

# Coherent State Transforms and Theta Functions

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Received February 2004

**Abstract**—Some aspects of the relations between analytic and geometric properties of theta functions, coherent state transforms for Lie groups, and geometric quantization are reviewed. In this context, the classical theta functions on abelian varieties and nonabelian theta functions for vector bundles on elliptic curves are studied. Some applications of these ideas to rank-2 nonabelian theta functions for genus  $g \geq 2$  are also discussed.

## 1. INTRODUCTION

It is an honor to contribute to this volume, dedicated to the memory of our friend and collaborator Professor Andrei Tyurin. He dedicated some of his last works to the study of theta functions [41–44], a subject notoriously marked by the interdisciplinarity of different areas of mathematics, which, as it seems to us, Andrei Tyurin very much appreciated.

Although of algebraic geometric origin, theta functions can also be studied from the points of view of analysis, symplectic geometry, and geometric quantization, and even from the viewpoint of quantum field theory. Of central interest is the Knizhnik–Zamolodchikov–Bernard–Hitchin (KZBH) connection [32, 7, 8, 27, 2, 47], which allows the comparison of theta functions for different complex structures. (For the conformal field theory approach, see also [11, 20, 21].) In this contribution, we will give the structure of an analytic theory for theta functions, with a view toward present and future applications to the nonabelian case.

Below, we will review our work on classical theta functions on abelian varieties [14] and on nonabelian theta functions for vector bundles on elliptic curves [15]. During the development of this work, we had the privilege of having many discussions with Andrei Tyurin, especially during his extended stays at our institute in Lisbon in the fall of 2000 and spring 2001. We were deeply touched by his constant enthusiasm, encouragement, and friendship.

In Section 2, we review the classical theory of theta functions and the algebro-geometric setting of nonabelian theta functions. In Section 3, we explore the relation between classical theta functions and a heat equation on the compact Lie group  $U(1)^g$ . Here, we emphasize the role of the distributional initial conditions and their relation to geometric quantization. Section 4 contains an introduction to coherent state transforms for Lie groups as well as some of their generalizations. In Section 5, in which we state the main results, the development of an analytic theory for theta functions is proposed in the framework of the coherent state transform. We treat the classical case of theta functions on abelian varieties and the higher rank theta functions for vector bundles over elliptic curves. An important role is played by the Schottky map for  $SL(n, \mathbb{C})$ .

In the final Section 6, we present some generalizations of the previous ideas to the much more difficult problem of constructing an analytic theory of nonabelian theta functions for genus  $g \geq 2$ .

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Many of these ideas were explored in collaboration with Andrei Tyurin, and some of the results stated will be presented in a work in preparation [16].

## 2. THETA FUNCTIONS

Riemann’s classical theta functions are holomorphic functions on the product  $\mathbb{C}^g \times \mathbb{H}_g$ , where  $\mathbb{H}_g$  is the Siegel upper half-space consisting of complex symmetric  $g \times g$  matrices with positive definite imaginary part. They depend on a nonnegative integer  $k$ , called the level, and can be defined as functions  $\theta(z, \Omega)$  that satisfy the following quasiperiodicity conditions:

$$\begin{aligned} \theta(z + \lambda_j, \Omega) &= \theta(z, \Omega), \\ \theta(z + \tilde{\lambda}_j, \Omega) &= e^{-2\pi i k z_j - \pi i k \Omega_{jj}} \theta(z, \Omega), \end{aligned} \tag{1}$$

where  $\{\lambda_j\}_{j=1, \dots, g}$  denotes the standard basis in  $\mathbb{C}^g$  and  $\tilde{\lambda}_j = \Omega \lambda_j$ . Due to the integer periods, the theta functions can be regarded as holomorphic functions in the variables  $w_j = e^{2\pi i z_j} \in \mathbb{C}^*$ , and it is easy to see that holomorphic functions on  $(\mathbb{C}^*)^g$  satisfying (1) form a vector space  $\mathcal{H}_{\Omega, k}$ , of dimension  $k^g$ , for which an explicit basis is [34, 31]

$$\theta_l(z, \Omega) = \sum_{n \in \mathbb{Z}^g} e^{\pi i (l + kn) \cdot \frac{\Omega}{k} (l + kn)} e^{2\pi i (l + kn) \cdot z}, \quad l \in (\mathbb{Z}/k\mathbb{Z})^g. \tag{2}$$

These functions have the following geometric interpretation. The lattice  $\Lambda_\Omega = \mathbb{Z}^g \oplus \Omega \mathbb{Z}^g$  acts naturally on the trivial line bundle over  $V = \mathbb{C}^g$  as

$$\lambda \cdot (z, t) = (z + \lambda, e_\lambda(z)t), \quad (z, t) \in V \times \mathbb{C}, \tag{3}$$

where  $e_{\lambda_j}(z) = 1$  and  $e_{\tilde{\lambda}_j}(z) = e^{-2\pi i k z_j - \pi i k \Omega_{jj}}$ , and this action defines a holomorphic line bundle

$$L = (V \times \mathbb{C})/\Lambda_\Omega \tag{4}$$

over the principally polarized abelian variety

$$M_\Omega = V/\Lambda_\Omega = \mathbb{C}^g/(\mathbb{Z}^g \oplus \Omega \mathbb{Z}^g).$$

It is then clear that theta functions of level  $k$  in (1) are holomorphic sections of  $L^k$ :

$$\mathcal{H}_{\Omega, k} = H^0(M_\Omega, L^k). \tag{5}$$

By the standard correspondence between line bundles and divisors,  $L$  can also be defined by the locus of points in  $M_\Omega$  where a given section vanishes. In the important case when  $M_\Omega$  is the Jacobian  $J(X)$  of a compact Riemann surface  $X$  of genus  $g$  and period matrix  $\Omega$ , this locus, called the *theta divisor*, can also be described as follows. In this case,  $M_\Omega = J(X)$  is the moduli space of holomorphic line bundles of degree 0 on  $X$ , and for a fixed line bundle  $\mathfrak{m}$  of degree  $g - 1$  on  $X$ , the set

$$\Theta_{\mathfrak{m}} = \{\mathfrak{n} \in J(X) : H^0(X, \mathfrak{m} \otimes \mathfrak{n}) \neq 0\}$$

is the zero set of a section of  $L \rightarrow J(X)$  [34].

One generalization of this theory to the nonabelian setting can be obtained by considering a reductive Lie group  $G_{\mathbb{C}}$  and by replacing the Jacobian (which is also the moduli space of  $\mathbb{C}^* = GL(1, \mathbb{C})$  principal bundles over  $X$ ) by a suitable moduli space of principal  $G_{\mathbb{C}}$ -bundles over  $X$ . In the case  $G_{\mathbb{C}} = SL(n, \mathbb{C})$ , which is our main example, one deals with the moduli space of semistable vector bundles of rank  $n$  and trivial determinant over  $X$ ,  $\mathcal{M}_n = \mathcal{M}_n(X)$ . This is a projective variety with singularities, except in the cases of genus 1, where  $\mathcal{M}_n \cong \mathbb{P}^{n-1}$ , and the case  $g = n = 2$ , for

which it is  $\mathbb{P}^3$  [36, 37]. For a line bundle  $\mathfrak{m} \in J^{g-1}(X)$  as before, the set

$$\Theta_{\mathfrak{m}} = \{E \in \mathcal{M}_n : H^0(X, E \otimes \mathfrak{m}) \neq 0\} \tag{6}$$

can be proved to be a divisor on  $\mathcal{M}_n$  [10] and defines a line bundle  $L \rightarrow \mathcal{M}_n$  that is independent of  $\mathfrak{m}$ . Moreover, this bundle is canonical, and any other line bundle on  $\mathcal{M}_n$  is a power of  $L$ . Therefore, one naturally defines a nonabelian theta function of level  $k$  as a section of  $L^k$ ,

$$\theta \in H^0(\mathcal{M}_n, L^k). \tag{7}$$

However, there are important differences and technical difficulties in the nonabelian case. First of all, since  $\pi_1(\mathcal{M}_n)$  is trivial, we cannot expect to realize  $L$  as a quotient of some action as in (3); this constitutes a problem when trying to obtain explicit expressions for nonabelian theta functions; on the other hand, if one considers embedding  $\mathcal{M}_n$  into a projective space to obtain such expressions, it is clear that they will depend on a particular embedding, which in turn depends on the complex structure of  $X$ . Another problem concerns the Kähler quantization of  $\mathcal{M}_n(X)$ , for which it is still an open question whether one can define naturally a unitary projectively flat connection on the bundle of conformal blocks. For a survey of these questions, see [4, 5].

The Verlinde formula [45], which computes the dimension of the space of nonabelian theta functions  $H^0(\mathcal{M}_n, L^k)$  in terms of  $g$ ,  $n$ , and  $k$ , is by now well established from the mathematical point of view [39, 12, 6]. Also, for some particular values of these integers, relations have been found between abelian and nonabelian theta functions.

### 3. ABELIAN THETA FUNCTIONS AND THE HEAT EQUATION

As mentioned above, abelian theta functions of level  $k$  on the abelian variety

$$M_{\Omega} = \mathbb{C}^g / (\mathbb{Z}^g \oplus \Omega \mathbb{Z}^g), \quad \Omega \in \mathbb{H}^g, \tag{8}$$

form a  $k^g$ -dimensional vector space of holomorphic functions on  $(\mathbb{C}^*)^g \cong \mathbb{C}^g / \mathbb{Z}^g$  with a basis given by (2). These functions are the pullback by

$$p : (\mathbb{C}^*)^g \rightarrow M_{\Omega} = (\mathbb{C}^*)^g / \Omega \mathbb{Z}^g \tag{9}$$

of holomorphic sections of  $L^k$ , where  $L$  is the holomorphic line bundle over  $M_{\Omega}$  defined in (4).

