On Schottky vector bundles over Riemann surfaces

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Let X be a compact Riemann surface of genus $g \geq 2$. Narasimhan and Seshadri's theorem [NS2], identifies the moduli space $\mathcal{M}^{st}(n)$, of flat *stable* holomorphic vector bundles of rank n over X, with the complex manifold formed by all n-dimensional irreducible unitary representations of the fundamental group of the surface. We can think of this result as being the vector bundle analog of the classical Fuchsian uniformization theorem for compact Riemann surfaces, since it states that any such vector bundle can be obtained as a quotient of the trivial bundle on the upper half plane, by a unitary representation of $\pi_1(X)$. Therefore, one can hope that, for vector bundles, there is also an analog of the Schottky uniformization of a Riemann surface, which would determine the class of vector bundles obtained from Schottky representations.

In this dissertation, we study vector bundles over X, associated to a general linear representation of the Schottky group of the surface; these will be called *Schottky vector bundles*. It is easy to see that, contrary to the case of unitary vector bundles, every Schottky bundle admits infinitely many non-conjugate Schottky representations. Moreover, there are examples of Schottky vector bundles which do not admit unitary representations. After discussing analytic structures on the space of conjugacy classes of Schottky representations, we consider the canonical map from this space to $\mathcal{M}^{st}(n)$ and prove that it is a local diffeomorphism on a nonempty open set of Schottky representations. From this, it follows that an open set in the moduli space of flat stable bundles will consist of Schottky vector bundles.

Contents

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Chapter 0

Introduction.

Holomorphic vector bundles over a compact Riemann surface have been studied from several points of view. Weil was the first to consider flat vector bundles: those obtained from linear representations of the fundamental group of the surface [W]. Since spaces of representations can admit nice analytic or algebraic structures, it is tempting to use this correspondence between vector bundles and representations to construct a moduli space for flat vector bundles. Mumford noticed that a nice moduli space for vector bundles can only be obtained if we restrict the set of bundles, and consider for instance, only stable or semistable ones [Mu].

Narasimhan and Seshadri [NS2] found that the algebro-geometric concept of stability corresponded exactly to irreducible unitary representations under Weil's correspondence. In other words they proved that, given a flat stable vector bundle E of rank n, over a Riemann surface X of genus $g \geq 2$, there is an irreducible unitary representation of $\pi_1(X)$, ρ (unique up to conjugation), such that E can be represented as the quotient $(\mathbb{H} \times \mathbb{C}^n)/\rho$, where ρ acts diagonally (on $\mathbb{C}^n$ via ρ, and on $\mathbb{H}$ via the Fuchsian group Γ uniformizing X). Hence, they identified the moduli space of flat stable vector bundles $\mathcal{M}^{st}(n)$, with the complex open manifold of n-dimensional irreducible unitary representations, which will be denoted by $\mathcal{U}^{st}(n)$. Stated in this way, this theorem resembles the classical Fuchsian uniformization of compact Riemann surfaces, which says that every such surface of genus $g \geq 2$ can be represented as a quotient $X = \mathbb{H}/\Gamma$, where $\Gamma \cong \pi_1(X)$ is a Fuchsian group.

There is another uniformization theorem for Riemann surfaces: the Schottky uniformization, which represents every compact Riemann surface as another quotient $X = \Omega/\Sigma$, where now Σ is a Schottky group: a finitely generated strictly loxodromic free Kleinian group, and Ω is a connected region in the Riemann sphere. Therefore it is natural, from this point of view, to look for properties of bundles constructed from n-dimensional linear representations of a Schottky group and to try to relate the Schottky representation space to the moduli space of vector bundles. A very simple count of the number of parameters involved indicates that both the moduli space of rank n flat stable vector bundles and the space of n-dimensional representations of a Schottky group modulo conjugation, have the same dimension: $n^2(g-1)+1$.

On a Riemann surface X realized as Ω/Σ for a Schottky group Σ, we can therefore consider the problem of determining whether every flat vector bundle E over X can be represented as the quotient $(\Omega \times \mathbb{C}^n)/\rho$, where ρ is an n-dimensional linear representation of the free group Σ, acting diagonally as before; a bundle E admitting such a realization will be called a *Schottky vector bundle*. This question, which can be viewed as a Schottky uniformization problem for vector bundles, is still open, to the best of our knowledge.

