = Q_2$, is a group. Several results of the previous sections can thus be elegantly translated into group theoretic terms. How to take advantage of this formulation to advance the study of RH problems and Toeplitz operators is an open and very interesting question.

2. The $\mathfrak{Q}$-classes C_{Q_1,Q_2}

In what follows, we abbreviate $L_\infty(\mathbb{R})$ to L_∞. Let $Q \in \mathcal{G}L_\infty^{2\times 2}$ be a symmetric matrix function of the form

$$Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix} \tag{2.1}$$

and let

$$q := -\det Q. \tag{2.2}$$

Let $\rho \in \mathcal{G}L_\infty$ be such that $\rho^2 = q$, choosing $\rho = q_2$ if $q_1 q_3 = 0$. Assume also that either $q_1 \in \mathcal{G}L_\infty$ or $q_1 = 0$ (as in [15]).

We denote by $\mathfrak{Q}$ the class of all matrices Q satisfying the above conditions.

To each $Q \in \mathfrak{Q}$ we associate

$$S_Q = \begin{bmatrix} q_1 & q_2 + \rho \\ 1 & \frac{q_3}{q_2+\rho} \end{bmatrix}, \tag{2.3}$$

$$H_\rho = \begin{bmatrix} 1 & 1 \\ \rho^{-1} & -\rho^{-1} \end{bmatrix}, \tag{2.4}$$

$$D_q = \operatorname{diag}(1, -q). \tag{2.5}$$

Remark that, with our assumptions,

$$\frac{q_3}{q_2 + \rho} \in \mathcal{G}L_\infty \tag{2.6}$$

since $q_3/(q_2 + \rho) = (q_2 - \rho)/q_1$ if $q_1 \in \mathcal{G}L_\infty$, and $\rho = q_2 \in \mathcal{G}L_\infty$ if $q_1 = 0$. Thus S_Q, H_ρ and D_q are in $\mathcal{G}L_\infty^{2\times 2}$. We have

$$\det S_Q = -2\rho, \quad \det H_\rho = -2\rho^{-1}, \quad \det D_q = -q = \det Q. \tag{2.7}$$

For $Q = D_q$,

$$S_{D_q} = \begin{bmatrix} 1 & \rho \\ 1 & -\rho \end{bmatrix} = 2H_\rho^{-1}. \tag{2.8}$$

Defining

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \widetilde{I} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ \widetilde{J} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.9}$$

we have

$$\widetilde{I}J = \widetilde{J} = -J\widetilde{I}, \quad \widetilde{I}\widetilde{J} = J = -\widetilde{J}\widetilde{I} \quad J\widetilde{J} = \widetilde{I} = -\widetilde{J}J. \tag{2.10}$$

It is easy to see that the following equalities hold:

$$H_\rho^{-1} = \frac{1}{2}\rho\widetilde{I}H_\rho^T\widetilde{J} \quad , \quad H_\rho\widetilde{I}H_\rho^{-1} = -\rho^{-1}JD_q \ , \tag{2.11}$$

$$H_\rho^T D_q H_\rho = 2\widetilde{J} \quad , \quad H_\rho^T\widetilde{J}H_\rho = 2\rho^{-1}\widetilde{I}, \tag{2.12}$$

$$S_Q^T\widetilde{J}S_Q = 2Q \quad , \quad JS_Q^T\widetilde{J} = -2\rho S_Q^{-1}\widetilde{I}. \tag{2.13}$$

Definition 2.1. We denote by $\mathfrak{Q}^{(2)}$ the set of all pairs $(Q_1, Q_2) \in \mathfrak{Q}^2$ with $\det Q_1 = \det Q_2$.

We now introduce the classes whose study will be the central topic in this paper. In what follows, we use the notation

$$Q_j = \begin{bmatrix} q_{j1} & q_{j2} \\ q_{j2} & q_{j3} \end{bmatrix}.$$

Definition 2.2. For each pair $(Q_1, Q_2) \in \mathfrak{Q}^{(2)}$, let

$$C_{Q_1,Q_2} = \{G \in \mathcal{G}L_\infty^{2\times 2} : G^T Q_1 G = \det G \cdot Q_2\}.$$

The classes C_{Q_1,Q_2} will be called $\mathfrak{Q}$-*classes* and, if $Q_1 = Q_2 = Q$, we abbreviate C_{Q_1,Q_2} to C_Q.

Several well known classes of functions are $\mathfrak{Q}$-classes. In particular,

$$C_{\widetilde{J}} = \mathcal{D} := \{D \in \mathcal{G}L_\infty^{2\times 2} : D \text{ is diagonal}\}, \tag{2.14}$$

$$C_I = \{G \in \mathcal{G}L_\infty^{2\times 2} : G^T = \operatorname{adj} G\} \tag{2.15}$$

where $\operatorname{adj} G$ denotes the adjugate of G. In particular, C_I includes all invertible anti-symmetric and all SO_2-valued matrix functions.

We present now some simple properties of these classes.

Proposition 2.3. *The following relations hold:*

- **(i).** $C_{Q_1,Q_2} \cdot C_{Q_2,Q_3} \subset C_{Q_1,Q_3}$;
- **(ii).** $G \in C_{Q_1,Q_2} \Leftrightarrow G^{-1} \in C_{Q_2,Q_1} \Leftrightarrow G^T \in C_{Q_2^{-1},Q_1^{-1}}$;
- **(iii).** $C_{Q_1,Q_2} = C_{fQ_1,fQ_2}$ *for all* $f \in \mathcal{G}L_\infty$;
- **(iv).** $G \in C_{Q_1,Q_2} \Rightarrow fG \in C_{Q_1,Q_2}$ *for all* $f \in \mathcal{G}L_\infty$.

As an immediate consequence of (2.12)-(2.13), we have:

$$H_\rho \in C_{D_q,-\rho\widetilde{J}} \cap C_{\widetilde{J},-\widetilde{I}},$$
$$S_Q \in C_{-\rho\widetilde{J},Q}. \tag{2.16}$$

It is also easy to see that every $\mathfrak{Q}$-class is non-empty. In fact, if $(Q_1, Q_2) \in \mathfrak{Q}^{(2)}$ with $\det Q_1 = \det Q_2 = -q$, defining

$$X_{Q_1,Q_2} := S_{Q_1}^{-1}S_{Q_2}, \quad Y_{Q_1,Q_2} := \rho S_{Q_1}^{-1}\widetilde{I}S_{Q_2}, \tag{2.17}$$

and abbreviating X_{Q_1,Q_2} and Y_{Q_1,Q_2} to X and Y, respectively, whenever the corresponding pair (Q_1, Q_2) is clear, we have:

Theorem 2.4. $X, Y \in C_{Q_1,Q_2}$, *and the following relations hold:*

$$YX^{-1}Y = qX, \tag{2.18}$$

$$Q_1 = JYX^{-1}, \quad Q_2 = qJY^{-1}X. \tag{2.19}$$

Proof. By (2.13), and observing that $\det X = 1$, $\det Y = -q$, we have

$$X^T Q_1 X = S_{Q_2}^T (S_{Q_1}^{-1})^T Q_1 S_{Q_1}^{-1} S_{Q_2} = \frac{1}{2} S_{Q_2}^T \widetilde{J} S_{Q_2} = Q_2,$$

so that $X \in C_{Q_1,Q_2}$, and analogously

$$Y^T Q_1 Y = \frac{\rho^2}{2} S_{Q_2}^T \widetilde{I}\widetilde{J}\widetilde{I} S_{Q_2} = -\frac{q}{2} S_{Q_2}^T \widetilde{J} S_{Q_2} = -qQ_2$$

where we took (2.10) into account. The equality (2.18) is straightforward, and (2.19) follows from (2.10) and (2.13). □

We define moreover, for each $Q \in \mathfrak{Q}$,

$$\widetilde{S}_Q = \frac{1}{2} \tilde{q}_1^{-1} H_\rho \operatorname{diag}(1, \tilde{q}_1) S_Q \tag{2.20}$$

where

$$\tilde{q}_1 = q_1 \text{ if } q_1^{-1} \in L_\infty, \quad \tilde{q}_1 = 1 \text{ if } q_1 = 0.$$

We have

$$\tilde{S}_Q = q_1^{-1} \begin{bmatrix} q_1 & q_2 \\ 0 & 1 \end{bmatrix}, \text{ if } q_1^{-1} \in L_\infty, \tag{2.21}$$

$$\widetilde{S}_Q = \frac{1}{2} \begin{bmatrix} 1 & q_2(2 + \frac{q_3}{2q_2^2}) \\ -q_2^{-1} & 2 - \frac{q_3}{2q_2^2} \end{bmatrix}, \text{ if } q_1 = 0 \tag{2.22}$$

(remark that, if $q_1 = 0$, then $Q \in \mathcal{G}L_\infty^{2\times 2}$ implies that $q_2 \in \mathcal{G}L_\infty$), and in both cases

$$\det \widetilde{S}_Q = \tilde{q}_1^{-1}.$$

From (2.13), using (2.12), we obtain

$$Q = \tilde{q}_1 \widetilde{S}_Q^T D_q \widetilde{S}_Q \tag{2.23}$$

and thus

$$\widetilde{S}_Q \in C_{D_q,Q}. \tag{2.24}$$

From (2.13) and (2.23) we see that $S_Q^T(\tilde{q}_1^{-1}\widetilde{J})S_Q = 2\widetilde{S}_Q^T D_Q \widetilde{S}_Q$. This is a particular case of the following relation, which can be checked straightforwardly by using the definition (2.20) and the relation (2.12).