It is a well-known classical fact, which goes back to Jacobi, that these functions are the analytic continuation to  $(\mathbb{C}^*)^g$  of the solutions of the following heat equation on  $U(1)^g$ :

$$\frac{\partial u}{\partial t} = - \sum_{\alpha, \beta=1}^g \frac{i}{4\pi} \Omega_{\alpha\beta} \frac{\partial^2 u}{\partial x_{\alpha} \partial x_{\beta}}, \tag{10}$$

evaluated at  $t = \frac{1}{k}$  and with (special) distributional initial conditions for  $t = 0$ ,

$$\varphi_l(x) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i(l+kn) \cdot x} = \sum_{l' \in (\mathbb{Z}_k)^g} \delta\left(x - \frac{l'}{k}\right) e^{2\pi i \frac{l' \cdot l}{k}}. \tag{11}$$

These distributions define fixed  $k^g$ -dimensional subspaces of the space of distributions  $C^{\infty}(U(1)^g)'$ ,

$$V_{k,g}^{U(1)} = \text{span}_{\mathbb{C}} \left\{ \delta\left(x - \frac{l'}{k}\right), l' \in (\mathbb{Z}_k)^g \right\} \subset C^{\infty}(U(1)^g)'. \tag{12}$$

Note that the unitary matrix  $S$  given by

$$S_{l,l'} = k^{-\frac{g}{2}} e^{2\pi i \frac{l' \cdot l}{k}}, \tag{13}$$

which appears in (11), characterizes the modular properties of the integrable representations of the affine Lie algebra  $\widehat{U}(1)_k^g$ . A similar remark applies to the nonabelian case.

A related observation is that the distributions in  $V_{k,g}^{U(1)}$  are supported precisely at the points of order dividing  $k$ ,

$$\left(\frac{1}{k}\mathbb{Z}_k\right)^g \subset U(1)^g = (\mathbb{R}/\mathbb{Z})^g, \tag{14}$$

on the “real part” of  $M_\Omega$ , which are also the  $k$ -Bohr–Sommerfeld points of a real polarization in the geometric quantization of the symplectic manifold underlying this abelian variety.

Let us recall the general notion of a real polarization and of Bohr–Sommerfeld points. Consider a symplectic manifold  $M$  of dimension  $2m$  with a symplectic form  $\omega$ , and let  $L \rightarrow M$  be a Hermitian line bundle with a connection  $\nabla$  that has curvature  $F_\nabla = 2\pi i\omega$ . A real polarization of the data  $(M, \omega, L, \nabla)$  (usually called prequantization data) is a surjective map  $\pi: M \rightarrow B$  onto a manifold  $B$  of dimension  $m$  such that  $\omega|_{\pi^{-1}(b)} = 0$  for all  $b \in B$ . Then, for a generic  $b$ , the fiber  $M_b = \pi^{-1}(b)$  will be a Lagrangian submanifold, and the restriction of  $L$  to  $M_b$  will have a connection with zero curvature. The holonomy of this connection gives rise to a homomorphism

$$H_b: \pi_1(M_b) \rightarrow U(1).$$

One defines  $b \in B$  to be a  $k$ -Bohr–Sommerfeld ( $k$ -BS) point and  $\pi^{-1}(b)$  to be a  $k$ -Bohr–Sommerfeld fiber if  $(H_b)^k = 1$ . The quantization of this system is then provided by the vector space  $H^n(M, \mathcal{S}_\pi)$  with its natural Hilbert space structure [38]. Here,  $\mathcal{S}_\pi$  denotes the sheaf of local sections of  $L^k$  that are covariantly constant along the fibers. Below, following [46, 29], we will also consider nonsmooth real polarizations with singular Bohr–Sommerfeld fibers, for which the notion of covariantly constant section still makes sense.

When  $M$  is a Kähler manifold and  $L$  is a positive holomorphic line bundle on  $M$ , there is an alternative path for the quantization of  $M$ . In this case, one uses a complex polarization and defines the quantum Hilbert space as  $H^0(M, L^k)$ . It is expected that, in most cases, there should be a natural unitary isomorphism between Hilbert spaces corresponding to different complex structures on  $M$ . Moreover, it is natural to expect that the cardinality of the  $k$ -Bohr–Sommerfeld set  $B_{k\text{-BS}} \subset B$  coincides with the dimension of this space (see [46, 29, 43, 44]).

When  $M_\Omega$  is our abelian variety and  $L^k$  are the bundles in (4), the quantizations in the real and complex polarizations can be compared in the following way. The symplectic form on  $M_\Omega$  is given by

$$\omega = \sum_{j=1}^g d\eta_j \wedge d\xi_j,$$

where  $\eta_1, \dots, \eta_j, \xi_1, \dots, \xi_j$  are the coordinates on  $V$  dual to the generators of the lattice  $\Lambda_\Omega$ , which are periodic coordinates on  $M_\Omega$  and related to the complex ones by

$$z = x + iy = \eta + \Omega\xi.$$

The canonical projection

$$\pi: M_\Omega \rightarrow (\mathbb{R}/\mathbb{Z})^g, \quad z \mapsto \eta$$

defines a real polarization of  $M_\Omega$  such that a connection 1-form of  $L^k$  restricted to the fiber  $M_\eta$  over  $\eta$  is  $\alpha = k \sum_{j=1}^g \eta_j d\xi_j$ . It is easy to see that the holonomy map  $H_\eta$  is trivial if and only if  $k\eta \in \mathbb{Z}^g$ , so that the  $k$ -Bohr–Sommerfeld points are the points of order dividing  $k$  on the real torus

$$U(1)_{k\text{-BS}}^g = \left(\frac{1}{k}\mathbb{Z}_k\right)^g.$$

In the next section, we will describe an analytic tool that will allow us to study theta functions, the heat equation, and geometric quantization in a unified way.

4. COHERENT STATE TRANSFORMS

**4.1. CST for Lie groups.** In this section, we describe the coherent state transforms (CSTs) for Lie groups introduced by Hall [22]. For simplicity, we will restrict ourselves to the case when  $K$  is simple. The general compact case can be treated in a similar way. In particular,  $K = U(1)^g$  will be considered in Sections 4.2 and 5.1. Let  $K$  be a compact connected simple Lie group,  $K_{\mathbb{C}}$  be its complexification (see [28]), and  $\Delta_K$  be the Laplacian on  $K$  associated to an Ad-invariant inner product on its Lie algebra  $\text{Lie}(K)$ .

For each  $f \in L^2(K, dx)$ , where  $dx$  is the normalized Haar measure on  $K$ , the image of  $f$  by the CST,  $C_t f$ , is the analytic continuation to  $K_{\mathbb{C}}$  of the solution of the heat equation

$$\frac{1}{\pi} \frac{\partial u}{\partial t} = \Delta_K u \tag{15}$$

with the initial condition given by  $u(0, x) = f(x)$ . The  $K$ -averaged coherent state transform is then defined by

$$C_t: L^2(K, dx) \rightarrow \mathcal{H}(K_{\mathbb{C}}), \quad f \mapsto C_t(f) = \mathcal{C} \circ e^{\pi t \Delta_K}(f), \tag{16}$$

where  $\mathcal{H}(K_{\mathbb{C}})$  denotes the space of holomorphic functions on  $K_{\mathbb{C}}$  and  $\mathcal{C}$  denotes the analytic continuation.

Let  $d\nu_t$  be the  $K$ -averaged heat kernel measure on  $K_{\mathbb{C}}$  defined in [22–26]. It coincides with the Haar measure in the compact directions (corresponding to  $K$ ) and is asymptotically Gaussian in the imaginary directions. Then, the following result holds.

**Theorem 1** (Hall). *For each  $t > 0$ , the mapping  $C_t$  defined in (16) is a unitary isomorphism from  $L^2(K, dx)$  onto the Hilbert space  $L^2(K_{\mathbb{C}}, d\nu_t) \cap \mathcal{H}(K_{\mathbb{C}})$ .*

To obtain a more explicit description of this CST, consider the expansion of  $f \in L^2(K, dx)$  given by the Peter–Weyl theorem,

$$f(x) = \sum_{\mathbb{R}} \text{tr}(\mathbb{R}(x)A_{\mathbb{R}}), \tag{17}$$

where the sum is taken over the set of (equivalence classes of) irreducible representations of  $K$ , and  $A_{\mathbb{R}} \in \text{End } V_{\mathbb{R}}$  is given by

$$A_{\mathbb{R}} = (\dim V_{\mathbb{R}}) \int_K f(x) \mathbb{R}(x^{-1}) dx, \tag{18}$$

$V_{\mathbb{R}}$  being the representation space for  $\mathbb{R}$ . Let  $\{X_i, i = 1, \dots, \dim K\}$  be an orthonormal basis for the Ad-invariant inner product on  $\text{Lie}(K)$  for which the longest root has squared norm 2. Viewing  $\text{Lie}(K)$  as the space of left-invariant vector fields on  $K$ , we have

$$\Delta_K = \sum_{i=1}^{\dim K} X_i X_i,$$

and one obtains

$$C_t f(g) = \sum_{\mathbb{R}} e^{-\pi t c_{\mathbb{R}}} \text{tr}(\mathbb{R}(g)A_{\mathbb{R}}), \tag{19}$$

where  $c_{\mathbb{R}} \geq 0$  is the eigenvalue of  $-\Delta_K$  on functions of the type

$$\text{tr}(A\mathbb{R}(x)), \quad A \in \text{End}(V_{\mathbb{R}}).$$

For the application to nonabelian theta functions on an elliptic curve of the form  $X = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$  with  $\tau \in \mathbb{H}_1$ , it is convenient to consider the operator  $-i\tau\Delta_K$  instead of  $\Delta_K$ . The extension of Theorem 1 to this case is straightforward. We denote the extended CST by the same symbol  $C_t$

and write

$$C_t f(g) = \sum_{\mathbb{R}} e^{i\pi t \tau c_{\mathbb{R}}} \operatorname{tr}(\mathbb{R}(g)A_{\mathbb{R}}). \tag{20}$$

Similarly, for classical theta functions on  $M_{\Omega}$ , the relevant operator is the one in (10) [14].