Returning to the uniformization of compact Riemann surfaces, it is known that the Fuchsian and Schottky uniformizations are related through a potential for the Weil-Peterson metric on Teichmüller space (see Zograf and Takhtajan [ZT1]). A Schottky uniformization for vector bundles could possibly result in a generalization of this result for the Kähler metric on the moduli space of vector bundles ([T], [ZT2]). Another reason to consider this problem is related to the so called generalized theta functions, which are sections of the determinant line bundle over $\mathcal{M}^{st}(n)$. Since the space of equivalence classes of Schottky representations of stable bundles $\mathcal{S}^{st}(n)$, is a Stein manifold, if the canonical map from this space to $\mathcal{M}^{st}(n)$ were surjective, by pulling back the Poincaré bundle over $\mathcal{M}^{st}(n)$ we would obtain a trivial vector bundle over $\mathcal{S}^{st}(n)$, thereby describing the generalized theta functions as holomorphic functions (see Beauville [Be]).

In the present work, we analyze Schottky vector bundles and compare them to the general case of flat vector bundles, and in particular to the unitary ones. In chapter 1, we present the important relationship between representations of the fundamental group and vector bundles, as introduced by Weil and Grothendieck and summarize what is known for the case of unitary representations. The statement of Schottky uniformization is recalled, and we define what is meant by a Schottky vector bundle. In chapter 2, simple cases of Schottky vector bundles are considered: the case of line bundles and the case of vector bundles over an elliptic curve. In both these cases, it turns out that every flat vector bundle admits a Schottky representation. These simple cases already allow us to see that, contrary to the case of unitary vector bundles, every Schottky vector bundle admits infinitely many non-conjugate Schottky representations. In chapter 3, we study complex analytic structures on some spaces of conjugacy classes of representations and relations among them, and consider the canonical maps from these representation spaces to the moduli space of flat stable bundles. The differential of one of these maps is in turn related to periods of certain differentials on the Riemann surface. Finally in chapter 4, we compare unitary and Schottky vector bundles: we find examples of (semistable) Schottky vector bundles which do not admit a unitary representation. Then, we use some properties of the periods of differentials to prove that there is an open set in the moduli space of stable bundles consisting of Schottky vector bundles.

Chapter 1

Preliminaries.

In this first chapter, we recall the fundamental relationship between vector bundles over a compact Riemann surface and representations of its fundamental group. We mention the theorems of Weil [W] and Narasimhan and Seshadri [NS2], recall the notion of stability, and discuss the case of unitary representations. We also review briefly how a Riemann surface can be uniformized by a Schottky group (the classical retrosection theorem of Koebe), and define Schottky vector bundles.

1.1 Representations of $\pi_1(X)$ and their associated vector bundles.

Let X be a compact Riemann surface of genus g. One way of constructing holomorphic vector bundles over X is the following. Let $\tilde{X} \times \mathbb{C}^n$ be the trivial complex vector bundle, of rank n, over the universal covering surface, $\tilde{X}$, of X. The fundamental group of X acts on $\tilde{X}$ by deck transformations. For any representation ρ, of $\pi_1(X)$ in the vector space $\mathbb{C}^n$ (which we will write as $\rho \in \mathrm{Hom}(\pi_1(X), GL(n,\mathbb{C}))$) we have an action of $\pi_1(X)$ in the trivial bundle $\tilde{X} \times \mathbb{C}^n$, as follows:

$$\gamma \cdot (z,v) = (\gamma \cdot z, \rho(\gamma)v) \qquad \forall \gamma \in \pi_1(X), (z,v) \in \tilde{X} \times \mathbb{C}^n$$