Proposition 2.5. *For any* $(Q_1, Q_2) \in \mathfrak{Q}^{(2)}$,

$$S_{Q_1}^T \begin{bmatrix} 0 & \tilde{q}_{11}^{-1} \\ \tilde{q}_{21}^{-1} & 0 \end{bmatrix} S_{Q_2} = 2\widetilde{S}_{Q_1}^T D_q \widetilde{S}_{Q_2}$$

with $q = -\det Q_1 = -\det Q_2$.

Using these results we can now obtain various descriptions for the matrix functions belonging to a given $\mathfrak{Q}$-class.

Theorem 2.6. *Let* $(Q_1, Q_2) \in \mathfrak{Q}^{(2)}$. *Then the following are equivalent:*

(i). $G \in C_{Q_1,Q_2}$

(ii). $G = S_{Q_1}^{-1} D S_{Q_2}$ *for some diagonal matrix* $D \in \mathcal{G}L_\infty^{2\times 2}$

(iii). $G = \widetilde{S}_{Q_1}^{-1} \hat{G} \widetilde{S}_{Q_2}$ *with* $\hat{G} \in C_{D_q}$

(iv). $G = \alpha X + \beta Y$ *with* $\alpha, \beta \in L_\infty$ *such that* $\alpha^2 - q\beta^2 \in \mathcal{G}L_\infty$.

Proof. (i)⇒(ii) Let $D := S_{Q_1} G S_{Q_2}^{-1}$; by (2.16) and Proposition 2.3, we have $D \in C_{-\rho\widetilde{J}} = C_{\widetilde{J}}$, therefore D is a diagonal matrix belonging to $\mathcal{G}L_\infty^{2\times 2}$ (see (2.14)). Conversely, if $D \in C_{\widetilde{J}}$ then $S_{Q_1}^{-1} D S_{Q_2} \in C_{Q_1,Q_2}$. The equivalence (i)⇔(iii) can be proved analogously, taking (2.24) into account. On the other hand, if $D = \mathrm{diag}(d_1, d_2)$ then we can write

$$D = \alpha I + \beta\rho\widetilde{I} \tag{2.25}$$

with $\alpha = \frac{d_1+d_2}{2}$, $\beta = \frac{d_1-d_2}{2\rho}$, so that $\alpha^2 - q\beta^2 = d_1 d_2 = \det D = \det G \in \mathcal{G}L_\infty^{2\times 2}$, and

$$G = S_{Q_1}^{-1} D S_{Q_2} = \alpha X + \beta Y. \tag{2.26}$$

Conversely, it is clear that if G takes the form (2.26) with $\alpha^2 - q\beta^2 \in \mathcal{G}L_\infty$, then $G \in \mathcal{G}L_\infty^{2\times 2}$ with

$$G^{-1} = \frac{1}{\alpha^2 - q\beta^2}(\alpha X^{-1} - q\beta Y^{-1}) \in C_{Q_2,Q_1} \tag{2.27}$$

and $G \in C_{Q_1,Q_2}$ by Proposition 2.3, (ii). Thus (ii) is equivalent to (iv). □

It is easy to see that the representations (ii)-(iv) in Theorem 2.6 are unique, for each G and each pair (Q_1, Q_2). We call $\hat{G}$ the *normal form* of G (with respect to C_{Q_1,Q_2}) if the relation (iii) of Theorem 2.6 holds; G is said to be of normal form if $G \in C_{D_q}$, for some $q \in \mathcal{G}L_\infty$, i.e.,

$$G = \alpha I + \beta R, \quad \text{with } R = Y_{D_q} = \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix}. \tag{2.28}$$

The case where $Q_1 = Q_2 = Q$ is of particular interest. The well-known class of Daniele-Khrapkov matrix functions ([17, 26]) is of this type; these matrices appear in problems from the areas of diffraction theory, acoustics, elastodynamics and integrable systems and have attracted a fair amount of interest in the literature ([8, 9, 11, 12, 13, 15, 20, 21, 22, 24, 27, 29, 30, 32]). In this case $X = I$, $Y = -JQ$, $\mathrm{tr}\, Y = 0$ and $Y^2 = qI$, and the results of Theorem 2.6 yield:

Corollary 2.7. *Let $Q \in \mathfrak{Q}$. Then the following are equivalent:*

$G \in C_Q$

$$G = S_Q^{-1} D S_Q \text{ for some diagonal } D \in \mathcal{G}L_\infty^{2\times 2} \tag{2.29}$$

$$G = \widetilde{S}_Q^{-1} \hat{G} \widetilde{S}_Q \text{ with } \hat{G} \in C_{D_q} \tag{2.30}$$

$$G = \alpha I + \beta Y \text{ with } \alpha, \beta \in L_\infty \text{ such that } \alpha^2 - q\beta^2 \in \mathcal{G}L_\infty. \tag{2.31}$$

It is clear from (2.31) that, defining

$$\mathcal{I} = \{\alpha I : \alpha \in \mathcal{G}L_\infty\}, \tag{2.32}$$

we have $\mathcal{I} \subsetneq C_Q$, for all $Q \in \mathfrak{Q}$. The following result shows that no other diagonal matrices belong to C_Q, unless Q has a particular form.

Theorem 2.8. *Let $G \in \mathcal{G}L_\infty^{2\times 2}$ be diagonal, $G \notin \mathcal{I}$. Then $G \in C_Q$ if and only if $Q = q_2\widetilde{J}$ with $q_2 \in \mathcal{G}L_\infty$.*

Proof. Let $G = \mathrm{diag}(a, b) \in C_Q$, with $a \neq b$. Then, for Q given by (2.1), the relation $G^T QG = \det G \cdot Q$ implies that

$$\begin{bmatrix} a^2q_1 & abq_2 \\ abq_2 & b^2q_3 \end{bmatrix} = ab \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix},$$

so that we must have $aq_1(a - b) = 0$ and $bq_3(a - b) = 0$. Since $a \neq b$, $a \neq 0$, $b \neq 0$ a.e., we must have $q_1 = q_3 = 0$ and, taking into account that $\det Q = -q_2^2 \in \mathcal{G}L_\infty$, it follows that $Q = q_2\widetilde{J}$ with $q_2 \in \mathcal{G}L_\infty$. Conversely, if $Q = q_2\widetilde{J}$, then $G \in C_Q$ by (2.14) and Proposition 2.3(iii). □

C_Q is also related with the space of solutions of the equation

$$L^T Q + QL = 0 \tag{2.33}$$

which is relevant in the study of Lie algebras and is studied in some recent papers ([18, 25]).

We have the following.

Theorem 2.9. *Let $\mathcal{L} \subset L_\infty^{2\times 2}$ be the space of solutions of (2.33) for a given $Q \in \mathfrak{Q}$. Then $\exp \mathcal{L} \subset C_Q$. If, in addition, $L \in \mathcal{G}L_\infty^{2\times 2}$, then $L \in C_Q$.*

Proof. Let (2.33) hold. Then QL is a skew-symmetric matrix function, i.e., $QL = aJ$ for some $a \in L_\infty$. On the other hand, from the equality

$$AJA^T = \det A \cdot J, \tag{2.34}$$

valid for any 2×2 matrix A, we have $q^{-1}JQJ = Q^{-1}$, so that

$$L = Q^{-1}(QL) = -q^{-1}aJQ$$

and, for all $n \in \mathbb{N}$,

$$L^{2n} = q^{-n}a^{2n}I, \quad L^{2n+1} = -q^{-n}a^{2n+1}JQ.$$

Thus,

$$\exp L = \sum_{n=0}^{\infty} \frac{L^n}{n!} = \sum_{n=0}^{\infty} \frac{q^{-n}a^{2n}}{(2n)!} I - \sum_{n=0}^{\infty} \frac{q^{-n-1}a^{2n+1}}{(2n+1)!} JQ$$

$$= \cosh(\rho^{-1}a)I + \rho^{-1}\sinh(\rho^{-1}a)Y$$

which is of the form (2.31) with $\alpha^2 - q\beta^2 = 1 \in \mathcal{G}L_\infty$. Thus, by Corollary 2.7, $\exp L \in C_Q$.

Since $QL = aJ$ and $L^T J = \det L \cdot JL^{-1}$ by (2.34), for $L \in \mathcal{G}L_\infty^{2\times 2}$, we have $L^T QL = \det L \cdot Q$; thus $L \in C_Q$. □

Remark 2.10. Matrix functions belonging to a family of exponentials of rational matrices, of the form $\exp(tL)$ with $t \in \mathbb{R}$ and $L \in \mathcal{R}^{2\times 2}$ satisfying an equality of the form (2.33), appear in the study of finite dimensional integrable systems defined by certain Lax equations, see for instance [14, 15].

3. Intersection and equality of $\mathfrak{Q}$-classes

It is clear from Proposition 2.3 (iii) that different pairs (Q_1, Q_2) and $(\widetilde{Q}_1, \widetilde{Q}_2)$ belonging to $\mathfrak{Q}^{(2)}$ may define equal classes C_{Q_1,Q_2} and $C_{\widetilde{Q}_1,\widetilde{Q}_2}$. A natural question is whether two $\mathfrak{Q}$-classes corresponding to different pairs (Q_1, Q_2) and $(\widetilde{Q}_1, \widetilde{Q}_2)$, for which there exists no $f \in \mathcal{G}L_\infty$ such that $Q_j = f\widetilde{Q}_j$, $j = 1, 2$, may be equal and, if not, how to describe their intersection.

