Gauge theories in dimension 4

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THE MAIN SUBJECT — gauge theories in 4 - dimensional Diff Geometry:

the Donaldson theory (instantons)
the Seiberg - Witten theory (abelian monopoles)
and
something between (moduli space of B - monopoles, constructed by A. Tyurin and V. Pidstrigach)

The gauge theories came to Diff Geom from Mathematical Physics: the main principle — GAUGE INVARIANCE — imposes

— **the equations of motion should be gauge invariant**
— **instead of a single solution — a class of gauge equivalent solutions**

The Gauge Invariance implies that Gauge Theories give some INVARIANTS of smooth structures
and
in real DIMENSION 4 one needs some invariants
since
4 - geometry admits different smooth structures for homeomorphic manifolds

Example: $\mathbb{R}^4$ admits continuum nonequivalent smooth structures (S.K. Donladson)
so it is a geometry indeed

Gauge theories in Dim 4 are Originated in Dim 2 = Geometry of Riemannian surfaces

Dim 2 : complex geometry coincides with conformal geometry
$\Longrightarrow$
all Gauge theories in Dim 2 belong to complex (Kahler, algebraic) geometry
Example: vortex equation on Riemannian surface, etc.

BUT: in Dim 4 we havn't such good coincidences

AT THE SAME TIME: both the theories (Donaldson and SW) have some specified frameworks where the notions turn to be extremely familiar and the computations are reduced to some standard schemes, namely

the Donaldson theory — holomorphic (algebraic) geometry

the Seiberg - Witten theory — symplectic geometry

MIRROR SYMMETRY: AG and SG are "dual" so (roughly) if X_1 and X_2 are conjugated by Mirror Symmetry then $AG(X_1)$ should "correspond" to $SG(X_2)$ and *vice versa*

Example: in Homological Mirror Symmetry (M. Kontsevich) "correspond" means that some category derived from AG of X_1 (the derived category of coherent sheaves) is isomorphic to some category derived from SG of X_2 (Fukaya category)

thus any RELATIONSHIP between (Donaldson) and (Seiberg - Witten) should reflect some kind of DUALITY connecting the theories

such a relationship was detected indeed
in Diff Geom setup: by A.N. Tyurin and V. Pidstrigach (nonabelian or B - monopoles)
in Theor Phys setup: by E. Witten (certain N = 2 SYM theory)
in the last one both the theories are realised as certain infro-red and ultra-violet limits (so it is duality indeed!)

Below we'd like to discuss Diff Geom aspects of the duality so
– first, we remind some basic constructions of the Donladson theory and the Seiberg - Witten theory
— second, we remind the construction of B - monopole moduli spaces following A.N. Tyurin and V. Pidstrigach
— third, in connection with the main background idea (Duality) we add some remarks on complex and symplectic geometry underlaing the constructions

IN WHAT FOLLOWS: M is a smooth compact connected orientable 4 - manifold
g is a fixed riemannian metric on M
$d\mu$ is the volume form given by a fixed orientation
For this metric g and orientation $d\mu$ one has
Hodge star operator $* : \Omega^i \to \Omega^{4-i}$
defined by the standard formula

$$\beta \wedge *\alpha \equiv < \beta, \alpha >_g d\mu \quad \forall \beta \in \Omega^i$$

for $i = 2$ it is AN INVOLUTION: $*^2 = id$
thus one has the decomposition

$$\Omega^2 = \Omega^+ \oplus \Omega^- \qquad | \quad *_{\Omega^\pm} = \pm id$$

on *selfdual* and *antiselfdual* 2 - forms
The Hodge theory of harmonic forms $\Longrightarrow$

$$H^\pm = \{\alpha \in \Omega^\pm \quad | \quad d\alpha = 0\}$$

(the spaces of selfdual and antiselfdual *harmonic forms*) have finite dimensions $b_2^\pm$ and the signature of *the intersection form* Q_M equals