A vector bundle with such an action is called a $\pi_1(X)$-vector bundle over $\tilde{X}$. By identifying all points under this action: $\gamma \cdot (z,v) = (z,v)$ for all $\gamma \in \pi_1(X)$ and $(z,v) \in \tilde{X} \times \mathbb{C}^n$, we get a holomorphic vector bundle over X, which will be denoted by $\tilde{X} \times_\rho \mathbb{C}^n$, or $V(\rho)$, for short (see [Gu] for details on this construction). André Weil was the first to study the vector bundles obtained in this way [W]. He showed that they have degree zero (the **degree** of a rank n vector bundle E, being defined as usual, as the first Chern class evaluated on the fundamental cycle: $\deg(E) = c_1(\wedge^n E)[X] \in \mathbb{Z}$) and that a certain converse holds. Let us say that a vector bundle E is **flat** if it arises from the above construction, i.e, if there is a $\rho \in \mathrm{Hom}(\pi_1(X), GL(n,\mathbb{C}))$ such that $E \cong V(\rho)$. A holomorphic vector bundle E is called **indecomposable** if it cannot be written as a direct sum $E = E_1 \oplus E_2$ of holomorphic sub-bundles. Unless otherwise specified, the words *bundle* or *vector bundle* will always mean *holomorphic vector bundle.*

Theorem 1.1 (Weil [W]) *A vector bundle E over X is flat if and only if all of its indecomposable components have degree 0.* $\square$

Two holomorphic vector bundles E and F are called **isomorphic** when there is a global holomorphic bundle map from E to F, restricting to an isomorphism on each fiber. This amounts to a global holomorphic section of the vector bundle $E^* \otimes F$, where E^* denotes the dual of the vector bundle E. Given two representations $\rho_1, \rho_2 \in \mathrm{Hom}(\pi_1(X), GL(n, \mathbb{C}))$, it is natural to ask when the vector bundles $V(\rho_1)$ and $V(\rho_2)$ are isomorphic. Of course, conjugate representations will give rise to isomorphic bundles, but in general, non-conjugate representations can produce isomorphic vector bundles. Since an isomorphism between $V(\rho_1)$ and $V(\rho_2)$ is, in particular, a global holomorphic section of $V(\rho_1)^* \otimes V(\rho_2) = V({}^t\rho_1^{-1} \otimes \rho_2)$, we are interested in describing sections of bundles associated to representations. As $V(\rho)$ is constructed from a $\pi_1(X)$-bundle, the vector space of sections of $V(\rho)$, $H^0(X, V(\rho))$, can be viewed as the space of "ρ-equivariant" holomorphic functions on $\tilde{X}$:

$$H^0(X, V(\rho)) = \left\{ \begin{array}{c} \text{holomorphic } f : \tilde{X} \to \mathbb{C}^n : \\ f(\gamma z) = \rho(\gamma) f(z) \quad \forall z \in \tilde{X}, \gamma \in \pi_1(X) \end{array} \right\}$$

The same principle applies to sections of other bundles. An important case is the sections of $\mathrm{End}V(\rho) \otimes K$ where K is the canonical line bundle on X whose sections are holomorphic 1-forms on X. Noting that $\mathrm{End}V(\rho) = V(\rho)^* \otimes V(\rho) = V({}^t\rho^{-1} \otimes \rho) = V(\mathrm{Ad}_\rho)$, where Ad_ρ denotes the adjoint representation, we see that:

$$H^0(X, \mathrm{End}V(\rho) \otimes K) = \left\{ \begin{array}{c} \text{holomorphic} \quad \omega : \tilde{X} \to \mathrm{Mat}_{n \times n}(\mathbb{C}) \\ \omega(\gamma z)\gamma'(z) = \rho(\gamma)\omega(z)\rho(\gamma)^{-1} \\ \forall z \in \tilde{X}, \gamma \in \pi_1(X) \end{array} \right\}$$

where the prime $'$ stands for $\frac{d}{dz}$. We have the following.