**4.2. CST for distributions.** Let, first,  $K$  be a simple group. CST (16) can be extended to the space of distributions  $C^{\infty}(K)'$  [14, 15]. As we saw in the abelian case (see (11) and (12)), this is a necessary step if we want to apply the CST to the study of theta functions. The distributions in  $C^{\infty}(K)'$  can be written in the form

$$f = \sum_{\lambda \in \Lambda_W^+} \operatorname{tr}(\mathbb{R}_{\lambda}A_{\lambda}), \tag{21}$$

where we labeled irreducible representations of  $K$  by the highest weights  $\lambda$  in the set of dominant weights  $\Lambda_W^+$ . The Fourier series (21) defines a distribution if the coefficients  $A_{\lambda}$  are such that there exists an integer  $N > 0$  satisfying

$$\lim_{\|\lambda\| \rightarrow \infty} \frac{\|A_{\lambda}\|}{(1 + \|\lambda\|^2)^N} = 0, \tag{22}$$

where  $\|A_{\lambda}\|$  is the operator norm of the endomorphism  $A_{\lambda} \in \operatorname{End}(\mathbb{R}_{\lambda})$ .

The Laplace operator  $\Delta_K$  and its powers act as continuous linear operators on the space  $C^{\infty}(K)'$  and define the action of the operator  $e^{-\pi i t \tau \Delta_K}$  on it, where  $t \geq 0$  and  $\tau \in \mathbb{H}_1$ . For  $f$  of the form (21), we have

$$e^{-\pi i t \tau \Delta_K} f = \sum_{\lambda \in \Lambda_W^+} e^{i\pi t \tau c_{\lambda}} \operatorname{tr}(\mathbb{R}_{\lambda}A_{\lambda}). \tag{23}$$

**Proposition 1** [15]. *If  $f = \sum_{\lambda \in \Lambda_W^+} \operatorname{tr}(\mathbb{R}_{\lambda}A_{\lambda}) \in C^{\infty}(K)'$ , then the series*

$$\sum_{\lambda \in \Lambda_W^+} e^{i\pi t \tau c_{\lambda}} \operatorname{tr}(\mathbb{R}_{\lambda}(g)A_{\lambda}), \tag{24}$$

where  $g \in K_{\mathbb{C}}$ , defines a holomorphic function on  $K_{\mathbb{C}} \times \mathcal{T}_1$ , which we denote by  $C_t f$ . Here,  $\mathcal{T}_1 = \mathbb{H}_1$  stands for the Teichmüller space of genus-1 Riemann surfaces.

Therefore, we have the following commutative diagram:

$$\begin{array}{ccc} L^2(K, dx) & \xrightarrow[\sim]{C_t} & \mathcal{H}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, d\nu_t) \\ \downarrow & & \downarrow \\ C^{\infty}(K)' & \xrightarrow{C_t} & \mathcal{H}(K_{\mathbb{C}}) \end{array} \tag{25}$$

where  $\xrightarrow{\sim}$  stands for an isomorphism and  $\hookrightarrow$ , for an inclusion.

For the case  $K = U(1)^g$  and for a complex Laplacian as in (10), the result is

**Proposition 2** [14]. *If  $f(x) = \sum_{n \in \mathbb{Z}^g} A_n e^{2\pi i n x} \in C^{\infty}(U(1)^g)'$ , then the series*

$$\sum_{n \in \mathbb{Z}^g} A_n e^{i\pi t n \cdot \Omega n} e^{2\pi i n x}, \tag{26}$$

where  $g \in (\mathbb{C}^*)^g$ , defines a holomorphic function on  $(\mathbb{C}^*)^g \times \mathcal{T}_g$ , which we denote by  $C_t f$ , where  $\mathcal{T}_g$  is the Teichmüller space of genus- $g$  Riemann surfaces.

An analogue of the diagram above is now the commutative diagram

$$\begin{CD}
 L^2(U(1)^g, dx) @>C_t>> \mathcal{H}((\mathbb{C}^*)^g) \cap L^2((\mathbb{C}^*)^g, d\nu_t) \\
 @VVV @VVV \\
 C^\infty(U(1)^g)' @>C_t>> \mathcal{H}((\mathbb{C}^*)^g)
 \end{CD} \tag{27}$$

The bottom rows of diagrams (27) and (25) preserve the injectivity of the CST of the upper rows. However, the surjectivity and unitarity are lost when taking the space of distributions. In the next section, we will see how to remedy these problems by looking at theta functions.

### 5. THETA FUNCTIONS AND GENERALIZED CST

**5.1. Abelian varieties.** As mentioned above, the maps  $C_t$  in the bottom rows of diagrams (27) and (25) are neither surjective nor unitary. Considering first the abelian case in (27), we face the following questions.

- Q1. How can the unitarity of  $C_t$  be recovered in a natural way taking into account the fact that, on the right-hand side of (27), we are interested not in  $(\mathbb{C}^*)^g$  but in  $M_\Omega \cong (\mathbb{C}^*)^g/\Omega\mathbb{Z}^g$ ?
- Q2. How to select in a natural way the subspaces  $V_{k,g}^{U(1)} \subset (C^\infty(U(1)^g))'$  of dimensions  $\dim V_{k,g}^{U(1)} = h^0(M_\Omega, L^k) = k^g$  such that their image by the CST corresponds to theta functions?

We have already answered Q2 in Section 3. On the other hand, we want to emphasize that, due to the “hidden” properties of the averaged heat kernel measure  $d\nu_t$  on  $(\mathbb{C}^*)^g$ , the consideration of question Q1 leads *independently* to the same answer to question Q2. Motivated by the map  $p: (\mathbb{C}^*)^g \rightarrow M_\Omega$ , we consider the modified inner product

$$\langle\langle F_1, F_2 \rangle\rangle := \int_D \overline{F_1} F_2 d\nu_t, \tag{28}$$

where  $D \subset (\mathbb{C}^*)^g$  is a fundamental domain for the quotient under  $p$ . This definition makes sense only if the holomorphic functions  $F_1$  and  $F_2$  are such that the integrand in (28),

$$\overline{F_1} F_2 \nu_t, \tag{29}$$

where  $\nu_t$  is the Gaussian function in  $d\nu_t = \nu_t dz$  and  $dz$  denotes the Lebesgue measure in  $\mathbb{C}^g$ , is invariant with respect to the action of  $\Omega\mathbb{Z}^g$  so that (28) does not depend on the choice of the fundamental domain  $D$ . The crucial property of  $\nu_t$  is stated in the following proposition.

**Proposition 3** [14]. *The function  $\nu_{1/k}$  on  $(\mathbb{C}^*)^g$  is the pullback by  $p: (\mathbb{C}^*)^g \rightarrow M_\Omega$  of an Hermitean structure on  $L^k$ .*

This implies the main result.

**Theorem 2** [14]. (i) *Nontrivial functions  $F$  such that the inner products (28) do not depend on the choice of  $D$  exist if and only if  $t = \frac{1}{k}$ ,  $k \in \mathbb{N}$ . For  $t = \frac{1}{k}$ , the set of such functions is the union of vector spaces intersecting only at  $F = 0$ . The space of classical theta functions  $H^0(M_\Omega, L^k)$  is one of these spaces, the others also corresponding to theta functions but translated with respect to (2).*

(ii) *The space  $H^0(M_\Omega, L^k)$  is in the image of  $(C^\infty(U(1)^g))'$  by  $C_{1/k}$  in the second row of (27). Its inverse image coincides with the space  $V_{k,g}^{U(1)}$  in (12).*

It is now clear that, in the process of addressing Q1, we were led to  $C_{1/k}$  and then to an alternative path to the answer to question Q2, expressed in (ii) above. Moreover, we obtain in this way a natural inner product on  $V_{k,g}^{U(1)}$ . We see from (11) and (12) that there is a natural isomorphism from a subspace  $U_{k,g}^{U(1)}$  of  $L^2(U(1)^g, dx)$ ,

$$U_{k,g}^{U(1)} = \text{span}_{\mathbb{C}}\{e^{2\pi i l \cdot x} \mid l: 0 \leq l_\alpha < k, 1 \leq \alpha \leq g\}, \tag{30}$$

to  $V_{k,g}^{U(1)}$  defined by

$$\Phi_k: U_{k,g}^{U(1)} \rightarrow V_{k,g}^{U(1)}, \quad f_l(x) = e^{2\pi i l \cdot x} \mapsto \varphi_l(x) = \sum_{\tilde{l} \in [l]_k} e^{2\pi i \tilde{l} \cdot x}, \tag{31}$$

where  $[l]_k = l + k\mathbb{Z}^g$ . We define on  $V_{k,g}^{U(1)}$  the inner product  $(\cdot, \cdot)$  for which  $\Phi_k$  is unitary. Therefore, the basis  $\{\varphi_l \mid l: 0 \leq l_\alpha < k\}$  is orthonormal. The answer to Q1 is then given by

**Theorem 3** [14]. *The restriction of the CST  $C_{1/k}$  to  $V_{k,g}^{U(1)}$  is an isometric isomorphism from  $(V_{k,g}^{U(1)}, (\cdot, \cdot))$  onto  $(H^0(M_\Omega, L^k), \langle\langle \cdot, \cdot \rangle\rangle)$ .*

Therefore, for principally polarized abelian varieties, diagram (27) can be completed to the commutative diagram in which the top and bottom horizontal arrows are unitary isomorphisms:

$$\begin{array}{ccc} L^2(U(1)^g, dx) & \xrightarrow[\sim]{C_{1/k}} & \mathcal{H}((\mathbb{C}^*)^g) \cap L^2((\mathbb{C}^*)^g, d\nu_t) \\ \downarrow & & \downarrow \\ C^\infty(U(1)^g)' & \xrightarrow{C_{1/k}} & \mathcal{H}((\mathbb{C}^*)^g) \\ \uparrow & & \uparrow \\ V_{k,g}^{U(1)} & \xrightarrow[\sim]{C_{1/k}} & H^0(M_\Omega, L^k) \end{array} \tag{32}$$

**5.2. The Schottky map.** As we have seen above, there are two crucial ingredients that allow the use of a CST for  $K = U(1)^g$  to study abelian theta functions. These are the surjective map (9) from the complex group  $K_{\mathbb{C}} = (\mathbb{C}^*)^g = (U(1)^g)_{\mathbb{C}}$  to the abelian variety  $M_\Omega$  and the quasiperiodicity properties of the averaged heat kernel measure with respect to the same map (9).