These questions were addressed, and partially answered, in [15]. If $Q_1 = Q_2$ and $\widetilde{Q}_1 = \widetilde{Q}_2$, we have the following.

Theorem 3.1. *([15]) For any $Q, \widetilde{Q} \in \mathfrak{Q}$, the classes C_Q and $C_{\widetilde{Q}}$ are not disjoint. We have $\mathcal{I} \subset C_Q \cap C_{\widetilde{Q}}$, and either $C_Q \cap C_{\widetilde{Q}} = \mathcal{I}$ or $C_Q = C_{\widetilde{Q}}$. The latter equality holds if and only if $Q = f\widetilde{Q}$ with $f \in \mathcal{G}L_\infty$.*

Thus a matrix function cannot belong to two different classes C_Q and $C_{\widetilde{Q}}$, unless it is a scalar multiple of the identity. This situation changes when we consider C_{Q_1,Q_2} with $Q_1 \neq Q_2$. To prove this we use the following result.

Theorem 3.2. *([13]) Let G_0 be any element of C_{Q_1,Q_2}. Then $C_{Q_1,Q_2} = C_{Q_1} \cdot G_0 = G_0 \cdot C_{Q_2}$.*

Theorem 3.3. *Let (Q_1, Q_2) and $(\widetilde{Q}_1, \widetilde{Q}_2) \in \mathfrak{Q}^{(2)}$. Then one and only one of the following propositions is true:*

- **(i).** $C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2} = \emptyset$;
- **(ii).** *There exists $G_0 \in \mathcal{G}L_\infty^{2\times 2}$ such that $C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2} = \{fG_0 : f \in \mathcal{G}L_\infty\}$;*
- **(iii).** $C_{Q_1,Q_2} = C_{\widetilde{Q}_1,\widetilde{Q}_2}$.

Proof. Suppose that (i) is not true, and there exists $G_0 \in C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2}$. Then, by Theorem 3.2,

$$C_{Q_1,Q_2} = C_{Q_1} \cdot G_0 = G_0 \cdot C_{Q_2}$$

$$C_{\widetilde{Q}_1,\widetilde{Q}_2} = C_{\widetilde{Q}_1} \cdot G_0 = G_0 \cdot C_{\widetilde{Q}_2}.$$

Thus, if $\widetilde{G}_0$ is any element of $C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2}$, by Proposition 2.3 we have

$$\widetilde{G}_0 G_0^{-1} \in C_{Q_1} \cap C_{\widetilde{Q}_1}, \quad G_0^{-1}\widetilde{G}_0 \in C_{Q_2} \cap C_{\widetilde{Q}_2}.$$

It follows from Theorem 3.1 that either $\widetilde{G}_0$ is of the form $\widetilde{G}_0 = fG_0$ with $f \in \mathcal{G}L_\infty$ and (ii) holds, or we have $Q_1 = f_1\widetilde{Q}_1$, $Q_2 = f_2\widetilde{Q}_2$ with $f_1, f_2 \in \mathcal{G}L_\infty$. In the latter case, the relations

$$G_0^T Q_1 G_0 = \det G_0 \cdot Q_2 \quad , \quad G_0^T \widetilde{Q}_1 G_0 = \det G_0 \cdot \widetilde{Q}_2$$

imply that $f_1 = f_2$, and it follows from Proposition 2.3, (iii) that $C_{Q_1,Q_2} = C_{\widetilde{Q}_1,\widetilde{Q}_2}$. On the other hand, (i) obviously cannot hold simultaneously with either (ii) or (iii), while (ii) and (iii) cannot hold simultaneously because $X, Y \in C_{Q_1,Q_2}$ (cf. Theorem 2.4) and X, Y cannot be both of the form fG_0 with $f \in \mathcal{G}L_\infty$, for the same matrix function G_0. □

The following example shows that, if $Q_1 \neq Q_2$ or $\widetilde{Q}_1 \neq \widetilde{Q}_2$, we can indeed have $C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2} = \emptyset$ as in Theorem 3.3 (i). Take Q_1 and Q_2 such that $S_{Q_1} S_{Q_2}^{-1} \notin \mathcal{D}$, where we denote by $\mathcal{D}$ the class of all diagonal matrices in $\mathcal{G}L_\infty$. Then

$$G \in C_{Q_1} \cap C_{Q_1,Q_2} \Leftrightarrow G = S_{Q_1}^{-1} D S_{Q_1} = S_{Q_1}^{-1} \widetilde{D} S_{Q_2}$$

with $D, \widetilde{D} \in \mathcal{D}$ by Theorem 2.6. It follows that $D^{-1}\widetilde{D} = S_{Q_1} S_{Q_2}^{-1}$, which is impossible because $S_{Q_1} S_{Q_2}^{-1} \notin \mathcal{D}$ by assumption. Therefore C_{Q_1} and C_{Q_1,Q_2} must be disjoint.

From Theorem 3.3 we now obtain necessary and sufficient conditions for two $\mathfrak{Q}$-classes to be equal.

Theorem 3.4. *If (Q_1, Q_2) and $(\widetilde{Q}_1, \widetilde{Q}_2)$ belong to $\mathfrak{Q}^{(2)}$, then the following are equivalent:*

- **(i).** $C_{Q_1,Q_2} = C_{\widetilde{Q}_1,\widetilde{Q}_2}$
- **(ii).** $X_{Q_1,Q_2}, Y_{Q_1,Q_2} \in C_{\widetilde{Q}_1,\widetilde{Q}_2}$
- **(iii).** *there exist $G, \widetilde{G} \in C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2}$ with $G^{-1}\widetilde{G} \notin \mathcal{I}$*
- **(iv).** $\widetilde{Q}_1^{-1} Q_1 = \widetilde{Q}_2^{-1} Q_2 = fI$ *with $f \in \mathcal{G}L_\infty$.*

Proof. From Theorem 2.6, (i)⇒(ii)⇒(iii). Conversely, if (iii) holds, then by Theorem 3.2 we have

$$C_{Q_2} = G^{-1} \cdot C_{Q_1,Q_2}, \quad C_{\widetilde{Q}_2} = G^{-1} \cdot C_{\widetilde{Q}_1,\widetilde{Q}_2}. \tag{3.1}$$

By Proposition 2.3 (i), $G^{-1}\widetilde{G} \in C_{Q_2} \cap C_{\widetilde{Q}_2}$ and, since $G^{-1}\widetilde{G} \notin \mathcal{I}$, we have $C_{Q_2} = C_{\widetilde{Q}_2}$ by Theorem 3.1. Thus we conclude from (3.1) that $C_{Q_1,Q_2} = C_{\widetilde{Q}_1,\widetilde{Q}_2}$, i.e., (iii)⇒(i). On the other hand, (iv)⇒(i) by Proposition 2.3, (iii), and we can show that (i)⇒(iv) as follows. $Y_{Q_1,Q_2} X_{Q_1,Q_2}^{-1} = -JQ_1 \notin \mathcal{I}$, but $Y_{Q_1,Q_2} X_{Q_1,Q_2}^{-1} \in C_{Q_1} \cap C_{\widetilde{Q}_1}$, so that $C_{Q_1} = C_{\widetilde{Q}_1}$ and $Q_1 = f_1 \widetilde{Q}_1$ with $f_1 \in \mathcal{G}L_\infty$ by Theorem 3.1. We conclude analogously that $Q_2 = f_2 \widetilde{Q}_2$ with $f_2 \in \mathcal{G}L_\infty$. Following the same reasoning as in the proof of Theorem 3.3, with G_0 replaced by $Y_{Q_1,Q_2} X_{Q_1,Q_2}^{-1}$, we conclude that $f_1 = f_2$. □

Remark that, given two pairs (Q_1, Q_2) and $(\widetilde{Q}_1, \widetilde{Q}_2)$ in $\mathfrak{Q}^{(2)}$ such that $C_{Q_1,Q_2} \neq C_{\widetilde{Q}_1,\widetilde{Q}_2}$, the intersection $C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2}$ can be determined using (ii) in Theorem 2.6. In fact, using the notation

$$A_1 = S_{\widetilde{Q}_1} S_{Q_1}^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \widetilde{A} = S_{\widetilde{Q}_2} S_{Q_2}^{-1} = \begin{bmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{bmatrix},$$

the elements of $C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2}$ are determined by the solutions $D, \widetilde{D} \in \mathcal{D}$ of

$$S_{Q_1}^{-1} D S_{Q_2} = S_{\widetilde{Q}_1}^{-1} \widetilde{D} S_{\widetilde{Q}_2}. \tag{3.2}$$

For $D = \operatorname{diag}(d_1, d_2)$ and $\widetilde{D} = \operatorname{diag}(\widetilde{d}_1, \widetilde{d}_2)$, and taking $d_1, d_2, \tilde{d}_1, \tilde{d}_2$ as unknowns, we have then

$$\begin{bmatrix} a_{11} & a_{12} & -\tilde{a}_{11}-\tilde{a}_{12} & 0 \\ a_{21} & a_{22} & 0 & -\tilde{a}_{21}-\tilde{a}_{22} \\ a_{11} & 0 & -\tilde{a}_{11} & 0 \\ 0 & a_{22} & 0 & -\tilde{a}_{22} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = A_1^{-1} \begin{bmatrix} \tilde{a}_{11}+\tilde{a}_{12} & 0 \\ 0 & \tilde{a}_{21}+\tilde{a}_{22} \end{bmatrix} \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix}$$

$$\left(\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} A_1^{-1} \begin{bmatrix} \tilde{a}_{11}+\tilde{a}_{12} & 0 \\ 0 & \tilde{a}_{21}+\tilde{a}_{22} \end{bmatrix} - \begin{bmatrix} \tilde{a}_{11} & 0 \\ 0 & \tilde{a}_{22} \end{bmatrix} \right) \begin{bmatrix} \tilde{d}_1 \\ \tilde{d}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

4. C_{Q_1,Q_2} and the product equation

It was shown in [13] that if $G \in C_{Q_1,Q_2}$ then we have

$$\det G \cdot (\phi_+)^T Q_2 \phi_+ = (\phi_-)^T Q_1 \phi_- \tag{4.1}$$

for any ϕ_+, ϕ_- such that $G\phi_+ = \phi_-$. The equality (4.1) was called *product equation*, since it could be obtained by multiplying both sides of two equalities that were derived using the representation (ii) of Theorem 2.6, and it was used in [9, 11, 12, 13] to study several factorization and RH problems.