$$\sigma(M) = \sigma(Q_M) = b_2^+ - b_2^-$$

Donaldson theory

Let $E \to M$ be $SU(2)$ - bundle defined by an integer number (the second Chern class)
then $\mathcal{A}_h(E)$ = the hermitian connection space = the configuration space
$\mathcal{G}_h(E)$ = the gauge group = fiberwise hermitian automorphisms of E
if $a \in \mathcal{A}_h(E)$ then $F_a \in \Omega^2(adE)$ — the CURVATURE tensor = adE - valued 2 - form and since the Hodge star decomposition can be extended to $\Omega^2(adE) = \Omega^+(adE) \oplus \Omega^-(adE)$ one has a natural equation = Yang - Mills equation

$$F_a^+ = 0$$

it is GAUGE INVARIANT indeed (one can easily see it from the LAGRANGIAN description:

$$L_{YM} = \int_M |F_a|^2 d\mu$$

- this Lagrangian is gauge invariant)
(The YM - equation is originated in classical electrodynamics = NONABELIAN GENERALIZATION of the Maxwell equation)

The moduli space of INSTANTONS (ASD - connections)
$\mathcal{M}_{asd} \in \mathcal{A}_h/\mathcal{G}_h =$ (the space of solutions)/(the gauge group action)
— generically smooth (non compact!) orientable manifold

$$v.dim\mathcal{M}_{asd} = 8c_2(E) - 3(b_2^+ + 1)$$

(for $SU(2)$ - bundles over simply connected manifolds)
reflects the properties of the underlaing Smooth Structure

Namely: if $b_2^+(M) > 1$ then the topology of $\mathcal{M}_{asd}$ for generic metric DOESN'T DEPEND on the choice of g = DEPENDS only on the given smooth structure!!!

Why $b_2^+(M) > 1$? ABELIAN instantons (REDUCIBLE solutions) doesn't exist for generic g if $b_2^+ > 0$ and two generic metrics g_1, g_2 can be joint by a path g_t without reducible solutions if $b_2^+ > 1$

(if $b_2^+ = 1 \quad \Longrightarrow$ some chamber structure in the metric space)
BUT: the moduli space $\mathcal{M}_{asd}$ is not compact — it follows from the CONFORMAL INVARIANCE of the Yang - Mills equation

How to derive topological information: DONALDSON POLINOMIALS

$$q_k(\Sigma_1, ..., \Sigma_m) \qquad \text{f}or \quad \Sigma_i \in H_2(M, \mathbb{Z})$$

and for some other classes

The value is given by pairing

$$< \mu(\Sigma_1) \cdot ... \cdot \mu(\Sigma_i); [\mathcal{M}_{asd}] >$$

where $\mu(\Sigma) \in H^2(\mathcal{B}^*, \mathbb{Z})$ and some other terms for other classes

The main observations which come through the studing of the theory are the following

For Kahler metric anti self duality = holomorphic bundle stable with respect to the principle polarization given by the Kahler form

If M admits the structure of algebriac surface then the corresponding smooth structure admits non trivial polinomial q_k

If M is decomposable into a connected sum $X_1 \sharp X_2$ such that both X_i have positive b_2^+ then all the polinomials are trivial

from these one can easily constract an example of two homeomorphic BUT NOT diffeomorphic 4 - manifolds (namely an algebraic surface and its topomodel)

on the other hand one can see that the theory has a good reduction to some special case:

it is reduced to the theory of STABLE VECTOR BUNDLES OVER ALGEBRAIC SURFACES

so algebraic geometry suggests such a good interpretation for the Donaldson theory!

BUT it is just a half of the story:
one has another natural transformation of the notions — due to the celebrated **Penrose twistor programme** (*in this way one can explain what does it mean "mathematical instanton" — another subject of Andrey Tyurin's work*)
Twistor bundles defined by the metric

over $x \in M$:
the set of all complex structures on T_xM compatible with g and $d\mu$ = projective line $\mathbb{P}_x^+$
the same set for g and $-d\mu$ = projective line $\mathbb{P}_x^-$
GLOBILIZING over whole M:
two projective bundles $\mathbb{P}^\pm \to M$ = twistor bundles