Our aim is to apply a CST for  $K = SU(n)^g$  to the study of  $SL(n, \mathbb{C})$  nonabelian theta functions. As we saw in Section 2, for a genus- $g$  Riemann surface  $X$ , the  $SL(n, \mathbb{C})$  theta functions are holomorphic sections of the line bundles  $L^k$  over the moduli space,  $\mathcal{M}_n(X)$ , of rank- $n$  semistable holomorphic vector bundles with trivial determinant. We will consider the Schottky map

$$\tilde{S}: SL(n, \mathbb{C})^g / SL(n, \mathbb{C}) \rightarrow \mathcal{M}_n(X), \tag{33}$$

which is, in many respects, in the Jacobian case, a higher rank analogue of the map  $p$  in (9), and study the pullback of theta functions to

$$SL_g(n, \mathbb{C}) \equiv SL(n, \mathbb{C})^g / SL(n, \mathbb{C}).$$

In fact, this map is not defined everywhere in  $SL_g(n, \mathbb{C})$ , and it is not known whether it is surjective. However, it is known that the Schottky map is locally (in a neighborhood of  $SU_g(n) \equiv SU(n)^g / SU(n) \subset SL_g(n, \mathbb{C})$ ) biholomorphic onto a neighborhood in the moduli space of stable bundles [13]. It is also conjectured that its image is dense in  $\mathcal{M}_n(X)$ . We will try to

identify the properties that characterize a holomorphic function on  $SL_g(n, \mathbb{C})$  whose restriction to the domain of the Schottky map is the pullback of a nonabelian theta function.

Let us now define the Schottky map for a complex linear subgroup  $G_{\mathbb{C}}$  of  $GL(n, \mathbb{C})$  in the case of semistable bundles. There is a well-known bijection between the space of flat  $G_{\mathbb{C}}$ -bundles over  $X$ ,  $H^1(X, G_{\mathbb{C}})$ , and the space of  $G_{\mathbb{C}}$ -representations of the fundamental group of  $X$ ,  $\pi_1(X)$ , modulo overall conjugation  $\text{Hom}(\pi_1(X), G_{\mathbb{C}})/G_{\mathbb{C}}$ ; it is explicitly given by

$$V: \text{Hom}(\pi_1(X), G_{\mathbb{C}})/G_{\mathbb{C}} \rightarrow H^1(X, G_{\mathbb{C}}), \quad \rho \mapsto V_{\rho} := \tilde{X} \times_{\rho} G_{\mathbb{C}}, \tag{34}$$

where the notation means that  $\pi_1(X)$  acts diagonally through  $\rho$  on the trivial  $G_{\mathbb{C}}$ -bundle over the universal cover  $\tilde{X}$  of  $X$  and  $V_{\rho}$  is the quotient with respect to this action.

Let us fix a canonical basis of  $\pi_1(X)$ : elements  $a_1, \dots, a_g, b_1, \dots, b_g$  that generate  $\pi_1(X)$ , subject to the single relation  $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$ . Then, an element  $B = (B_1, \dots, B_g) \in G_{\mathbb{C}}^g$  induces a representation class

$$\rho_B \in \text{Hom}(\pi_1(X), G_{\mathbb{C}})/G_{\mathbb{C}} \tag{35}$$

defined by  $\rho_B(a_i) = 1$  and  $\rho_B(b_i) = B_i$ . The Schottky map is then

$$S: (G_{\mathbb{C}}^g)^{\circ} \rightarrow \mathcal{M}_n, \quad B \mapsto [V_{\rho_B}], \tag{36}$$

where  $[V_{\rho_B}]$  denotes (the isomorphism class of) the holomorphic vector bundle of rank  $n$  over  $X$  defined by the flat  $G_{\mathbb{C}}$ -bundle  $V_{\rho_B}$ . Here,  $(G_{\mathbb{C}}^g)^{\circ}$  denotes the open dense subset of  $G_{\mathbb{C}}^g$  that corresponds to semistable bundles. Note that, for  $G_{\mathbb{C}} = SL(n, \mathbb{C})$ ,  $[V_{\rho_B}]$  has always degree 0. In other words, this map sends a  $g$ -tuple of  $n \times n$  invertible matrices  $(B_1, \dots, B_g) \in G_{\mathbb{C}}^g \subset GL(n, \mathbb{C})^g$  to the flat rank- $n$  holomorphic vector bundle determined by the holonomies  $(1, \dots, 1, B_1, \dots, B_g)$  around the loops  $(a_1, \dots, a_g, b_1, \dots, b_g)$ , respectively.

In the case of line bundles, the situation is simple,  $GL(1, \mathbb{C}) = \mathbb{C}^*$  is abelian and the degree-0 line bundles in  $H^1(X, \mathcal{O}^*)$  form an abelian variety, the Jacobian  $J(X)$  of  $X$ . The Schottky map reduces then to the map  $p$  in (9) (for more detail, see [15]).

**5.3.  $SU(n)$  and  $g = 1$ .** Let us now consider the Schottky map for the case of an elliptic curve  $X_{\tau} = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$  and for semistable bundles of rank  $n$  with trivial determinant over  $X_{\tau}$ . This is a map

$$S: SL(n, \mathbb{C}) \rightarrow \mathcal{M}_n(X_{\tau}).$$

From geometric invariant theory (see [17, 35]), we know that, under the adjoint action,  $SL(n, \mathbb{C})$  has a good quotient, which is a map  $SL(n, \mathbb{C}) \rightarrow T_{\mathbb{C}}/W$ , where  $T_{\mathbb{C}}$  is the maximal torus of  $SL(n, \mathbb{C})$  and  $W$  is the Weyl group; this map sends a matrix to the unordered set of its eigenvalues. Therefore, the Schottky map factors through  $T_{\mathbb{C}}/W$ , as in the following diagram:

$$\begin{array}{ccc} SL(n, \mathbb{C}) & \xrightarrow{q} & T_{\mathbb{C}}/W \\ & \searrow S & \downarrow \tilde{S} \\ & & \mathcal{M}_n(X_{\tau}) \end{array}$$

Therefore, our task reduces to describing  $\tilde{S}$ . Fixing the following Cartan subalgebra of  $sl(n, \mathbb{C})$ ,

$$\mathfrak{h} = \{A \in sl(n, \mathbb{C}) : A \text{ is diagonal}\},$$

and using the corresponding coroot lattice  $\check{\Lambda}_R$ , define  $M$  to be the abelian variety

$$M = \check{\Lambda}_R \otimes X_{\tau} = \check{\Lambda}_R \otimes (\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}) = \mathfrak{h}/(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R).$$

The Weyl group  $W$  acts naturally on  $M$ , via the usual action on  $\check{\Lambda}_R$ , and, as shown in [18, 19, 33],  $\mathcal{M}_n(X_\tau)$  is isomorphic to the  $l = (n - 1)$ -dimensional complex projective space  $\mathbb{P}^l(\mathbb{C})$  obtained as the quotient under this action,

$$\mathcal{M}_n(X_\tau) \cong M/W \cong \mathbb{P}^{n-1}.$$

Let now

$$p: \mathbb{C}^* \rightarrow J(X_\tau) \cong X_\tau$$

be the abelian Schottky map (9) for genus one, where, as usual, we identify the elliptic curve with its Jacobian. It is easy to see that we have

**Proposition 4** [15]. *The following diagram is commutative:*

$$\begin{array}{ccccc}
 SL(n, \mathbb{C}) & \xrightarrow{q} & T_{\mathbb{C}}/W & \xleftarrow{\tilde{\pi}} & T_{\mathbb{C}} \hookrightarrow (\mathbb{C}^*)^n \\
 & \searrow s & \downarrow \tilde{S} & & \downarrow p^n|_{T_{\mathbb{C}}} \\
 & & \mathcal{M}_n(X_\tau) & \xleftarrow{\pi} & M \hookrightarrow J(X_\tau)^n
 \end{array} \tag{37}$$

We can write explicitly the map  $p^n|_{T_{\mathbb{C}}}$  as

$$p^n|_{T_{\mathbb{C}}}: \mathfrak{h}/\check{\Lambda}_R \cong T_{\mathbb{C}} \rightarrow M = \mathfrak{h}/(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R), \quad v + \check{\Lambda}_R \mapsto v + \check{\Lambda}_R + \tau\check{\Lambda}_R. \tag{38}$$

**Proposition 5.** *The space  $H^0(\mathcal{M}_n(X_\tau), L^k)$  of level- $k$  nonabelian theta functions on the moduli space of semistable rank- $n$  holomorphic vector bundles with trivial determinant,  $\mathcal{M}_n(X_\tau)$ , is naturally identified with the space  $\mathcal{H}_{k,\tau}^+$  of holomorphic Weyl invariant functions on  $\mathfrak{h}$  satisfying the quasiperiodicity conditions*

$$\theta(v + \check{\alpha} + \tau\check{\beta}) = e^{-2\pi i k \beta(v) - \pi i k \tau \langle \beta, \beta \rangle} \theta(v), \quad \alpha, \beta \in \Lambda_R; \tag{39}$$

*i. e.,*

$$H^0(\mathcal{M}_n(X_\tau), L^k) \xrightarrow{\sim} \mathcal{H}_{k,\tau}^+ \equiv \{\theta \in \mathcal{H}(\mathfrak{h}) : \theta \text{ satisfies (39), } w\theta = \theta \ \forall w \in W\}.$$

Before we can formulate analogues of questions Q1 and Q2 in Section 5.1, we need an Ad-invariant version of the  $SU(n)$  CST or, more precisely, an Ad-invariant version of diagram (25). Ad-invariant distributions have decomposition (21) with  $A_\lambda = a_\lambda I_\lambda$ , where  $I_\lambda$  denotes the identity operator in  $V_{R_\lambda}$  and the coefficients  $a_\lambda \in \mathbb{C}$  satisfy the growth conditions implied by (22). For such a distribution, we have

$$f = \sum_{\lambda \in \Lambda_W^+} a_\lambda \chi_\lambda, \tag{40}$$

where  $\chi_\lambda$  denotes the character corresponding to  $\lambda$ .