We will prove here that (4.1), considered in a more general setting, is in fact a necessary and sufficient condition for G to belong to C_{Q_1,Q_2}, showing through examples that it can indeed be used to answer the question of how to characterize $\mathfrak{Q}$-classes containing a given matrix function G.

Theorem 4.1. *$G \in C_{Q_1,Q_2}$ if and only if the equality*

$$\det G \cdot \phi^T Q_2 \phi = \psi^T Q_1 \psi \tag{4.2}$$

holds for every (ϕ, ψ) such that

$$G\phi = \psi. \tag{4.3}$$

Proof. If $G \in C_{Q_1,Q_2}$ then, for all (ϕ, ψ) such that (4.3) holds,

$$\psi^T Q_1 \psi = \phi^T G^T Q_1 G \phi = \det G \cdot \phi^T Q_2 \phi.$$

Conversely, suppose that (4.2) holds for every (ϕ, ψ) such that (4.3) holds. We have $G \cdot \operatorname{adj} G = \det G \cdot I$ so that, denoting by Φ_1 and Φ_2 the two columns of $\operatorname{adj} G$,

$$G\Phi_1 = \det G \cdot \Psi_1, \quad G\Phi_2 = \det G \cdot \Psi_2$$

with $\Psi_1 = [1\ 0]^T$, $\Psi_2 = [0\ 1]^T$. Thus $G(\Phi_1 + \Phi_2) = \det G \cdot (\Psi_1 + \Psi_2)$ and, taking $\phi = \Phi_1 + \Phi_2$ and $\psi = \det G \cdot (\Psi_1 + \Psi_2)$ in (4.2), we have

$$\det G \cdot (\Psi_1 + \Psi_2)^T Q_1 (\Psi_1 + \Psi_2) = (\Phi_1 + \Phi_2)^T Q_2 (\Phi_1 + \Phi_2)$$

which, taking into account that we also have, from (4.3),

$$\det G \cdot \Psi_1^T Q_1 \Psi_1 = \Phi_1^T Q_2 \Phi_1, \quad \det G \cdot \Psi_2^T Q_1 \Psi_2 = \Phi_2^T Q_2 \Phi_2, \tag{4.4}$$

implies that

$$\det G\cdot(\Psi_1^T Q_1\Psi_2+\Psi_2^T Q_1\Psi_1)=\Phi_1^T Q_2\Phi_2+\Phi_2^T Q_2\Phi_1. \qquad (4.5)$$

Now, $\Psi_1^T Q_1\Psi_2$ is scalar and Q_1 is symmetric, so $\Psi_1^T Q_1\Psi_2=\Psi_2^T Q_1\Psi_1$ and, analogously, $\Phi_1^T Q_2\Phi_2=\Phi_2^T Q_2\Phi_1$. Thus, it follows from (4.5) that

$$\det G\cdot\Psi_1^T Q_1\Psi_2=\Phi_1^T Q_2\Phi_2=\Phi_2^T Q_2\Phi_1=\det G\cdot\Psi_2^T Q_1\Psi_1. \qquad (4.6)$$

Recalling that $[\Psi_1\ \Psi_2]=I$ and $[\Phi_1\ \Phi_2]=\operatorname{adj}G$, we have from (4.4)-(4.6):

$$\det G\cdot Q_1=\det G\cdot[\Psi_1\ \Psi_2]^T Q_1[\Psi_1\ \Psi_2]=\det G\cdot\begin{bmatrix}\Psi_1^T Q_1\Psi_1 & \Psi_1^T Q_1\Psi_2\\ \Psi_2^T Q_1\Psi_1 & \Psi_2^T Q_1\Psi_2\end{bmatrix}=$$

$$=\begin{bmatrix}\Phi_1^T Q_2\Phi_1 & \Phi_1^T Q_2\Phi_2\\ \Phi_2^T Q_2\Phi_1 & \Phi_2^T Q_2\Phi_2\end{bmatrix}=(\operatorname{adj}G)^T Q_2(\operatorname{adj}G).$$

Therefore $\det G\cdot Q_1=(\det G)^2(G^{-1})^T Q_2 G^{-1}$, i.e. $G^T Q_1 G=\det G\cdot Q_2$. □

Example. Take, for example,

$$G=\begin{pmatrix}1 & -s\\ s^{-1} & 1\end{pmatrix} \qquad (4.7)$$

with $s=is_-r^{k+\frac{m}{2}}s_+$, where

$$k\in\mathbb{Z},\ m\in\{0,1\},\ s_-^{\pm1}\in H_\infty^-,\ s_+^{\pm1}\in H_\infty^+, \qquad (4.8)$$

and r is given by (1.4). Matrix functions of this form were studied, for instance in [12, 24, 27].

It is clear, by (2.28), that

$$G\in C_{D_q}\quad\text{with } q=-s^2. \qquad (4.9)$$

On the other hand, for $\phi=(\phi_1,\phi_2)$ and $\psi=(\psi_1,\psi_2)$,

$$G\phi=\psi\Leftrightarrow\begin{cases}\phi_1-s\phi_2=\psi_1\\ s^{-1}\phi_1+\phi_2=\psi_2\end{cases}\Leftrightarrow\begin{cases}\phi_1-s\phi_2=\psi_1\\ s_+^{-1}(\phi_1+s\phi_2)=ir^{k+\frac{m}{2}}s_-\psi_2\end{cases}.$$

Multiplying the last two equations we obtain

$$ir^{k+\frac{m}{2}}s_-\psi_1\psi_2=s_+^{-1}(\phi_1^2-s^2\phi_2^2), \qquad (4.10)$$

which is equivalent to

$$\psi^T(ir^{k+\frac{m}{2}}s_-\widetilde{J})\psi=\det G\cdot\phi^T(s_+^{-1}\widetilde{I}D_q)\phi\,,$$

taking into account that $s^2=-q$ and $\det G=2$. Thus we also have, from Theorem 4.1,

$$G\in C_{Q_1,Q_2}\quad\text{with } Q_1=ir^{k+\frac{m}{2}}s_-\widetilde{J},\ Q_2=s_+^{-1}\widetilde{I}D_q. \qquad (4.11)$$

Remarking that $G\phi = \psi \Leftrightarrow \phi = G^{-1}\psi$ and $G^{-1} = \frac{1}{2}\widetilde{I}G\widetilde{I}$, we have moreover, replacing ϕ by $\widetilde{I}\psi$ and ψ by $2\widetilde{I}\phi$ in (4.3),

$$\begin{aligned} G\phi = \psi \Leftrightarrow\ & G\widetilde{I}\psi = 2\widetilde{I}\phi \\ \Leftrightarrow\ & \phi^T(2r^{k+\frac{m}{2}}s_-\widetilde{I}\widetilde{J}\widetilde{I})\phi = \psi^T(-is_+^{-1}\widetilde{I}\widetilde{I}D_q\widetilde{I})\psi \\ \Leftrightarrow\ & \psi^T(s_-^{-1}\widetilde{I}D_q)\psi = \det G\cdot\phi^T(-ir^{k+\frac{m}{2}}s_+\widetilde{J})\phi\,, \end{aligned} \tag{4.12}$$

so that

$$G \in C_{\widetilde{Q}_1,\widetilde{Q}_2}, \quad \text{with } \widetilde{Q}_1 = s_-^{-1}\widetilde{I}D_q,\ \widetilde{Q}_2 = -ir^{k+m/2}s_+\widetilde{J}. \tag{4.13}$$

Thus, from (4.9), (4.11) and (4.13),

$$G \in C_{D_q} \cap C_{Q_1,Q_2} \cap C_{\widetilde{Q}_1,\widetilde{Q}_2}.$$

Remark that, for any pair $\mathcal{C}_1, \mathcal{C}_2$ of elements in the set $\{C_{D_q}, C_{Q_1,Q_2}, C_{\widetilde{Q}_1,\widetilde{Q}_2}\}$, we have $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$ and $\mathcal{C}_1 \neq \mathcal{C}_2$ (since (iv) in Theorem 3.4 obviously doesn't hold). Therefore we conclude by Theorem 3.3 that $\mathcal{C}_1 \cap \mathcal{C}_2 = \{fG : f \in \mathcal{G}L_\infty\}$ (note that the same result would be obtained by solving (3.2)).

5. The $\mathfrak{Q}$-classes and factorization

We now study several possible representations of the elements of a $\mathfrak{Q}$-class as products, having in mind the factorization theory in the context of which this study arose.

We begin with a simple but fundamental relation.