Lemma 1.2 *Let $\rho_1, \rho_2 \in Hom(\pi_1(X), GL(n, \mathbb{C}))$ and $z_0 \in \tilde{X}$ be a basepoint. The following three conditions are equivalent:*

1) $V(\rho_1) \cong V(\rho_2)$

2) there is a holomorphic function $f : \tilde{X} \to GL(n, \mathbb{C})$ such that

$$f(\gamma z) = \rho_2(\gamma) f(z) \rho_1(\gamma)^{-1} \qquad \forall \gamma \in \pi_1(X), z \in \tilde{X}$$

3) There exists $\omega \in H^0(X, EndV(\rho_1) \otimes K)$ such that

$$\rho_2(\gamma) = f_\omega(\gamma z)\rho_1(\gamma) f_\omega(z)^{-1} \qquad \forall \gamma \in \pi_1(X), z \in \tilde{X},$$

where $f_\omega(z)$ is the unique solution of the differential equation $f^{-1}df = \omega, f(z_0) = I$.

proof: ($1 \Leftrightarrow 2$): As we noted above, an isomorphism between $V(\rho_1)$ and $V(\rho_2)$ is a holomorphic global section F of $V(\rho_1)^* \otimes V(\rho_2) = V({}^t\rho_1^{-1} \otimes \rho_2)$, which consists of invertible matrices. So it corresponds to a holomorphic $f : \tilde{X} \to GL(n, \mathbb{C})$ such that $f(\gamma \cdot z) = ({}^t\rho_1^{-1} \otimes \rho_2)(\gamma) f(z) =$

$\rho_2(\gamma)f(z)\rho_1(\gamma)^{-1}$ for all $\gamma \in \pi_1(X), z \in \tilde{X}$. ($2 \Leftrightarrow 3$): If we have f as in 2), and define $\omega(z) := f(z_0)f(z)^{-1}df(z)f(z_0)^{-1}$, then ω will be an endomorphism valued differential, with the required properties. To see this, differentiate 2):

$$0 = d\rho_2(\gamma) = d(f \circ \gamma)\rho_1(\gamma)f^{-1} + (f \circ \gamma)\rho_1(\gamma)d(f^{-1}) \Leftrightarrow$$
$$\Leftrightarrow ((f^{-1}df) \circ \gamma)\gamma' = \rho_1(\gamma)(f^{-1}df)\rho_1(\gamma)^{-1}$$

This means that $f^{-1}df$ belongs to $H^0(X, \mathrm{End}V(\rho_1) \otimes K)$, and therefore, so does ω, because ρ_1 and $f(z_0)\rho_1 f(z_0)^{-1}$ define the same bundle. Conversely, if we have 3), then clearly f_ω verifies 2). Note that the differential equation $f^{-1}df = \omega, f(z_0) = I$ has a single-valued solution since $\tilde{X}$ is simply connected. □

Let us now recall some properties of holomorphic vector bundles that will be used later on, and in particular the notion of stability. The **slope** of a vector bundle E is the rational number $\mu(E) = \deg(E)/\mathrm{rk}(E)$. E is called **stable** (respectively **semistable**) if, for all proper holomorphic sub-bundles $F \subset E$, we have $\mu(F) < \mu(E)$ (resp. $\mu(F) \leq \mu(E)$). We say that E is **simple** if the only automorphisms of E are scalar, i.e, the only global holomorphic sections of $\mathrm{End}E = \mathrm{Hom}(E, E)$ are the constant multiples of the identity endomorphism: $\dim H^0(X, \mathrm{End}E) = 1$. Homomorphisms between stable or semistable vector bundles have properties similar to homomorphisms between line bundles, as we see below, justifying in part, the introduction of the concept of stability (the following results appear in several references, for example [NS2] and [NR], but are included for convenience).

Proposition 1.3 *Given two semistable vector bundles F, E with $\mu(F) > \mu(E)$, there are no nonzero homomorphisms from F to E, i.e, $H^0(X, Hom(F, E)) = 0$. Similarly, if E, F are stable vector bundles and $\mu(F) \geq \mu(E)$, then either $H^0(X, Hom(F, E)) = 0$ or $E \cong F$; moreover $dim H^0(X, Hom(E, E)) = 1$, so that a stable bundle is simple.*

proof: Let $f : F \to E$ be a bundle homomorphism. It is a standard fact, that if K denotes the kernel bundle of f and H is the bundle "generically generated by the image of f" (see [NS2], for instance), there is a commutative diagram, with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & F & \to & F/K & \to & 0 \\ & & & & f\downarrow & & g\downarrow & & \\ 0 & \leftarrow & E/H & \leftarrow & E & \leftarrow & H & \leftarrow & 0 \end{array}$$