By restricting the coherent state transform (16) to the closed subspace of Ad-invariant functions on  $SU(n)$ , we easily obtain

**Proposition 6** [15]. *The restriction  $C_t^{\text{Ad}}$  of CST (16) to the Hilbert space  $L^2(SU(n), dx)^{\text{Ad}_{SU(n)}}$  is an isometric isomorphism onto the Hilbert space  $L^2(SL(n, \mathbb{C}), d\nu_t)^{\text{Ad}_{SL(n, \mathbb{C})}} \cap \mathcal{H}(SL(n, \mathbb{C}))$ .*

The extension to distributions is straightforward and can be summarized as the following Ad-invariant version of (25).

**Theorem 4** [15]. *The following diagram is commutative:*

$$\begin{array}{ccc}
 L^2(SU(n), dx)^{\text{Ad}} & \xrightarrow{\sim} C_t^{\text{Ad}} & \mathcal{H}(SL(n, \mathbb{C}))^{\text{Ad}} \cap L^2(SL(n, \mathbb{C}), d\nu_t) \\
 \downarrow & & \downarrow \\
 C^\infty(SU(n))^{\prime\text{Ad}} & \xrightarrow{C_t^{\text{Ad}}} & \mathcal{H}(SL(n, \mathbb{C}))^{\text{Ad}}
 \end{array} \tag{41}$$

We can now repeat the questions of Section 5.1 in the following form:

- Q1. How can the unitarity of  $C_t^{\text{Ad}}$  in the second row of (41) be recovered in a natural way taking into account the fact that, on the right-hand side, we are interested not in  $SL(n, \mathbb{C})/SL(n, \mathbb{C}) \cong \mathfrak{h}/W \triangleright \check{\Lambda}_R$  but in  $\mathcal{M}_n(X_\tau) \cong [\mathfrak{h}/(\check{\Lambda}_R + \tau\check{\Lambda}_R)]/W$ ?
- Q2. How to select in a natural way the subspaces

$$V_{k,1}^{SU(n)} \subset (C^\infty(SU(n)))^{\prime\text{Ad}}$$

of dimensions given by the Verlinde numbers, such that their image by the CST corresponds to  $g = 1$  theta functions?

Again, we will see that an answer to Q1 leads to an answer to Q2. As before, consider a modified inner product on  $\mathcal{H}(SL(n, \mathbb{C}))^{\text{Ad}_{SL(n, \mathbb{C})}}$ ,

$$\langle\langle F_1, F_2 \rangle\rangle = \int_{q^{-1}([\mathfrak{h}_0])} \overline{F_1} F_2 d\nu_t, \tag{42}$$

where  $[\mathfrak{h}_0]$  is a fundamental domain for the action of  $W \triangleright (\check{\Lambda}_R + \tau\check{\Lambda}_R)$ .

This definition makes sense only if we consider it for functions for which the inner product does not depend on the choice of  $[\mathfrak{h}_0]$ . The key technical result here is the following integration formula, which is an analogue of the Weyl integration formula for heat kernel measures. Let  $\sigma$  be the denominator in the Weyl character formula,  $\sigma_{\mathbb{C}}$  be its analytic continuation to  $SL(n, \mathbb{C})$ , and  $d\nu_t^{\text{ab}}$  denote the averaged heat kernel measure on  $T_{\mathbb{C}}$  corresponding to the Laplacian

$$\Delta^{\text{ab}} = \sum_{i,j=1}^{n-1} \frac{1}{4\pi^2} C^{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

where  $C^{ij}$  is the inverse Cartan matrix and  $x_i$  are periodic coordinates on  $T$ .

**Theorem 5** [15]. *If  $f \in L^1(SL(n, \mathbb{C}), d\nu_t)$  is a class function, then, for all  $t > 0$ ,*

$$\int_{SL(n, \mathbb{C})} f(g) \nu_t(g) dg = \frac{e^{-2t\pi\|\rho\|^2}}{|W|} \int_{T_{\mathbb{C}}} f(h) |\sigma_{\mathbb{C}}(h)|^2 \nu_t^{\text{ab}}(h) dh. \tag{43}$$

By (43), the independence of (42) with respect to the choice of  $[\mathfrak{h}_0]$  corresponds to considering functions  $F \in \mathcal{H}(SL(n, \mathbb{C}))^{\text{Ad}_{SL(n, \mathbb{C})}}$  such that

$$|F|^2 |\sigma_{\mathbb{C}}|^2 \nu_t^{\text{ab}} \tag{44}$$

is  $\tau\check{\Lambda}_R$ -invariant as a function on  $\mathfrak{h}$ . Crucial here are, as in the abelian case of Proposition 3, the quasiperiodicity properties satisfied by  $\nu_t^{\text{ab}}$ .

**Proposition 7** [15]. *The function  $\nu_t^{\text{ab}}$  satisfies the following quasiperiodicity conditions:*

$$\begin{aligned} \nu_t^{\text{ab}}(v + \check{\alpha}) &= \nu_t^{\text{ab}}(v), \\ \nu_t^{\text{ab}}(v + \tau\check{\alpha}) &= |e^{2\pi i \frac{1}{t} \alpha(v)}|^2 |e^{\pi i \frac{1}{t} \tau \langle \alpha, \alpha \rangle}|^2 \nu_t^{\text{ab}}(v) \quad \text{for all } \check{\alpha} \in \check{\Lambda}_R. \end{aligned} \tag{45}$$

These imply the following

**Corollary 1.** *The function  $\nu_{1/k'}^{\text{ab}}$ ,  $k' \in \mathbb{N}$ , considered as a function on  $T_{\mathbb{C}}/W$ , is the pullback by  $\tilde{S}: T_{\mathbb{C}}/W \rightarrow \mathcal{M}(X_\tau)$  of an Hermitean structure on  $L^k$ .*

Let us consider first the case  $n > 2$ . From Theorem 5, Proposition 7, and Corollary 1, we obtain

**Theorem 6** [15]. (i) *For  $K = SU(n)$ ,  $n > 2$ , nontrivial functions  $F$  such that the inner products (42) do not depend on the choice of  $[\mathfrak{h}_0]$  exist if and only if  $t = \frac{1}{k+n}$ ,  $k \in \mathbb{N} \cup \{0\}$ . For  $t = \frac{1}{k+n}$ , the set of such functions forms a vector space  $\mathcal{H}_{k,\tau} \subset \mathcal{H}(SL(n, \mathbb{C}))^{\text{Ad}}$  naturally isomorphic to  $H^0(M_n(X_\tau), L^k)$  with an isomorphism given by*

$$\varphi: \mathcal{H}_{k,\tau} \xrightarrow{\sim} H^0(M_n(X_\tau), L^k), \quad F \mapsto \frac{e^{i\pi\tau \frac{\|\rho\|^2}{k+n}}}{|W|} \sigma_{\mathbb{C}} F|_{T_{\mathbb{C}}}. \tag{46}$$

(ii) *The space  $\mathcal{H}_{k,\tau}$  is in the image of  $(C^\infty(SU(n, \mathbb{C}))')^{\text{Ad}}$  by  $C_{1/(k+n)}^{\text{Ad}}$  in the second row of (41). Its inverse image defines the space  $V_{k,1}^{SU(n)}$  as in Q2.*

**Proof** (of (i)). If the function  $|F|^2 |\sigma_{\mathbb{C}}|^2 \nu_t^{\text{ab}}$  is  $(\check{\Lambda}_R \oplus \tau\check{\Lambda}_R)$ -invariant, then, from Proposition 7 and the holomorphicity in  $\mathfrak{h} \times \mathcal{T}_1$ , it follows that  $F$  must also satisfy the following quasiperiodicity conditions:

$$(F\sigma_{\mathbb{C}})(v + \tau\check{\alpha}) = e^{-2\pi i \langle d, \alpha \rangle} e^{-2\pi i \frac{1}{t} \alpha(v)} e^{-\pi i \frac{1}{t} \tau \langle \alpha, \alpha \rangle} (F\sigma_{\mathbb{C}})(v), \tag{47}$$

where  $d \in \mathfrak{h}_{\mathbb{R}}^*$ . Nonzero holomorphic functions on  $\mathfrak{h}/\check{\Lambda}_R$  satisfying (47) do not exist if  $1/t \notin \mathbb{N}$ . This results from the fact that if  $1/t \notin \mathbb{N}$ , then the automorphy factors in (47) are not invariant under  $v \mapsto v + \check{\beta}$ ,  $\check{\beta} \in \check{\Lambda}_R$ , which makes it impossible to solve (47) on  $\mathfrak{h}/\check{\Lambda}_R$ . We see from (39) that the functions satisfying (47) with  $\frac{1}{t} = k' \in \mathbb{N}$  are level- $k'$  theta functions. Since  $\sigma_{\mathbb{C}}$  is  $W$ -anti-invariant and the functions  $F$  are  $W$ -invariant, the functions  $F\sigma_{\mathbb{C}}$  satisfying (47) are  $W$ -anti-invariant theta functions that exist only for level  $k' \geq n$  [33].  $\square$