Theorem 5.1. *Let $(Q_1, Q_2) \in \mathfrak{Q}^{(2)}$ and $G \in C_{Q_1,Q_2}$. Then*

$$GJQ_2 = JQ_1G \in C_{Q_1,Q_2}. \tag{5.1}$$

Proof. By Theorems 2.4 and 2.6,

$$GJQ_2 = (\alpha X + \beta Y)J(qJY^{-1}X) = -q\alpha XY^{-1}X - \beta qX = -\alpha Y - \beta qX.$$

Analogously,

$$JQ_1G = J(JYX^{-1})(\alpha X + \beta Y) = -\alpha Y - \beta YX^{-1}Y = -\alpha Y - \beta qX.$$

□

Corollary 5.2. *For all $Q \in \mathfrak{Q}$, we have*

$$\widetilde{S}_QJQ = JD_q\widetilde{S}_Q. \tag{5.2}$$

Proof. This is a direct consequence of Theorem 5.1, taking into account that $\widetilde{S}_Q \in C_{D_q,Q}$ by (2.24). □

Corollary 5.3. *Let $G \in C_{Q_1,Q_2}$ and let $G\phi = \psi$. Then*

$$G(JQ_2\phi) = JQ_1\psi. \tag{5.3}$$

Thus, if we have two 2×1 vector functions ϕ, ψ such that $G\phi = \psi$, then we can write

$$G[\phi \ \ -JQ_2\phi] = [\psi \ \ -JQ_1\psi].$$

Let

$$M_\phi^{Q_2} := [\phi \ \ -JQ_2\phi], \quad M_\psi^{Q_1} := [\psi \ \ -JQ_1\psi]. \tag{5.4}$$

Using the relation (2.34), valid for any 2×2 matrix, and taking into account that, if A_1 and A_2 are two columns in A,

$$A_1^T JA_2 = -A_2^T JA_1 = \det A, \quad A_1^T JA_1 = A_2^T JA_2 = 0,$$

we obtain

$$\det(M_\phi^{Q_2}) = \phi^T J(-JQ_2\phi) \ = \ \phi^T Q_2\phi \tag{5.5}$$

$$\det(M_\psi^{Q_1}) = \psi^T J(-JQ_1\psi) \ = \ \psi^T Q_1\psi. \tag{5.6}$$

Furthermore, we have the following:

Theorem 5.4. *For every $G \in C_{Q_1,Q_2}$, there exists a solution to*

$$G\phi = \psi \tag{5.7}$$

with $\phi, \psi \in (L_\infty)^2$, such that

$$\alpha := \psi^T Q_1\psi = \det G \cdot \phi^T Q_2\phi \ \in \mathcal{G}L_\infty. \tag{5.8}$$

Proof. From (2.23) we have

$$(\widetilde{S}_{Q_1}^{-1})^T Q_1 \widetilde{S}_{Q_1}^{-1} = \widetilde{q}_1 D_q,$$

so taking ψ equal to the first column of $\widetilde{S}_{Q_1}^{-1}$, we have $\alpha = \psi^T Q_1\psi = \tilde{q}_1 \in \mathcal{G}L_\infty$ and, for $\phi = G^{-1}\psi$, we see that (5.7) and (5.8) are satisfied. □

Theorem 5.5. *If $G \in C_{Q_1,Q_2}$ and ϕ, ψ satisfy (5.7) and (5.8), then*

$$G = M_\psi^{Q_1}(M_\phi^{Q_2})^{-1},$$

where $M_\psi^{Q_1}$ and $M_\phi^{Q_2}$ are given by (5.4) and

$$M_\psi^{Q_1} \in C_{Q_1,D_q}, \quad M_\phi^{Q_2} \in C_{Q_2,D_q}.$$

Proof. Let $G_{01} := \widetilde{S}_{Q_1}M_\psi^{Q_1}$, $G_{02} := \widetilde{S}_{Q_2}M_\phi^{Q_2}$. We have $\widetilde{S}_{Q_1}M_\psi^{Q_1} = [\widetilde{S}_{Q_1}\psi \ \ -JD_q\widetilde{S}_{Q_1}\psi]$ from Corollary 5.2. On the other hand, if $\widetilde{S}_{Q_1}\psi = (s_1, s_2)$ then

$$[\widetilde{S}_{Q_1}\psi \ \ -JD_q\widetilde{S}_{Q_1}\psi] = s_1 I + s_2 R$$

with $R = -JD_q$. Therefore $G_{01} \in C_{D_q}$ by (2.28). We conclude that $G_{02} \in C_{D_q}$ analogously.

Since $G_{01} \in C_{D_q}$ and $\widetilde{S}_{Q_1} \in C_{D_q,Q_1}$, we have $M_\psi^{Q_1} \in C_{Q_1,D_q}$ by Proposition 2.3 and, similarly, $M_\phi^{Q_2} \in C_{Q_2,D_q}$. The factorization for G now follows from (5.1), (5.3) and (5.5)-(5.8). □

From this we conclude that, not only $C_{Q_1,D_q} \cdot C_{D_q,Q_2} \subset C_{Q_1,Q_2}$, as it follows from Proposition 2.3, but also the converse inclusion is true, i.e.

$$C_{Q_1,Q_2} = C_{Q_1,D_q} \cdot C_{D_q,Q_2}$$

Moreover, since for every $Q_3 \in \mathfrak{Q}$ such that $\det Q_3 = -q = \det Q_j$ $(j = 1, 2)$ we have $\widetilde{S}_{Q_3} \in C_{D_q,Q_3}$, we see that every $G \in C_{Q_1,Q_2}$ can be represented as a product

$$G = (M_\psi^{Q_1} \widetilde{S}_{Q_3})(\widetilde{S}_{Q_3}^{-1}(M_\phi^{Q_2})^{-1}) \tag{5.9}$$

with $M_\psi^{Q_1} \widetilde{S}_{Q_3} \in C_{Q_1,Q_3}$ and $\widetilde{S}_{Q_3}^{-1}(M_\phi^{Q_2})^{-1} \in C_{Q_3,Q_2}$. We have thus proved the following.

Theorem 5.6. *Let $Q_1, Q_2, Q_3 \in \mathfrak{Q}$ be such that* $\det Q_1 = \det Q_2 = \det Q_3$. *Then every $G \in C_{Q_1,Q_2}$ admits a factorization* (5.9) *with $M_\psi^{Q_1} \widetilde{S}_{Q_3} \in C_{Q_1,Q_3}$ and $\widetilde{S}_{Q_3}^{-1}(M_\phi^{Q_2})^{-1} \in C_{Q_3,Q_2}$, and we have*

$$C_{Q_1,Q_2} = C_{Q_1,Q_3} \cdot C_{Q_3,Q_2}. \tag{5.10}$$

Finally, we present here a factorization result whose meaningfulness will become apparent in section 7.

Theorem 5.7. *Every $G \in C_{Q_1,Q_2}$ admits a factorization*

$$G = (M_\psi^{Q_1} H_\rho S_{Q_2}) X^{-1} (M_\phi^{Q_2} H_\rho S_{Q_1})^{-1}$$

with $M_\psi^{Q_1} H_\rho S_{Q_2}$, $(M_\phi^{Q_2} H_\rho S_{Q_1})^{-1} \in C_{Q_1,Q_2}$ and

$$C_{Q_1,Q_2} = C_{Q_1,Q_2} \cdot X^{-1} \cdot C_{Q_1,Q_2}. \tag{5.11}$$

Proof. We have from (5.9)

$$G = (M_\psi^{Q_1} H_\rho S_{Q_2})(S_{Q_2}^{-1} S_{Q_1})(M_\phi^{Q_2} H_\rho S_{Q_1})^{-1} =$$

$$= (M_\psi^{Q_1} H_\rho S_{Q_2}) X^{-1} (M_\phi^{Q_2} H_\rho S_{Q_1})^{-1}$$

and, since $M_\psi^{Q_1} H_\rho S_{Q_2}$, $(M_\phi^{Q_2} H_\rho S_{Q_1})^{-1} \in C_{Q_1,Q_2}$, (5.11) holds. □

6. $\mathfrak{MQ}$-classes and applications

Having in mind the application of the results of the previous sections to the study of RH problems - either of vectorial or of 2×2 matricial type, such as (1.6) and (1.1), respectively - and to the study of Toeplitz operators, we now consider Q_1 and Q_2 with entries in some concrete spaces of analytic or meromorphic functions. In what follows we assume that, unless stated otherwise,

$$Q_1 \in \mathcal{G}(\mathcal{M}_\infty^-)^{2\times 2}, \quad \text{with } q_{11} \in \mathcal{G}\mathcal{M}_\infty^- \text{ or } q_{11} = 0, \tag{6.1}$$

$$Q_2 \in \mathcal{G}(\mathcal{M}_\infty^+)^{2\times 2}, \quad \text{with } q_{21} \in \mathcal{G}\mathcal{M}_\infty^+ \text{ or } q_{21} = 0. \tag{6.2}$$

$\mathfrak{Q}$-classes C_{Q_1,Q_2} with Q_1, Q_2 satisfying those conditions are called $\mathfrak{MQ}$-classes.