where g is of maximal rank, i.e, $\wedge^m g : \wedge^m(F/K) \to \wedge^m H$ (where $m = \mathrm{rk}(F/K) = \mathrm{rk}(H)$) is a non-zero homomorphism of line bundles. By standard properties of line bundles $\deg(F/K) \leq \deg(H)$, and by the stability of F and E, we have:

$$\mu(F) \leq \mu(F/K) \leq \mu(H) \leq \mu(E),$$

which contradicts $\mu(F) > \mu(E)$. Similarly, if F and E are stable and non-isomorphic and f is nonzero, we will have:

$$\mu(F) < \mu(F/K) < \mu(H) < \mu(E).$$

Finally, if $E \cong F$ of rank n, then

$$\begin{aligned} 0 = \deg(E \otimes E^*) = \deg(K \otimes E^*) + \deg(H \otimes E^*) = \\ = \deg(K)\ n - \deg(E)\ \mathrm{rk}(K) + \deg(H)\ n - \deg(E)\ \mathrm{rk}(H) = \\ = [\mu(K) - \mu(E)]n\ \mathrm{rk}(K) + [\mu(H) - \mu(E)]n\ \mathrm{rk}(H) \end{aligned}$$

since E is stable, both summands in the last row are strictly negative, unless one of K or H are zero. So, we found that any automorphism of a stable bundle is either an isomorphism or it is 0. Now let λ be an eigenvalue of f. Then $h := f - \lambda I : E \to E$ is an automorphism of E which is not an isomorphism, so $f = \lambda I$, i.e, f is a scalar, and so E is simple as wanted. □

For clarity and completeness, let us establish a few relations between representations and their associated vector bundles. For a representation of $\pi_1(X)$, ρ, let $Z(\rho)$ denote the set of $n \times n$ complex matrices which commute with all $\rho(\gamma)$ for all $\gamma \in \pi_1(X)$. $Z(\rho)$ is a vector subspace of the set of all complex $n \times n$ matrices, $Mat_{n\times n}(\mathbb{C})$, and has complex dimension at least 1, because it always contains the scalar matrices.

Proposition 1.4 *Let $\rho \in Hom(\pi_1(X), GL(n, \mathbb{C}))$. Then:*

$$\begin{array}{ccccc} V(\rho)\ \textit{stable} & \overset{1}{\Rightarrow} & V(\rho)\ \textit{simple} & \overset{2}{\Rightarrow} & V(\rho)\ \textit{indecomposable} \\ \Downarrow 3 & & \Downarrow 4 & & \\ \rho\ \textit{irreducible} & \overset{5}{\Rightarrow} & dim Z(\rho) = 1 & & \end{array}$$

proof: (1) was proved in the last proposition. (2) If $E = E_1 \oplus E_2$ then clearly $\mathrm{End}E_1 \oplus \mathrm{End}E_2 \subset \mathrm{End}E$ so that $dim H^0(X, \mathrm{End}E) \geq 2$. (3) If ρ is reducible, preserving a vector space V, then we can decompose our space as $\mathbb{C}^n \cong V \oplus W$, and the vector bundle associated with $\rho|_V$, $\tilde{X} \times_\rho V$ is a proper holomorphic sub-bundle of $V(\rho)$ of degree 0, so $V(\rho)$ cannot be stable. (4) If there are more than just the scalar matrices commuting with all $\rho(\gamma)$, $\gamma \in \pi_1(X)$, then they will be non-scalar global sections of $\mathrm{End}V(\rho)$, so clearly dim $H^0(X, \mathrm{End}V(\rho)) \geq 2$. (5) This is a simple application of Schur's lemma: if ρ is irreducible then any matrix commuting with all matrices $\rho(\gamma)$ has to be a scalar matrix. □

1.2 Unitary representations.

Weil noted also the importance of unitary representations in this construction. For instance, the analogue of lemma 1.2 is much simpler in this case: *if ρ and σ are unitary representations then $V(\rho) \cong V(\sigma)$ if and only if ρ and σ are conjugate in $U(n)$* (see [NS1]). Suggested by this fact, Narasimhan and Seshadri [NS1], considered the space of conjugacy classes of irreducible n-dimensional unitary representations of $\pi_1(X)$, and proved that it is an open complex manifold, which we will denote by $\mathcal{U}^{st}(n)$. They found also that the vector bundles obtained from these representations are stable, and conversely, every stable vector bundle can be obtained from a unitary irreducible representation. See the original paper [NS2], for a proof.