Let  $D_k = \{\lambda \in \Lambda_W^+ \mid \langle \lambda, \hat{\alpha} \rangle \leq k\}$ , where  $\hat{\alpha}$  is the highest root of  $sl(n, \mathbb{C})$ . This set labels the integrable representations of the affine Kac–Moody algebra  $\widehat{sl}(n, \mathbb{C})_k$ . Consider also the affine Weyl group  $W^{\text{aff}} := W \triangleright \Lambda_R$ , and let us define the level- $(k+n)$  action of  $W^{\text{aff}}$  on  $\mathfrak{h}_{\mathbb{R}}^*$  by

$$\tilde{w} \cdot \lambda = w \cdot (\lambda + (n+k)\alpha), \quad \tilde{w} = (w, \alpha) \in W^{\text{aff}}. \tag{48}$$

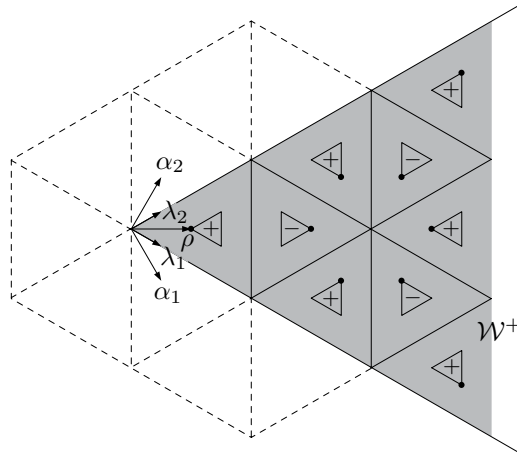
The vector space  $V_{k,1}^{SU(n)}$  has a basis given by

$$\varphi_{\gamma,k} = \sum_{\lambda + \rho \in [\gamma + \rho]_k \cap \Lambda_W^+} \epsilon_\lambda \chi_\lambda, \quad \gamma \in D_k, \tag{49}$$

where  $[\gamma + \rho]_k$  denotes the orbit of  $\gamma + \rho$  under the action (48) of  $W^{\text{aff}}$  and  $\epsilon_\lambda = \epsilon(w)$ , where  $\tilde{w}$  is a unique element of  $W^{\text{aff}}$  such that  $\lambda + \rho = \tilde{w} \cdot (\gamma + \rho)$ . The figure below shows an example of the orbit  $[\gamma + \rho]_k$  inside the fundamental Weyl chamber  $\mathcal{W}^+ = \mathbb{R}_{\geq 0} \cdot \Lambda_W^+$ .

As in Section 5.1, we see that there is a natural isomorphism from a subspace  $U_{k,1}^{SU(n)}$  of  $L^2(SU(n), dx)^{\text{Ad}}$ ,

$$U_{k,1}^{SU(n)} = \text{span}_{\mathbb{C}}\{\chi_\gamma, \gamma \in D_k\}, \tag{50}$$



The points  $\lambda + \rho \in [\gamma + \rho]_k \cap \Lambda_W^+$  in the case of  $SU(3)$  for  $k = 1$  and  $\gamma = 0$ , together with the corresponding sign  $\epsilon_\lambda$

to  $V_{k,1}^{SU(n)}$  defined by

$$\Phi_k: U_{k,1}^{SU(n)} \rightarrow V_{k,1}^{SU(n)}, \quad \chi_\gamma \mapsto \varphi_{\gamma,k}. \tag{51}$$

We define the inner product  $(\cdot, \cdot)$  on  $V_{k,1}^{SU(n)}$  for which  $\Phi_k$  is unitary. Therefore, the basis  $\{\varphi_\gamma, \gamma \in D_k\}$  is orthonormal. The answer to Q1 is then given by

**Theorem 7** [15]. *The restriction of the CST  $C_{1/(k+n)}^{Ad}$  to  $V_{k,1}^{SU(n)}$  is a unitary transform from  $(V_{k,1}^{SU(n)}, (\cdot, \cdot))$  onto  $(\mathcal{H}_{k,\tau}, \langle\langle \cdot, \cdot \rangle\rangle)$ .*

Therefore, for theta functions on the moduli space of holomorphic vector bundles on elliptic curves, we have the following commutative diagram instead of diagram (32):

$$\begin{array}{ccc}
 L^2(SU(n), dx)^{Ad} & \xrightarrow[\sim]{C_{1/(k+n)}^{Ad}} & \mathcal{H}(SL(n, \mathbb{C}))^{Ad} \cap L^2(SL(n, \mathbb{C}), dv_t) \\
 \downarrow & & \downarrow \\
 C^\infty(SU(n))^{Ad} & \xrightarrow{C_{1/(k+n)}^{Ad}} & \mathcal{H}(SL(n, \mathbb{C}))^{Ad} \\
 \uparrow & & \uparrow \\
 V_{k,1}^{SU(n)} & \xrightarrow[\sim]{C_{1/(k+n)}^{Ad}} & \mathcal{H}_{k,\tau} \\
 & & \downarrow \varphi \wr \\
 & & H^0(\mathcal{M}_n(X_\tau), L^k)
 \end{array} \tag{52}$$

where all the isomorphisms are unitary.

Note that the extended CST in Theorem 7 defines a unitary connection on the bundle of conformal blocks that coincides with the KZBH connection.

Consider now the geometric quantization in a real polarization of

$$\mathcal{M}_n(X_\tau) = [\mathfrak{h}/(\check{\Lambda}_R + \tau \check{\Lambda}_R)]/W$$

with the symplectic form given by the polarizing form of the abelian variety  $M = \mathfrak{h}/(\check{\Lambda}_R + \tau \check{\Lambda}_R)$ . We can view  $\mathcal{M}_n(X_\tau)$  as the symplectic manifold of flat  $SU(n)$  connections on  $X_\tau$  with its canonical symplectic structure [1]

$$\text{Hom}(\pi_1(X_\tau), SU(n))/SU(n).$$

Since  $\pi_1(X_\tau)$  is abelian, this space is isomorphic to the product of the maximal torus of  $SU(n)$  by itself, quotiented by  $W$  acting diagonally,  $(T \times T)/W$ , and then a (nonsmooth) real polarization is just given by the projection onto the first factor,

$$(T \times T)/W \rightarrow T/W$$

(see [46]). Under the canonical identification of  $T/W$  with the Weyl alcove

$$\mathfrak{A} = \{\nu \in \mathcal{W}^+ \mid \langle \nu, \alpha \rangle \leq 1\} \subset \mathfrak{h}_{\mathbb{R}}^*,$$

the Bohr–Sommerfeld points of level  $k$  are the points of the weight lattice divided by  $k + n$  inside the Weyl alcove

$$(T/W)_{k\text{-BS}} \cong \left(\frac{1}{k+n}\Lambda_W\right) \cap \mathfrak{A}.$$

As in the abelian case, we can relate the distributions  $\varphi_{\gamma,k}$  in (51) to the Dirac delta distributions supported on these Bohr–Sommerfeld points. In fact, for  $\gamma \in D_k \subset \Lambda_W^+$ , using additive notation for points on  $T/W \cong \mathfrak{A}$ , we can expand such a delta distribution as

$$\delta\left(v - \frac{\gamma+\rho}{k+n}\right) = \sum_{\lambda \in \Lambda_W^+} \overline{\chi_\lambda\left(\frac{\gamma+\rho}{k+n}\right)} \chi_\lambda(v) \tag{53}$$

since the characters form an orthonormal basis for  $L^2(SU(n), dx)^{\text{Ad}}$ . From (49), we obtain

$$\delta\left(v - \frac{\gamma+\rho}{k+n}\right) = \sum_{\gamma' \in D_k} \chi_{\gamma'}\left(-\frac{\gamma+\rho}{k+n}\right) \varphi_{\gamma'}(v). \tag{54}$$

Again, notice that the matrix

$$A_{\gamma,\gamma'} = \chi_{\gamma'}\left(-\frac{\gamma+\rho}{k+n}\right) \tag{55}$$

is, up to scale, a unitary  $S$ -matrix that characterizes the modular properties of the integrable representations of the affine Lie algebra  $\widehat{SU}(n)_k$  [30]. Note also that the delta distributions in (54) are supported on regular elements of the level-1 action of  $W^{\text{aff}}$  on  $\mathfrak{h}_{\mathbb{R}}^*$ . This justifies the consideration of Bohr–Sommerfeld points of level  $k + n$  in the level- $k$  quantization of  $\mathcal{M}_n(X_\tau)$ .

Finally, in the case when  $K = SU(2)$ , Theorem 6 is replaced by

**Theorem 8** [15]. (i) *For  $K = SU(2)$ , nontrivial functions  $F$  such that the inner product (42) does not depend on the choice of  $\mathfrak{h}_0$  exist if and only if  $t = \frac{2}{2k+3}$  or  $t = \frac{1}{k+2}$ ,  $k \in \mathbb{N} \cup \{0\}$ . For each of these values of  $t$ , such functions form the union of two vector spaces intersecting only at 0, with dimensions given by the Verlinde number  $k + 1$ . One of these vector spaces is naturally isomorphic to  $H^0(\mathcal{M}_2(X_\tau), L^k)$ . The other one corresponds to sections of orbifold line bundles over  $\mathcal{M}_2(X_\tau)$ .*

(ii) *The space  $H^0(\mathcal{M}_2(X_\tau), L^k)$  is in the image of  $(C^\infty(SU(2, \mathbb{C}))')^{\text{Ad}}$  by  $C_{1/(k+2)}^{\text{Ad}}$  in the second row of (41). Its inverse image defines the space  $V_{k,1}^{SU(2)}$  as in Q2.*

## 6. $SU(2)$ AND $g \geq 2$

**6.1. Unitary Schottky bundles and trinion decompositions.** It is known from the theorem of Narasimhan and Seshadri [36] that  $\mathcal{M}_n$  is the moduli space of flat  $SU(n)$  connections on  $X$ :

$$\mathcal{M}_n = \text{Hom}(\pi_1(X), SU(n))/SU(n).$$

In the higher genus case  $g \geq 2$ , we will restrict ourselves in this section to rank-2 bundles, since a suitable real polarization of  $\mathcal{M}_2$  was explicitly defined in [29]. To describe it, one uses a trinion

decomposition of  $X$ ,  $\alpha = (\alpha_1, \dots, \alpha_{3g-3})$ , where the  $\alpha_i$  form a maximal set of simple closed nonintersecting pairwise nonhomotopic curves on  $X$ . A trinion decomposition defines a trivalent graph  $\Gamma_\alpha$  (where each trinion (a three-holed sphere) corresponds to a vertex and each  $\alpha_i$ , to an edge) and also a 3-dimensional handlebody whose boundary is  $X$  and where the  $\alpha_i$  are homotopically trivial.