In this case, from (2.21) and (2.22) we also have

$$\widetilde{S}_{Q_1} \in \mathcal{G}(\mathcal{M}_\infty^-)^{2\times 2}, \quad \widetilde{S}_{Q_2} \in \mathcal{G}(\mathcal{M}_\infty^+)^{2\times 2} \tag{6.3}$$

and (cf. (2.2))

$$q \in \mathcal{R}, \tag{6.4}$$

since $q = -\det Q_1 = -\det Q_2$ with $\det Q_1 \in \mathcal{M}_\infty^-$ and $\det Q_2 \in \mathcal{M}_\infty^+$. Clearly, we will moreover have $S_{Q_1} \in \mathcal{G}(\mathcal{M}_\infty^-)^{2\times 2}, S_{Q_2} \in \mathcal{G}(\mathcal{M}_\infty^+)^{2\times 2}$ if $s = q^{1/2} \in \mathcal{R}$.

Theorem 6.1. *Let $G \in C_{Q_1,Q_2}$, with $\det Q_1 = \det Q_2 = -s^2 \in \mathcal{R}$. Then the representation $G = S_{Q_1}^{-1} D S_{Q_2}$ of Theorem 2.6 (ii) is an $\mathcal{M}$-bounded factorization for G and T_G is nearly Fredholm equivalent to T_D.*

Proof. It follows from (6.1), (6.2) and (2.3) that $S_{Q_1} \in \mathcal{G}(\mathcal{M}_\infty^-)^{2\times 2}$, $S_{Q_2} \in \mathcal{G}(\mathcal{M}_\infty^+)^{2\times 2}$; thus $G = S_{Q_1}^{-1} D S_{Q_2}$ is an $\mathcal{M}$-bounded factorization. Now from Theorem 3.10 in [28] (see also [20]), we conclude that T_G is nearly Fredholm equivalent to T_D. □

Daniele-Khrapkov matrices $G \in C_Q$ with $\det Q = -s^2$, $s \in \mathcal{R}$ ([20, 22, 30, 32]) satisfy the conditions of this theorem, and a meromorphic factorization for these matrix functions can easily be obtained by (scalar) *WH*-factorization of the diagonal elements in D. Remark, however, that Theorem 6.1 can also be applied in cases where G may not admit a *WH*-factorization, as shown in the following examples.

Example. Let

$$G = \begin{bmatrix} d_1 & q_+ d_1 + q_- d_2 \\ 0 & d_2 \end{bmatrix}, \tag{6.5}$$

with $d_1, d_2 \in \mathcal{G}L_\infty$, $q_\pm \in H_\infty^\pm$.

Since any solution to $G\phi = \psi$, with $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ satisfies the product equation

$$d_1 d_2 (\phi_1 + q_+ \phi_2) \phi_2 = (\psi_1 - q_- \psi_2) \psi_2, \tag{6.6}$$

which is equivalent to (4.2) with

$$Q_1 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & -q_- \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & q_+ \end{bmatrix}, \tag{6.7}$$

we see that $G \in C_{Q_1,Q_2}$ with $Q_1 \in \mathcal{G}(H_\infty^-)^{2\times 2}, Q_2 \in \mathcal{G}(H_\infty^+)^{2\times 2}$ (cf. Theorem 4.1). It follows from Theorem 2.6 (ii) that

$$G = \begin{bmatrix} q_- & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} d_2 & 0 \\ 0 & d_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & q_+ \end{bmatrix} \tag{6.8}$$

which is an $\mathcal{M}$-bounded factorization of G.

It is worth noting that, if $d_1 = e_\lambda$, $d_2 = e_{-\lambda}$ with $\lambda \in \mathbb{R}$, the Toeplitz operator T_G is equivalent, or at least closely related, to a finite interval convolution operator, λ being the length of the corresponding interval ([3, 23]). More generally, if $d_1, d_2 \in AP$ and additionally $q_\pm \in AP^\pm$, then (6.8) reduces the AP-factorization of G to that of d_1 and d_2.

We can see moreover, as a consequence of Theorem 2.6 (ii), that (6.5) gives the general form of all matrix functions in C_{Q_1,Q_2} with Q_1, Q_2 satisfying (6.1) and such that $q_{11} = q_{21} = 0$ as in (6.7).

Example. Let $G = \alpha D + \beta A$, where $\alpha, \beta \in L_\infty$ with $\alpha^2 + \beta^2 \in \mathcal{G}L_\infty$, and D and A are diagonal and anti-diagonal, respectively, of the form

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -d_1 d_2 \\ 1 & 0 \end{bmatrix}, \tag{6.9}$$

for some $d_1, d_2 \in \mathcal{G}L_\infty$ (see [13], section 3). The class of all matrices of this form includes the Daniele-Khrapkov subclass C_{D_q} with $q \in \mathcal{GR}$ (take $d_1 = d_2$ and $d_1^2 = q$).

Assume moreover that $d_1 \in \mathcal{GM}_\infty^-$, $d_2 \in \mathcal{GM}_\infty^+$. Then we have $G \in C_{Q_1,Q_2}$ with $Q_j = \mathrm{diag}(d_j^{-1}, d_j)$, $j = 1, 2$ where Q_1 and Q_2 satisfy (6.1). Remark that $\det Q_1 = \det Q_2 = 1$. From Theorem 2.6 (ii),

$$G = \frac{1}{2} \begin{bmatrix} d_1 & 1 \\ -i & i d_1^{-1} \end{bmatrix} \begin{bmatrix} (\alpha + i\beta) d_2 & 0 \\ 0 & (\alpha - i\beta) d_1 \end{bmatrix} \begin{bmatrix} d_2^{-1} & i \\ 1 & -i d_2 \end{bmatrix} \tag{6.10}$$

which yields an $\mathcal{M}$-bounded factorization for G.

The product representation of Theorem 2.6, (iii),

$$G = \widetilde{S}_{Q_1}^{-1} \hat{G} \widetilde{S}_{Q_2}, \quad \text{with } \hat{G} \in D_q, \tag{6.11}$$

also attains a new significance in this setting. It associates to each $G \in C_{Q_1,Q_2}$ a Daniele-Khrapkov matrix, the normal form $\hat{G} \in D_q$. The latter can be considered as being of the simplest kind preserving the function q ($q = -\det Q_1 = -\det Q_2 = -\det D_q$) and some properties of the associated Toeplitz operators. We have the following theorem, which generalizes some results of [8], Section 4.

Theorem 6.2. *If $G \in C_{Q_1,Q_2}$ and $\hat{G} \in C_{D_q}$ is its normal form with respect to (Q_1, Q_2), given by Theorem 2.6 (iii), then T_G is nearly Fredholm equivalent to $T_{\hat{G}}$; the two Toeplitz operators are strictly Fredholm equivalent if*

$$q_{11} \in \mathcal{G}H_\infty^-,\ q_{12} \in H_\infty^- \quad \text{or} \quad q_{11} = 0,\ q_{12}, q_{13} \in H_\infty^- \tag{6.12}$$

and

$$q_{21} \in \mathcal{G}H_\infty^+,\ q_{22} \in H_\infty^+ \quad \text{or} \quad q_{21} = 0,\ q_{22}, q_{23} \in H_\infty^+. \tag{6.13}$$

Proof. From Theorem 2.6 (iii) and (6.2), it follows that (1.7) holds ([28], Theorem 3.10). If (6.12) and (6.13) are satisfied, then $\widetilde{S}_{Q_1} \in \mathcal{G}(H_\infty^-)^{2\times 2}$ and $\widetilde{S}_{Q_2} \in \mathcal{G}(H_\infty^+)^{2\times 2}$, implying that indeed T_G and $T_{\hat{G}}$ are strictly Fredholm equivalent. □

Remark 6.3. Transformations of the form $G \mapsto \hat{G} = UGV$, with $U \in \mathcal{GM}_\infty^-$, $V \in \mathcal{GM}_\infty^+$, as in (6.11), play an important role in the study of WH-factorization and, more generally, Φ - factorization ([20, 28]). A systematic study of transformations of this type, with an emphasis on the case where U and V are invertible rational matrix functions, is undertaken in

[20]. Such transformations are taken as a basis for a classification scheme for a very large class of 2×2 matrix functions and a description of their normal forms. The authors study in particular the problem of determining those 2×2 matrix functions which can be transformed into Daniele-Khrapkov matrix functions, as in (6.11), and the existence of some significant invariants under those transformations, such as the so-called *deviator polynomial* ([30, 32]) corresponding, in the setting of this paper, to the function q.

Remark 6.4. If (6.12) and (6.13) are satisfied, the operators T_G and $T_{\hat{G}}$ satisfy in fact a much stronger equivalence relation than strict Fredholm equivalence. Namely they are (algebraically and topologically) equivalent, in the sense that there are invertible operators in $(H_p^+)^2$, $E = T_{\widetilde{S}^{-1}_{Q_1}}$ and $F = T_{\widetilde{S}_{Q_2}}$, such that $T_G = ET_{\hat{G}}F$ (see [2]).

The results of Sections 4 and 5 now yield the following theorem, which is a direct consequence of Theorem 5.4 and shows that it is enough to determine one solution to the RH problem $G\phi_+ = \phi_-$ satisfying certain conditions, in order to obtain a meromorphic factorization of G. A WH-factorization can then be obtained from the latter, as previously mentioned.

Theorem 6.5. *Let $G\phi_+ = \phi_-$, with $\phi_\pm \in (\mathcal{M}^\pm_\infty)^2$ such that $\phi_-^T Q_1\phi_- \in \mathcal{G}\mathcal{M}^-_\infty$ and $\phi_+^T Q_2\phi_+ \in \mathcal{G}\mathcal{M}^+_\infty$. Then G admits a meromorphic factorization $G = M_-(M_+)^{-1}$ with $M_- = M^{Q_1}_{\phi_-}$, $M_+ = M^{Q_2}_{\phi_+}$, defined by (*5.4*).*

Note that the product equation (4.2) corresponds in this case to the relation $\det M_- = \det G . \det M_+$. Remark also that it is possible to take advantage of the fact that G can belong to different $\mathfrak{M}\mathfrak{Q}$-classes, as shown in Section 3, to choose a pair (Q_1, Q_2) corresponding to "simple" meromorphic factors $M^{Q_1}_{\phi_-}$, $M^{Q_2}_{\phi_+}$. This will be illustrated in an example at the end of this section.