Theorem 1.5 (Narasimhan and Seshadri [NS2]). *If $\rho \in Hom(\pi_1(X), U(n))$ is irreducible, then $V(\rho)$ is a stable bundle. Conversely, if E is a stable flat bundle, then $E = V(\rho)$ for some unique (up to conjugation) irreducible, unitary representation ρ.* $\square$

This result identifies the complex manifold $\mathcal{U}^{st}(n)$ with the quasi-projective variety of stable flat vector bundles of rank n, $\mathcal{M}^{st}(n)$. Proposition 1.4 becomes now very simple in the case of unitary representations.

Proposition 1.6 *If $\rho \in Hom(\pi_1(X), U(n))$ then:*

$$V(\rho) \text{ stable} \Leftrightarrow V(\rho) \text{ simple} \Leftrightarrow V(\rho) \text{ indecomposable} \Leftrightarrow \rho \text{ irreducible.}$$

proof: According to proposition 1.4 we just need to prove that ρ irreducible implies $V(\rho)$ stable, but this is the easy part of Narasimhan and Seshadri's theorem above. $\square$

As a side remark, we mention that Donaldson reproved Narasimhan-Seshadri's theorem by minimizing a Yang-Mills type functional for unitary connections on holomorphic vector bundles [D].

1.3 Schottky uniformization of Riemann surfaces.

Let us recall how a Riemann surface is uniformized by a Schottky group. We start with a few definitions. Schottky groups are an important class of **Kleinian groups**: groups of Möbius transformations acting properly discontinuously on some domain of the Riemann sphere. A **Schottky group** is a strictly loxodromic finitely generated free Kleinian group (see Maskit [Ma]), and a **marked Schottky group of genus** g is a Schottky group Σ, together with a choice of g free generators $T_1, ..., T_g \in PSL(2, \mathbb{C})$ of Σ. There is a notion of equivalence between two marked Schottky groups: $(\Sigma, T_1, ..., T_g)$ is equivalent to $(\Sigma', T_1', ..., T_g')$ if there exists a Möbius transformation M such that $T_i' = MT_iM^{-1}$ for all $i = 1, ..., g$. The set of equivalence classes of marked Schottky groups of genus g is called the **Schottky space of genus** g. It is an open set in $\mathbb{C}^{3g-3}$ (see, for instance, Chuckrow [C]). For every Schottky group Σ, the domain of discontinuity of Σ is a region in the complex plane Ω_Σ whose complement is a Cantor set (see [C] or [AS]).

Schottky groups can be used to construct surfaces, because the action of Σ on Ω_Σ produces a compact Riemann surface $X := \Omega_\Sigma/\Sigma$. An important result is that every marked Schottky group $(\Sigma, T_1, ..., T_g)$ has a standard fundamental domain (see [C]), which is a connected region D, in the Riemann sphere, bounded by smooth closed curves $C_1, ..., C_g, C_1', ..., C_g'$, each lying on the outside of all the others, such that $T_i(C_i) = C_i'$ (of course D is not determined uniquely by the marked Schottky group). Under the canonical holomorphic map $\Omega_\Sigma \to X$, the boundary curves of a standard fundamental domain are mapped onto smooth non-intersecting simple closed curves $\alpha_1, ..., \alpha_g$; this motivates the following terminology, introduced by Bers ([Bs2]).

A **complete set of retrosections** on a Riemann surface of genus g is a choice of g smooth simple non-intersecting, homologically independent, closed curves $\alpha_1, ..., \alpha_g$.

Figure 1.1:

We see therefore, that a marked Schottky group together with the choice of a standard fundamental domain determines a Riemann surface with a complete set of retrosections. A much deeper statement is that every compact Riemann surface can be obtained in this way, which is the content of the classical retrosection theorem.