To any fixed trinion decomposition  $\alpha$ , we can associate the following map:

$$\pi_\alpha : \mathcal{M}_2 \rightarrow [0, 1]^{3g-3} \subset \mathbb{R}^{3g-3}, \quad A \mapsto \frac{1}{\pi} \arccos \left( \frac{1}{2} \operatorname{tr}(H_{\alpha_i}(A)) \right)_{i=1}^{3g-3},$$

where  $H_\gamma(A)$  denotes the holonomy of the flat connection  $A$  around the loop  $\gamma$  ( $\pi_\alpha$  does not depend on a choice of basepoint since the trace is conjugation invariant). This map  $\pi_\alpha$  defines a (nonsmooth) real polarization of  $\mathcal{M}_2$  whose image is a convex polytope  $\Delta_\alpha$  defined by the inequalities

$$\begin{aligned} \theta_{i_1} + \theta_{i_2} - \theta_{i_3} &\geq 0, \\ \theta_{i_1} + \theta_{i_2} + \theta_{i_3} &\leq 2, \end{aligned} \tag{56}$$

whenever  $\alpha_{i_1}$ ,  $\alpha_{i_2}$ , and  $\alpha_{i_3}$  are the three curves bounding a trinion, and where  $\theta_i$  denotes the  $i$ th coordinate in  $\mathbb{R}^{3g-3}$ . It is known that the fibers of  $\pi_\alpha$  are generically Lagrangean tori, although there are some singular fibers [29]. There is, in particular, a maximally degenerate fiber  $\pi_\alpha^{-1}(0)$  that consists of connections that are trivial on all of the  $\alpha_i$  and therefore define connections on the handlebody. Since the fundamental group of the handlebody is free on  $g$  generators, this fiber (which, of course, as a subset of  $\mathcal{M}_2$ , depends on the trinion decomposition  $\alpha$ ) consists of unitary Schottky representations; so, we define

$$\mathcal{S}_\alpha := \pi_\alpha^{-1}(0) \subset \mathcal{M}_2, \quad \mathcal{S}_\alpha \cong SU(2)^g/SU(2).$$

Then, we have the following

**Theorem 9.** *For any trinion decomposition  $\alpha$ , there is another (“dual”) trinion decomposition  $\beta$  such that the map*

$$\phi := \pi_\beta|_{\mathcal{S}_\alpha} : \mathcal{S}_\alpha \rightarrow \Delta_\beta$$

*is surjective. For  $g = 2$ ,  $\phi$  is a homeomorphism, and for  $g > 2$ , it is a finite cover of  $\Delta_\beta$ .*

We can make the following remarks.

1. One cannot hope that  $\phi$  is one-to-one and onto since it is known that the space  $SU(2)^g/SU(2)$  is not contractible in general (for instance,  $SU(2)^3/SU(2) \cong S^6$ , see [9]).

2. Clearly, the statement only depends on the graphs of the decompositions. But the roles of  $\alpha$  and  $\beta$  are not interchangeable since the graph of  $\beta$  cannot contain a separating arc  $\beta_2$  (equivalently, a loop  $\beta_1$ ), i.e., a configuration of the form



which has the shape of a “handle”  $X_1$  (a torus with one hole) in  $X$ . If this were the case, any other trinion decomposition  $\alpha$  would have a curve  $\alpha_1$  in  $X_1$  which would be homotopic either to  $\beta_1$  or to a curve that intersects  $\beta_1$  at one point and, together with  $\alpha_1$ , generates the fundamental group of the handle, and introducing the relation  $\beta_2 = [\alpha_1, \beta_1]$  in  $\pi_1(X)$ . Then, since any connection in the space  $\mathcal{S}_\alpha$  has holonomy 1 around  $\alpha_1$ , this would imply either  $B_1 = 1$  or  $B_2 = 1$  (according to either  $\alpha_1 = \beta_1$  or  $\beta_2 = [\alpha_1, \beta_1]$ ), which means that  $\phi$  cannot be onto.

3. It is not difficult to prove the theorem in the case  $g = 2$ . We have only two genus-2 trivalent graphs. Consider the trinion decomposition  $\alpha$  that gives a  $\Theta$ -shaped graph (the other one can be

treated in an analogous way). Assume that the corresponding theta-shaped handlebody is laid flat on a table, so that the three  $\alpha$ 's are like meridians and separate two  $\varepsilon$ -shaped trinions. This defines  $\pi_\alpha$  and the unitary Schottky set  $\mathcal{S}_\alpha \subset \mathcal{M}$ . Let  $\beta$  be a decomposition with three horizontal curves on the intersection of the surface with a horizontal plane cutting the handlebody into two pieces, the “equators,” so that, with respect to  $\beta$ , we have north and south trinions ( $\Gamma_\beta$  is also a theta graph, as it should be by remark 2). Let  $\theta = (\theta_1, \theta_2, \theta_3) \in \Delta_\beta$  be arbitrary numbers satisfying equations (56), and  $A \in \pi_\beta^{-1}(\theta)$ . Notice that, if  $B_i = H_{\beta_i}(A)$ ,  $i = 1, 2, 3$ , we have  $B_1 B_2 B_3 = 1$  (for an appropriate choice of orientations of the  $\beta$ 's). Note also that with two  $\alpha$ 's and two  $\beta$ 's (call them  $\alpha_1, \alpha_2, \beta_1, \beta_2$ ) we can form a symplectic basis of  $\pi_1(X) = \{\alpha_1, \alpha_2, \beta_1, \beta_2 : [\alpha_1, \beta_1][\alpha_2, \beta_2] = 1\}$ , so that

$$\mathcal{M} = \{(a_1, a_2, b_1, b_2) \in SU(2)^4 : [a_1, b_1][a_2, b_2] = 1\} / SU(2).$$

It is now clear that, under the above identification, the matrices  $a_1 = a_2 = 1$ ,  $b_1 = B_1$ , and  $b_2 = B_2$  define another connection  $\tilde{A}$  with the properties  $\pi_\beta(\tilde{A}) = \theta$  and  $\tilde{A} \in \mathcal{S}_\alpha$ , so that  $\phi$  is onto. From [29, Proposition 3.1], we also see that, given  $\theta = (\theta_1, \theta_2, \theta_3) \in \Delta_\beta$ , the choice of matrices  $B_i$  such that  $\text{tr}(B_i) = 2 \cos(\pi\theta_i)$  (and such that  $B_1 B_2 B_3 = 1$ ) is unique up to simultaneous conjugation, so that  $\phi$  is also injective.

The proof of Theorem 9 and further discussions will appear in [16]. From our point of view, the relevance of this result is related to the possibility of describing nonabelian theta functions as (a complexification of) solutions of a certain heat equation on  $SU(2)^g$ , as happens for genus 1 and for abelian theta functions.

From [29], we know that, given an integer  $k$ , the polytopes  $\Delta_\beta$  have special points labeling the Bohr–Sommerfeld fibers of the real polarization  $\pi_\beta$ . The number of these fibers is equal to the Verlinde number for rank 2 and level  $k$ . Hence, it is desirable to compare the polytopes with the unitary Schottky spaces and, in particular, to see if “Bohr–Sommerfeld” points on  $\mathcal{S}_\alpha$  have some special significance for the “initial conditions” of these theta functions.

**6.2. Spaces of distributions with Verlinde dimensions.** Let us recall three facts that were behind the success of the strategy of Section 5 in the following two cases:

- A. Abelian varieties, including moduli spaces of line bundles on curves of arbitrary genus, or  $G = U(1)$ , an arbitrary-genus case.
- B.  $G = SU(n)$ , a  $g = 1$  case.

The facts that are valid for cases A and B are the following.

**Fact 1.** By considering the Schottky map (see the definition in (36) and (9), (33))

$$K_{\mathbb{C}} \xrightarrow{S} \mathcal{M}(X), \tag{58}$$

where  $K_{\mathbb{C}} = (\mathbb{C}^*)^g$  for case A and  $K_{\mathbb{C}} = SL(n, \mathbb{C})$  for case B, one realizes the so-called vector bundle of conformal blocks on the Teichmüller space  $\mathcal{T}_g$  of genus- $g$  curves,

$$\tilde{\mathcal{H}}_{k,g} \rightarrow \mathcal{T}_g, \quad \tilde{\mathcal{H}}_{k|_X} := H^0(\mathcal{M}(X), L^k), \tag{59}$$

as a subbundle  $\mathcal{H}_k$  of the trivial bundle

$$\mathcal{H}_{k,g} \hookrightarrow \mathcal{T}_g \times \mathcal{H}(K_{\mathbb{C}}). \tag{60}$$

The fibers  $\mathcal{H}_{k,X}$  of  $\mathcal{H}_{k,g}$  are finite-dimensional subspaces of  $\mathcal{H}(K_{\mathbb{C}})$  that satisfy the quasiperiodicity (see (1) and (39)) and other properties, e.g., the Ad-invariance in the nonabelian case.