If $Q_1 = Q_2 = Q$, writing $G = (M_-\widetilde{S}_Q)(\widetilde{S}_Q^{-1}M_+^{-1})$ and taking into account that in this case $Q, \widetilde{S}_Q \in \mathcal{G}\mathcal{R}^{2\times 2}$ and $M_\pm\widetilde{S}_Q \in C_Q$ (cf. (5.9)), we also have:

Corollary 6.6. *If $G \in C_Q$, then G admits a meromorphic factorization with factors belonging to C_Q.*

Finally we discuss here the relations of the results of the previous sections with the equivalence of the RH problem on $\mathbb{R}$

$$G\phi_+ = \phi_- \ , \tag{6.14}$$

with $\phi_\pm$ belonging to certain spaces of functions analytic in $\mathbb{C}^\pm$ (such as $(C_\mu^\pm)^2$), respectively, to a scalar RH problem with respect to a contour on an associated Riemann surface Σ. This equivalence was established in [15], Proposition 2.19, assuming $\det Q_1 = \det Q_2 = -q$ with q of the form

$$q = q_0^2 p \, , \tag{6.15}$$

where $q_0 \in \mathcal{R}$ and p is a polynomial of degree $2(g+1)$, $g \geq 0$, with simple roots none of which on $\mathbb{R}$. In this case Σ is the Riemann surface of genus g described by the algebraic equation $\mu^2 = p(\lambda)$.

Since a given 2×2 matrix function G may belong to two different $\mathfrak{MQ}$-classes C_{Q_1,Q_2} and $C_{\widetilde{Q}_1,\widetilde{Q}_2}$ with $\det Q_1 = \det Q_2 = -q$ and $\det \widetilde{Q}_1 = \det \widetilde{Q}_2 = -\tilde{q}$, where q and $\tilde{q}$ are associated, via (6.15), to polynomials p and $\tilde{p}$ of different degrees, G can be associated to Riemann surfaces of different genuses.

Example. An example illustrating this situation is given by the class of matrix functions considered in the example of Section 4, see (4.7)-(4.8). In fact we have, from (4.9), (4.11) and (4.13), assuming for simplicity that $k = m = 0$:

$$\det D_q = s^2, \quad \det Q_1 = \det Q_2 = -s_-^2, \quad \det \widetilde{Q}_1 = \det \widetilde{Q}_2 = -s_+^2. \quad (6.16)$$

Assume that s^2 is the quotient of two polynomials with non-common and simple zeroes, and s_+^2 is a quotient of two polynomials of the first degree with different zeroes. Then, considering that $G \in C_{\widetilde{Q}_1,\widetilde{Q}_2}$, we see from the last equality in (6.16) that the RH problem (6.14) is equivalent to a scalar RH problem on the Riemann sphere, while considering that $G \in C_{D_q}$ leads to formulating an equivalent RH problem on a Riemann surface of higher genus g, depending on the number of zeroes and poles of s^2.

This raises the question whether other problems formulated on a Riemann surface of genus g using the results of [15] might have an equivalent, but simpler, formulation as RH problems on a Riemann surface of smaller genus, taking into account that the same 2×2 matrix function G may belong to different $\mathfrak{Q}$-classes.

The fact that G may belong to two different $\mathfrak{MQ}$-classes has yet another consequence which is, to the authors' knowledge, mentioned here for the first time. It consists in the possibility of solving RH problems of the form (6.14) by simple algebraic methods, directly using different product equations associated to the same matrix G. We illustrate this possibility with the following example.

Taking again G given by (4.7)-(4.8) as in the example of Section 4, with $s^2 \in \mathcal{GR}$ having simple zeroes and poles, consider the RH problem

$$G\phi_+ = \phi_-, \quad \phi_\pm \in \left(H_2^\pm\right)^2 \quad (6.17)$$

whose solutions characterize the kernel of the Toeplitz operator T_G. We have the following.

Theorem 6.7. *Let G satisfy* (4.7)-(4.8) *with $s^2 \in \mathcal{GR}$ having simple zeroes and poles. If in one of the half-planes $\mathbb{C}^+$ or $\mathbb{C}^-$ there are no more than two points which are zeroes or poles of s^2, then* $\ker T_G = \{0\}$.

Proof. Let us assume that s^2 has z_+ zeroes and p_+ poles in the upper-half plane, with $z_+ + p_+ \leq 2$. In this case, s^2 admits a Wiener-Hopf factorization $s^2 = r_+ r^{\tilde{k}} r_-$ with $r_\pm \in \mathcal{G}\left(\mathcal{R} \cap H_\infty^\pm\right)$, $\tilde{k} = z_+ - p_+$.

Suppose that $\tilde{k} \geq 0$ $(z_+ \geq p_+)$. We have

$$s = ir^{m/2} s_- r^k s_+, \text{ with } s_\pm \in \mathcal{G}H^\pm_\infty, \tag{6.18}$$

$$k = \frac{\tilde{k}}{2}, \; m = 0, \text{ if } \tilde{k} \text{ is even}, \tag{6.19}$$

$$k = \frac{\tilde{k}-1}{2}, \; m = 1, \text{ if } \tilde{k} \text{ is odd}. \tag{6.20}$$

We may choose for G the product equation (4.10) associated with the class C_{Q_1,Q_2} defined by (4.11). Thus, for every solution of (6.17), we have

$$\frac{1}{(\xi+i)^{m/2}} s_+^{-1} \left(\phi_{1+}^2 - s^2\phi_{2+}^2\right) = \frac{i}{(\xi-i)^{m/2}} r^{k+m} s_- \phi_{1-}\phi_{2-} = r_1 \in \mathcal{R} \cap L_1(\mathbb{R}) \tag{6.21}$$

where r_1 has at most a pole of order $k+m$ at $-i$ and p_+ poles in $\mathbb{C}^+$ (at the points which are poles of s^2 in $\mathbb{C}^+$).

On the other hand, every solution of (6.17) must also satisfy the product equation associated with the class C_{D_q} defined by (4.9):

$$2\left(\phi_{1+}^2 + s^2\phi_{2+}^2\right) = \left(\phi_{1-}^2 + s^2\phi_{2-}^2\right). \tag{6.22}$$

Taking into account that $\tilde{k} \geq 0$, the following situations may occur: either

$$p_+ = 0, \tilde{k} = z_+ \in \{0,1,2\}, \tag{6.23}$$

and in this case

$$\tilde{k} = 0 \Rightarrow z_+ = p_+ = 0 \Rightarrow k = m = 0 \Rightarrow r_1 = 0 \tag{6.24}$$

$$\tilde{k} = 1 \Rightarrow z_+ = 1 \Rightarrow k = 0, m = 1 \Rightarrow r_1 = 0 \tag{6.25}$$

$$\tilde{k} = 2 \Rightarrow z_+ = 2 \Rightarrow k = 1, m = 0 \Rightarrow r_1 = 0, \tag{6.26}$$

or

$$p_+ = 1, \tilde{k} = 0, \tag{6.27}$$

and in this case $k = m = 0$ and $r_1 = 0$.

Since $r_1 = 0$, it follows from (6.21) that $\phi_{1+}^2 = s^2\phi_{2+}^2$ and either $\phi_{1-} = 0$ or $\phi_{2-} = 0$. If $\phi_{1-} = 0$, then from (6.22) we have $4\phi_{2+}^2 = \phi_{2-}^2$ so that $\phi_{2+} = \phi_{2-} = 0$ and $\ker T_G = 0$. If $\phi_{2-} = 0$, we conclude analogously that $\ker T_G = 0$. Finally, the case when $\tilde{k} < 0$ can be treated analogously using (6.22) and the product equation (4.12) associated with $C_{\tilde{Q}_1,\tilde{Q}_2}$. □

Corollary 6.8. *With the same assumptions as in Theorem* 6.7*, if s is continuous in $\dot{\mathbb{R}}$, then T_G is invertible.*

Proof. Since $s^2 \in \mathcal{GR}$, if s is continuous in $\dot{\mathbb{R}}$ then we have $s \in \mathcal{G}C_\mu(\dot{\mathbb{R}})$ and $G \in (C_\mu(\dot{\mathbb{R}}))^{2\times 2}$; since $\det G = 2$ and T_G is injective, it follows that G admits a canonical WH-factorization and T_G is invertible ([31]). □

Remark that, if the assumptions of Corollary 6.8 are satisfied, then G admits a canonical *WH*-factorization $G = G_-(G_+)^{-1}$ which can also be obtained by using the different product equations associated with G. This is illustrated in the following example.

Example. Let G satisfy the conditions of Theorem 6.7, with $s^2(\xi) = -\frac{(\xi-i)(\xi-2i)}{(\xi+i)(\xi+2i)}$. In this case, $s = is_- r s_+$, where $s_-(\xi) = \sqrt{(\xi-2i)/(\xi-i)}$ and $s_+(\xi) = \sqrt{(\xi+i)/(\xi+2i)}$, $k = 1$ and $m = 0$ (as in (6.26)). The factorization of matrix functions of this form has been obtained in [11, 24]; the method that we use here to obtain it is considerably simpler.