Theorem 1.7 (Koebe [Kr]) *For every compact Riemann surface X of genus g with a complete set of retrosections $(\alpha_1, ..., \alpha_g)$, there exists a marked Schottky group of genus g, $\Sigma =< T_1, ...T_g >$ and a fundamental domain for Σ with $2g$ boundary curves $C_1, ..., C_g, C_1', ..., C_g'$, such that $X = \Omega_\Sigma/\Sigma$ and the map $\Omega_\Sigma \to X$ sends both C_i and C_i' to α_i. Moreover, the marked Schottky group is unique up to equivalence and up to $T_i \mapsto T_i^{-1}$.*

Sketch of proof: (Following Ahlfors and Sario [AS]) Let N denote the smallest normal subgroup of $\pi_1(X)$ containing the homotopy classes of $\alpha_1, ..., \alpha_g$. N is an infinitely generated subgroup of $\pi_1(X)$ whose quotient $\Sigma = \pi_1(X)/N$ is an abstract free group on g generators. Now, consider the abstract Riemann surface X_N which corresponds, via covering space theory, to the subgroup N of $\pi_1(X)$. It is clear that X is given as a quotient of X_N by a group of bijective self mappings of X_N, isomorphic to Σ. It is also easy to see that X_N will have planar character, and so, by Koebe's mapping theorem, it can be mapped onto a connected region of the plane $\Omega \subset \hat{\mathbb{C}}$ Second, we prove that $X_N = \Omega$ belongs to the class $\mathcal{O}_{AD}$, of Riemann surfaces with no non-constant analytic functions with finite Dirichlet integral (see Ahlfors and Sario [AS], Ch.IV §19F). It is easy to prove that any injective self map of a region in the class $\mathcal{O}_{AD}$ is a fractional linear transformation ([AS], Ch.IV §2D). Hence, X is actually given by a quotient of Ω by a free group of Möbius transformations, also denoted Σ, which is therefore a Schottky group. □

Another proof of the existence part of this theorem has been found by Bers [Bs2] using the theory of quasi-conformal mappings.

Remark 1.8 From the theorem, we see that a compact Riemann surface X with a complete set of retrosections $\alpha_1, ..., \alpha_g$ determines a region $\Omega_\Sigma \in \mathbb{H}$ (uniquely up to conjugation by a by a Möbius transformation) which is the domain of discontinuity of the corresponding marked Schottky group $\Sigma =< T_1, ..., T_g >$ and also a holomorphic covering map $p : \Omega_\Sigma \to X$, whose group of deck transformations is precisely Σ. We can form the following short exact sequence of groups:

$$1 \to N \to \pi_1(X) \to \Sigma \to 1,$$

where $N = \pi_1(\Omega)$ is the smallest normal subgroup of $\pi_1(X)$ containing $\alpha_1, ..., \alpha_g$ (If there is no danger of confusion, we will denote curves in X and their homotopy classes by the same symbols). We can therefore choose elements $\beta_1, ..., \beta_g$ of the fundamental group of X, such that their projection onto Σ is precisely the marked generators of Σ: $T_1, ..., T_g$, respectively. It is then easy to verify that $\pi_1(X)$ can be generated by $\alpha_1, ..., \alpha_g$ and $\beta_1, ..., \beta_g$ with the single relation $\prod_{i=1}^{g}[\alpha_i, \beta_i] = 1$. From now on, we will assume that X is a compact Riemann surface of genus g with a *fixed* complete set of retrosections $\alpha_1, ..., \alpha_g$. We also fix $\Sigma = \Sigma(X) =< T_1, ..., T_g >$, $\Omega = \Omega_\Sigma$ and $\beta_1, ..., \beta_g \in \pi_1(X)$ as described above, and use both uniformizations of X ($X = \Omega_\Sigma/\Sigma$ as before, and $X = \mathbb{H}/\Gamma$, where Γ is a Fuchsian group) interchangeably.