For the total space $Y = \mathbb{P}^- \to M$ one defines an almost complex structure (''twisted'' or ''twistor'' almost complex structure)

$$I_{tw} : TY \to TY \quad I_{tw}^2 = \mathrm{id}$$

defined by the picture

I_{tw} depends on THE CONFORMAL CLASS of g ONLY (as well as the instanton equation does)
Quite known fact: the structure I_{tw} is INTEGRABLE if and only if our metric g is SELF DUAL
for this case we get
complex threefold Y
vector bundle $\pi^*(E)$ over Y where $\pi : Y \to M$ is the canonical projection

Then: a is ASD over M, g if and only if $\pi^* a$ defines a HOLOMORPHIC structure on $\pi^* E$ so the last one is a HOLOMORPHIC BUNDLE (with some special properties coming with the lifting π^*)

Example: for S^4 with standard metric $Y = \mathbb{CP}^3$ with standard complex structure — "mathematical instantons" are holomorphic bundles over $\mathbb{CP}^3$ with some special properties

Thus algebraic geometry is the framework for the Donaldson theory (f.e. almost all examples in "The geometry of 4 - manifolds" by S. Donaldson and P. Kronheimer are from this framework)

Seiberg - Witten theory

From twistors to SPINOR bundles:
since $w_1(M) = 0$ (orientability) one can lift the twistor projective bundles to VECTOR BUNDLES; to do this one chooses

$$c \in H^2(M, \mathbb{Z}) \quad | \quad c = w_2(M) \quad (mod 2)$$

and then one gets $W^{\pm} \to M$ = SPINOR BUNDLES s.t.

$$c_1(W^{\pm}) = c; \qquad c_2(W^{+}) = \frac{1}{4}(c^2 - 2\chi - 3\sigma);$$

$$c_2(W^{-}) = \chi + c_2(W^{+})$$

Moreover, $W^{\pm}$ carry natural hermitian structures induced by g

choice of $c \in H^2(M, \mathbb{Z})$ = choice of $Spin^{\mathbb{C}}$ - structure on M

Properties of spinor bundles:

1. $Hom(W^-, W^+) \cong T^{\mathbb{C}}M$ moreover its real part TM is isomorphic to Hom compatible with both the hermitian structures

2. $adW^{\pm} \cong \Lambda^{\pm}$ so there is a QUADRATIC MAP

$$\phi \in \Gamma(W^+) \mapsto (\phi \otimes \overline{\phi})_0 \in \Omega^+$$

thus for the SW- theory

The configuration space: $\mathcal{A}_h(detW^+) \times \Gamma(W^+)$

The gauge group: $\mathcal{G} = Aut_h(detW^+)$

The equations

$$D_a(\phi) = 0, \qquad F_a^+ = -(\phi \otimes \overline{\phi})_0$$

are gauge invariant

NO CONFORMAL INVARIANCE $\Longrightarrow$ for a generic metric the moduli space if $b_2^+ > 0$ the moduli space of solutions is a **smooth compact orientable manifold**

$$v.dim\mathcal{M}_{mon} = \frac{1}{4}(c^2 - 2\chi - 3\sigma)$$

if $b_2^+ > 1$ then THE TOPOLOGY of $\mathcal{M}_{mon}$ depends on the smooth structure only!
AGAIN A INVARIANT of smooth structures
Computation, the simplest case:
let $c^2 = 2\chi - 3\sigma$, then the moduli space is a finite number of points
thus

$$N_SW(c) = \sum_{p_i \in \mathcal{M}_{mon}} \pm 1 \in \mathbb{Z}$$

where the signs are given by the orientation
BUT: zero dimensional moduli space $=$ c is the canonical class of an almost complex structure!

it seems that this theory is related to the complex geometry!!!