We start by determining a solution to the RH problem

$$G\phi_+ = \phi_-, \qquad \phi_\pm \in (C_\mu^\pm)^2 \tag{6.28}$$

such that $\phi_{2-}(-i) = 0$. From (4.11) it follows that $\phi_\pm$ satisfy the product equation

$$s_+^{-1}(\phi_{1+}^2 - s^2\phi_{2+}^2) = is_- r\phi_{1-}\phi_{2-} = K, \tag{6.29}$$

with $K \in \mathbb{C}$. On the other hand, from (4.9) we have

$$2(\phi_{1+}^2 + s^2\phi_{2+}^2) = \phi_{1-}^2 + s^2\phi_{2-}^2 = \frac{A_1\xi + A_0}{\xi + 2i}\ , \tag{6.30}$$

with $A_1, A_0 \in \mathbb{C}$. In addition, from (4.13) we have

$$-4is_+ r\phi_{1+}\phi_{2+} = s_-^{-1}(\phi_{1-}^2 - s^2\phi_{2-}^2) = \frac{B_1\xi + B_0}{\xi + 2i}, \tag{6.31}$$

with $B_1, B_0 \in \mathbb{C}$. From (6.29), (6.30) and (6.31), taking the condition $\phi_{2-}(-i) = 0$ into account, we obtain

$$\phi_{1+} = \sqrt{\frac{KK_1}{2}\left(\frac{\xi + i\sqrt{2}}{\xi + 2i}\right)\left[1 + \frac{\sqrt{(\xi+i)(\xi+2i)}}{K_1(\xi + i\sqrt{2})}\right]} \tag{6.32}$$

$$\phi_{2+} = -s^{-1}\sqrt{\frac{KK_1}{2}\left(\frac{\xi + i\sqrt{2}}{\xi + 2i}\right)\left[1 - \frac{\sqrt{(\xi+i)(\xi+2i)}}{K_1(\xi + i\sqrt{2})}\right]} \tag{6.33}$$

$$\phi_{1-} = \sqrt{KK_1\left(\frac{\xi + i\sqrt{2}}{\xi + 2i}\right)\left[1 + \frac{\sqrt{(\xi-i)(\xi-2i)}}{K_2(\xi + i\sqrt{2})}\right]} \tag{6.34}$$

$$\phi_{2-} = -i\sqrt{KK_1\left(\frac{(\xi + i\sqrt{2})(\xi+i)}{(\xi - 2i)(\xi - i)}\right)\left[1 - \frac{\sqrt{(\xi-i)(\xi-2i)}}{K_2(\xi + i\sqrt{2})}\right]} \tag{6.35}$$

with $K \in \mathbb{C}$ (we can take $K = 1$), $K_1 = 2\sqrt{3} - \sqrt{6}$ and $K_2 = -(2\sqrt{3} + \sqrt{6})$. By Theorem 6.5, and using the relation (4.13), we obtain the meromorphic factorization $G = M_-(M_+)^{-1}$ with $M_- = M_{\phi_-}^{\widetilde{Q}_1}$, $M_+ = M_{\phi_+}^{\widetilde{Q}_2}$:

$$M_- = \begin{bmatrix} \phi_{1-} & -B_1^{-1}\frac{\sqrt{(\xi-2i)(\xi-i)}}{\xi+i}\phi_{2-} \\ \phi_{2-} & B_1^{-1}\frac{\xi+2i}{\sqrt{(\xi-2i)(\xi-i)}}\phi_{1-} \end{bmatrix}$$

$$M_+ = \begin{bmatrix} \phi_{1+} & iB_1^{-1}s_+^{-1}\phi_{1+} \\ \phi_{2+} & -iB_1^{-1}s_+^{-1}\phi_{2+} \end{bmatrix}$$

with $B_1 = 2(2\sqrt{2} - 3)$.

Due to our choice of the relation (4.13), we have obtained $M_\pm \in \mathcal{G}(H_\infty^\pm)^{2\times 2}$, thus $G = M_-(M_+)^{-1}$ is a canonical WH-factorization for G.

7. $\mathfrak{Q}$-classes as groups

Let $(Q_1, Q_2) \in \mathfrak{Q}^{(2)}$ and let X and Y be given by (2.17). Defining

$$\mathcal{A} = \{\alpha X + \beta Y : \alpha, \beta \in L_\infty(\mathbb{R})\}, \tag{7.1}$$

let $*$ be the operation defined in $\mathcal{A}$ by

$$G_1 * G_2 = G_1 X^{-1} G_2. \tag{7.2}$$

for $G_1, G_2 \in \mathcal{A}$. From (2.18) we have

$$(\alpha X + \beta Y) * (\tilde{\alpha} X + \tilde{\beta} Y) = (\alpha\tilde{\alpha} + q\beta\tilde{\beta})X + (\alpha\tilde{\beta} + \tilde{\alpha}\beta)Y.$$

The operation $*$ is commutative and associative, with identity element X. Remark that, by Theorem 2.6, $C_{Q_1,Q_2} \subset \mathcal{A}$ and C_{Q_1,Q_2} is closed under the operation $*$, by Theorem 2.6 (iv) and Proposition 2.3 (ii), (iii). We have then the following:

Theorem 7.1. $(C_{Q_1,Q_2}, *)$ *is a commutative group with identity X. The inverse of G in this group is*

$$(G)_*^{-1} := X G^{-1} X \tag{7.3}$$

where G^{-1} denotes the usual inverse of G.

Choosing a notation similar to that used for the inverse in (7.3), we also define

$$(G)_*^2 := G * G, \tag{7.4}$$

and analogously for $(G_*)^n$, with $n \in \mathbb{Z}$. By (2.18) we have

$$(Y)_*^2 = qX. \tag{7.5}$$

If $G = \alpha X + \beta Y \in C_{Q_1,Q_2}$, then by Theorem 2.6 (iv) we have $\alpha^2 - q\beta^2 \in \mathcal{G}L_\infty$ and it is easy to see from (7.5) that

$$(G)_*^{-1} = \frac{1}{\alpha^2 - q\beta^2}(\alpha X - \beta Y). \tag{7.6}$$

In particular, $(Y)_*^{-1} = q^{-1}Y$. Of course, if $Q_1 = Q_2 = Q$, then $X = I$, $Y = -JQ$ and the operation $*$ reduces to the usual multiplication of matrices, meaning that C_Q is a group with the usual product.

Defining

$$a_l : C_{Q_1} \times C_{Q_1,Q_2} \to C_{Q_1,Q_2} \tag{7.7}$$

$$a_l(G_l, G) = G_l G, \tag{7.8}$$

it follows from (5.10) that (7.7) defines a map such that

$$a_l(G_{l_1} G_{l_2}, G) = a_l(G_{l_1}, a_l(G_{l_2}, G))$$
$$a_l(I, G) = G$$

for all $G_{l_1}, G_{l_2} \in C_{Q_1}$ and $G \in C_{Q_1,Q_2}$. Thus (7.7) defines a (left) group action of C_{Q_1} on C_{Q_1,Q_2}. We can define analogously a right group action of C_{Q_2} on C_{Q_1,Q_2}. Since C_{Q_1} is non-empty and, for any two elements G_1, G_2 in C_{Q_1,Q_2}, there exists a unique $A \in C_{Q_1}$ such that $AG_1 = G_2$ $(A = G_2 G_1^{-1})$ we have the following:

Theorem 7.2. *The group C_{Q_1} acts on C_{Q_1,Q_2} on the left by a_l. This left action is regular (transitive and free, therefore faithful), and C_{Q_1,Q_2} is a principal homogeneous space for C_{Q_1}. The orbit space $C_{Q_1,Q_2}/C_{Q_1}$ is a unit set; C_{Q_1,Q_2} is the orbit of X (or any other element of C_{Q_1,Q_2}) under the action of C_{Q_1}.*

The results of Section 3 can thus be interpreted in terms of orbits. Let $\mathcal{O}$ denote the set of all $\mathfrak{Q}$-classes and let us call any element of $\mathcal{O}$ an *orbit.*

Theorem 7.3. *If O_1, O_2 are two orbits in $\mathcal{O}$, then one and only one of the following propositions is true:*

(i) *O_1 and O_2 are disjoint*
(ii) *$O_1 = O_2$*
(iii) *There exist some $G \in \mathcal{G}(L_\infty)^{2\times 2}$ such that*

$$O_1 \cap O_2 = \{fG, f \in \mathcal{G}L_\infty\}.$$

The results of Section 5 can also be seen as connected to the question of existence of non trivial factorizations of $G \in C_{Q_1,Q_2}$ in the group $(C_{Q_1,Q_2}, *)$. In particular, from Theorem 5.6, we have the following.

Theorem 7.4. *Given $(Q_1, Q_2) \in \mathfrak{Q}^{(2)}$, every $G \in C_{Q_1,Q_2}$ admits a factorization*

$$G = G_1 * G_2 \tag{7.9}$$

with $G_1, G_2 \in C_{Q_1,Q_2}$. In particular (7.9) *holds with*

$$G_1 = M_\psi^{Q_1} H_\rho S_{Q_2}, \quad G_2 = \left(M_\phi^{Q_2} H_\rho S_{Q_1}\right)^{-1},$$

with $M_\psi^{Q_1}$ and $M_\phi^{Q_2}$ defined by (5.4)*, for any ϕ, ψ satisfying* (5.7)-(5.8).

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