**Fact 2.** The fibers  $\mathcal{H}_{k,X}$  for different curves  $X$  can be canonically identified via the CST, by taking the correspondence of a theta function for a given complex structure with a distribution corresponding to the degenerate complex structure  $\Omega = 0$ . In this context, the extended CST is unitary for a natural Hermitean structure and defines on  $\mathcal{H}_{k,g} \rightarrow \mathcal{T}_g$  a unitary flat connection that

coincides with the KZBH connection. Horizontal sections are solutions of

$$\begin{cases} \frac{\partial \theta}{\partial s} = \Delta \theta, \\ \theta(s = 0) \in V_{k,g}^G \subset (C^\infty(K))'^{\text{Ad}}, \end{cases} \tag{61}$$

where  $s = 0$  corresponds to  $\Omega = 0$ .

**Fact 3.** For the Haar measure on  $K$ , an orthonormal basis  $\{f_\gamma\}_{\gamma \in \Gamma}$  of  $L^2(K, dx)^{\text{Ad}}$  is labeled by a set  $\Gamma$  of  $n$ -tuples ( $n = g$ ) of irreducible representations of  $G$ . So,  $\Gamma = \widehat{U}(1)^g$  in case A and  $\Gamma = \widehat{SU}(n)$  in case B. Let  $\widetilde{\Gamma} = \Gamma$  in case A and  $\widetilde{\Gamma} = \Lambda_W \supset \Gamma$  in case B. For every level  $k \in \mathbb{N}$ , there is an action of an infinite discrete group  $P$  on  $\widetilde{\Gamma}$  ( $P = \mathbb{Z}^g$  in case A and  $P = W \triangleright \Lambda_R = W^{\text{aff}}$  for case B). The quotient  $\widetilde{\Gamma}/P_{k'}$  of  $\widetilde{\Gamma}$  by the level- $k'$  action of  $P$  ( $k' = k$  in case A and  $k' = k + n$  in case B) has a number of classes  $m$  that coincides with the Verlinde number in both cases. To every class  $[\gamma + \rho]_{k'}$  ( $\rho = 0$  in case A), there corresponds one distribution in  $V_{k,g}^G$ ,

$$\varphi_{\gamma,k} = \sum_{\tilde{\gamma} \in [\gamma]_k} f_{\tilde{\gamma}} \tag{62}$$

in case A and

$$\varphi_{\gamma,k} = \sum_{\tilde{\gamma} + \rho \in [\gamma + \rho]_{k'} \cap \Lambda_W^+} \epsilon_{\tilde{\gamma}} f_{\tilde{\gamma}} \tag{63}$$

in case B, where  $\epsilon_\gamma$  can take the values  $\pm 1$ . These distributions form a basis of  $V_{k,g}^G$  and are therefore the initial conditions for  $\theta$  functions in (61). Their CST image gives an orthonormal frame of the bundle  $\mathcal{H}_{k,g} \rightarrow \mathcal{T}_g$ .

For genus  $g \geq 2$  and rank  $n \geq 2$ , we still have the Schottky map (see (36)), so that we can still, in principle, consider the bundle of conformal blocks as a subbundle of the trivial bundle  $\mathcal{T}_g \times \mathcal{H}(SL(n, \mathbb{C})^g)^{\text{Ad}}$ . The problem here is that the Schottky map is rather complicated, and we do not know an analogue of the quasiperiodicity properties that the sections  $\theta$  have to satisfy. Also, even though there exists a flat connection on  $\mathcal{H}_{k,g} \rightarrow \mathcal{T}_g$ , the form of the horizontal sections is not known, nor is their behavior in the limit  $\Omega \rightarrow 0$ . However, Fact 3 above remains valid (almost) precisely as stated for cases A and B, except for the last part concerning the orthonormality of the corresponding frame (still conjectural). Namely, for  $n = 2$ , there continue to exist natural spaces  $V_{k,g}^{SU(2)} \subset (C^\infty(SU(2)^g))'^{\text{Ad}}$ , with  $\dim V_{k,g}^{SU(2)} = \text{Verlinde number}$ , obtained from the level- $(k + n)$  action of the affine Weyl group on  $\Lambda_W$ . We conjecture that these distributions are the boundary values of horizontal holomorphic sections of the KZBH connection for  $SL(2, \mathbb{C})$  and genus  $g$ , at particular singular boundary points of  $\mathcal{T}_g$ . Moreover, by choosing an orthonormal basis of these distribution spaces (see below), we should obtain nonabelian theta functions corresponding to orthonormal frames for  $\mathcal{H}_{k,g}$ , therefore fixing the Hermitean structure on this bundle.

These natural spaces  $V_{k,g}^{SU(2)}$  can be obtained as follows. Let  $X$  be a given genus- $g$  Riemann surface with a fixed trinion decomposition  $\alpha$  as in the previous section. As we will see, a basis for  $V_{k,g}^{SU(2)}$  can be given that corresponds to labeling the  $(3g - 3)$  edges of the graph  $\Gamma_\alpha$  with irreducible representations of  $SU(2)$ , subject to the condition that they are all level- $k$  integrable representations and that the tensor product of three representations meeting at a vertex must contain the trivial representation. As shown in [29], the set of these labelings,  $\Xi_k$ , has cardinality equal to the Verlinde number for genus  $g$ , rank 2, and level  $k$ . To describe these distributions, we will first consider the associated orthonormal spin network basis on  $L^2(SU(2)^g, dx)^{\text{Ad}}$ , where  $dx$  denotes the Haar measure. For the trivalent graph  $\Gamma_\alpha$ , there is a natural isomorphism

$$\frac{SU(2)^{3g-3}}{SU(2)^{2g-2}} \xrightarrow{\sim} \frac{SU(2)^g}{SU(2)}, \tag{64}$$

where the denominator in the first quotient acts by gauge transformations at the vertices of the graph, while that in the second quotient is the diagonal conjugation action of  $SU(2)$ . The advantage of the left-hand side in (64) is that it gives a natural orthonormal basis  $\{f_\gamma\}_{\gamma \in \Xi}$  on  $L^2(SU(2)^g, dx)^{\text{Ad}}$ , where  $\Xi = \bigcup_{k=1}^\infty \Xi_k$ , namely, the so-called spin network basis [3, 40]. Now,  $\Xi$  is the set of labelings of the edges of  $\Gamma_\alpha$  with irreducible representations of  $SU(2)$  such that the tensor product of three representations meeting at a vertex contains the trivial representation. Note that, for  $SU(2)$ , such tensor product contains only one copy of the trivial representation; i.e., there is only one invariant tensor. By assigning group elements to each of the  $(3g - 3)$  edges, taking their corresponding representation matrices, and producing this invariant tensor, one defines a function on the left-hand side of (64); this corresponds to an Ad-invariant function  $f_\gamma$  on  $SU(2)^g$ , which we can normalize so that  $\|f_\gamma\| = 1$  in  $L^2(SU(2)^g, dx)^{\text{Ad}}$ .

**Theorem 10** [3]. *The set  $\{f_\gamma, \gamma \in \Xi\}$  forms an orthonormal basis for  $L^2(SU(2)^g, dx)^{\text{Ad}}$ .*

Consider the level- $(k + 2)$  orbits of the action of the affine Weyl group  $W^{\text{aff}}$  on  $\Lambda_W$  and the spaces

$$U_{k,g}^{SU(2)} = \text{span}_{\mathbb{C}} \langle f_\gamma \in L^2(SU(2)^g, dx)^{\text{Ad}} : \gamma \in \Xi_k \rangle$$

as in (50). Consider the following map analogous to (51):

$$\Phi_k : U_{k,g}^{SU(2)} \rightarrow (C^\infty(SU(2)^g))^{\text{Ad}}, \quad f_\gamma \mapsto \varphi_{\gamma,k} = \sum_{\tilde{\gamma} + \hat{\rho} \in [\gamma + \hat{\rho}]_{k+2} \cap \Xi} \epsilon_{\tilde{\gamma}} f_{\tilde{\gamma}}, \quad (65)$$

where  $\hat{\rho} = (\rho, \dots, \rho) \in \Lambda_W^{3g-3}$ ,  $[\gamma + \hat{\rho}]_{k+2}$  denotes the orbit of the  $(3g - 3)$ -tuple of  $SU(2)$  representations  $\gamma + \hat{\rho}$  under  $(W^{\text{aff}})^{3g-3}$ , and  $\epsilon(\gamma) = \prod_{j=1}^{3g-3} \epsilon_{\gamma_j}$  with  $\epsilon_{\gamma_j}$  defined as in (49).

The map  $\Phi_k$  is well defined and is injective. By analogy with the previous cases, we denote the image of  $U_{k,g}^{SU(2)}$  by  $V_{k,g}^{SU(2)}$ . This reasoning, together with the results of the previous section and the situation in cases A and B above, leads us to the following natural conjectures.

**Conjecture 1.** *The space  $V_{k,g}^{SU(2)}$  is a linear span of Dirac delta functions supported on the points of  $\mathcal{S}_\alpha$  that are in the Bohr–Sommerfeld fibers of the moment map associated to the dual trinion decomposition  $\beta$ , as in the previous section.*

**Conjecture 2.** *The distributions in  $V_{k,g}^{SU(2)}$  are boundary values of the KZBH-horizontal non-abelian theta functions of rank 2 and genus  $g$  when the curve  $X$  degenerates to the graph  $\Gamma_\alpha$ .*

Further details and results concerning these conjectures will appear in [16].

### ACKNOWLEDGMENTS

The work was partially supported by the Center for Mathematics and Applications, IST, Lisbon (C.F. and J.M.), by the Center for Analysis, Geometry and Dynamical Systems, IST, Lisbon (J.P.N.), and also by the FCT (Portugal) via the program POCTI and FEDER and by the projects POCTI/33943/MAT/2000 and CERN/FIS/43717/2002.

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*This article was submitted by the authors in English*