The first results (E. Witten):

Algebraic surface has NONTRIVIAL SW - invariant

If $M = X_1 \sharp X_2$ with both positive b_2^+ then the invariant is TRIVIAL

— AGAIN one derives that an algebraic surface is not diffeomorphic to its topomodel

— AGAIN there is a special case: when M is an algebraic surface and g is the Kahler metric then every solution is a pair (holomorphic line bundle, holomorphic section) — holomorphic line bundle with Chern class $c = c_1(W^+)$

— thus in AG the Seiberg - Witten theory = complete linear systems

— the information is carried by the Picard lattice

BUT: algebraic geometry or even Kahler geometry doesn't exhaust general framework of SW - nontrivial cases

THEOREM (C. Taubes): *if M is a symplectic manifold with $b_2^+ > 1$ then the associated canonical class K has nontrivial SW - invariant*

Moreover, in SG one has GROMOV INVARIANT: in real dimension 4 the invariant is given by the number of PSEUDOHOLOMORPHIC CURVES which represent PD class of $2c - K$ (counted with signs)

Gromov invariant belongs exactly to SG — it is defined for any compact symplectic manifold

THEOREM (C. Taubes): *GR = SW*

thus SYMPLECTIC GEOMETRY is a good framework of the Seiberg - Witten theory (but may be a wider framework exists?)

(Kotschik, Morgan, Taubes) Nonsymplectic 4 - manifold with non trivial SW - invariant (but not simply connected)

TYURIN - PIDSTRIGACH CONSTRUCTION relates $\mathcal{M}_{asd}$ with $\mathcal{M}_{mon}$ by a COBORDISM FILM

Configuration space: $\mathcal{A}_h(adE) \times \mathcal{A}_\omega \times \Gamma(E \otimes W^+)$ where

$$\mathcal{A}_\omega = \{\nabla \in \mathcal{A}(det(E \otimes W^+)) \quad | \quad F_\nabla = \omega\}$$

The gauge group: $\mathcal{G} = \mathcal{G}_E = Aut_h(E)$

The equation of B - monopole

$$D_{a_0}^{\nabla_{det}}(\Phi) = 0$$

$$F_{a_0}^+ = -(\Phi\overline{\Phi})_{00}$$

"double traceless part"

REDUCIBLE SOLUTIONS: either ASD - connections (with TRIVIAL spinor part) or ABELIAN monopoles

Thus the TOPOLOGY of the moduli spaces is the same

(PROBLEM: TRANSVERSALITY)

BUT this coincidence seems to be PURE NUMERICAL — weak duality (Numerical Mirror Symmetry)
What about a stronger duality? —
for example: TWISTOR space for Seiberg - Witten equation?
does some appropriate construction exist?
Preliminary ideas: SW - moduli space depends on RIEMANNIAN METRIC — not only conformal class. What one derives from the classical twistor construction for riemannian metric?
returning to the TWISTOR CONSTRUCTION:
riemannian metric g defines on the twistor space

$$Y = \mathbb{P}^- \to M$$

a NONDEGENERATED 2 - FORM Ω_g
and
this 2 - form depends on g — and not only on its conformal class!

PROBLEM: when this 2- form IS CLOSED — in other words — when g defines a SYMPLECTIC STRUCTURE on the twistor space?
The answer SHOULD BE a natural condition on riemannian metric (Background: SG is dual to AG, don't forget it)
Another problem of Riemannian Geometry and a subject of Andrey Tyurin's lectures: what is canonical class of Riemannian Manifold?
f.e. if M is an algebraic surface then its canonical class together with the Picard lattice are distinguished topological data
now one gets
GENERALIZATION: basic classes of Riemannian manifold
cohomological 2 - class $\alpha \in H^2(M, \mathbb{Z})$ is called basic if the Seiberg - Witten invariant of α is nontrivial
thus $B \subset H^2(M, \mathbb{Z})$ — sublattice of BASIC classes — carries the information about the smooth structure — analog of the Picard lattice

PROBLEM: **HIT - conjecture**
for any Picard lattice *the Hodge Index Theorem* says that the signature of the restricted intersection form is $(1, \rho)$
— what about the restriction of the intersection form to B (basic classes)?

And finally (*what was interesting for the speaker*)

is there some more general condition than to be symplectic for nontriviality of the invariants for **almost complex 4 - manifolds**?

One wishes to continue the discussion one year later at the next VBAC meeting