1.4 Schottky vector bundles.

Since a Schottky group Σ is a subgroup of $PSL(2, \mathbb{C})$, the vector bundle analogue of the Riemann surface Ω/Σ is the quotient of a bundle over Ω by a representation of Σ in $GL(n, \mathbb{C})$. To build bundles like these, we generalize the construction of section 1, by allowing other coverings of X. Let $p : Y \to X$ be a (unbranched) covering, with group of deck transformations Γ. Let ρ be a **factor of automorphy for** p, i.e, a function $\rho : Y \times \Gamma \to GL(n, \mathbb{C})$, holomorphic in Y, verifying:

$$\rho(\gamma_1\gamma_2, y) = \rho(\gamma_1, \gamma_2 y)\rho(\gamma_2, y) \qquad \forall \gamma_1, \gamma_2 \in \Gamma, y \in Y$$

As before, we can make the trivial rank n vector bundle over Y, $Y \times \mathbb{C}^n$ into a Γ-vector bundle, by letting Γ act as:

$$\gamma \cdot (y, v) = (\gamma \cdot y, \rho(\gamma, y)v) \qquad \forall \gamma \in \Gamma, y \in Y, v \in \mathbb{C}^n.$$

The quotient vector bundle under this action, $Y \times_\rho \mathbb{C}^n$, will be denoted by $V_Y(\rho)$ and is then a vector bundle over X. In the case where $Y = \tilde{X}$ is the universal cover of X, $\Gamma = \pi_1(X)$ and ρ does not depend on Y (note that a constant factor of automorphy is just a representation of the group of deck transformations), we recover the initial construction: $V_{\tilde{X}}(\rho) = V(\rho)$.

It is easy to see that any vector bundle can be constructed from (non constant) factors of automorphy of the Schottky cover.

Proposition 1.9 *Let X be a compact Riemann surface with a complete set of retrosections and $p : \Omega \to X$ its Schottky cover. Then every vector bundle E over X is isomorphic to $V_\Omega(\rho)$, for some factor of automorphy $\rho : \Sigma \times \Omega \to GL(n, \mathbb{C})$.*

proof: Consider the pull-back bundle $F = p^*E$ over Ω. By definition of the pullback, if F_y denotes the fiber of F over the point $y \in \Omega$, we have canonical identifications: $F_y = F_{\gamma y}$, for all $\gamma \in \Sigma$. Since Ω is an open Riemann surface, and hence, a Stein space, p^*E is a holomorphically trivial bundle. Let $h : F \to \Omega \times \mathbb{C}^n$ be a holomorphic trivialization. Then, $h|_{F_y} : F_y \to \mathbb{C}^n$ is an isomorphism of vector spaces, and therefore $\rho(\gamma, y) := h|_{F_{\gamma y}} \circ (h|_{F_y})^{-1}$ is a well defined isomorphism from $\mathbb{C}^n$ to $\mathbb{C}^n$, i.e, and invertible matrix for all $y \in \Omega, \gamma \in \Sigma$, depending holomorphically on y. It is easy to check that ρ is indeed a factor of automorphy, and that $E = V_\Omega(\rho)$. $\square$.

Of course, the problem is to find vector bundles given by factors of automorphy that are, in fact, representations of Σ. Therefore, we make the following definition.

Definition 1.10 *Let X be a Riemann surface with a fixed complete set of retrosections. A vector bundle E over X is a* **Schottky vector bundle** *if there is a Schottky representation $\rho \in Hom(\Sigma, GL(n, \mathbb{C}))$, such that $E \cong V_\Omega(\rho)$.*

In practice, we will use the following alternative description of Schottky vector bundles, whose verification is an easy consequence of using both uniformizations of X.

Proposition 1.11 *A vector bundle E over X is Schottky if and only if $E = V(\rho)$, where $\rho \in Hom(\pi_1(X), GL(n, \mathbb{C}))$ is such that $\rho(\alpha_i) = \mathbf{1}$ for all $i = 1, ...g$.* □

In particular, we see that a Schottky vector bundle is flat.

Chapter 2

Simple examples of Schottky bundles.

In this chapter we describe Schottky vector bundles in the simplest cases. When the rank of the bundle is one, i.e, in the case of line bundles, we find that every line bundle of degree 0 is Schottky, although the Schottky representations of each bundle are not unique. When the underlying Riemann surface is the Riemann sphere or a surface of genus one, we see that again every flat vector bundle is Schottky.

2.1 Schottky line bundles.

Schottky line bundles can be described in complete detail. Again, let $\alpha_1, ..., \alpha_g, \beta_1, ..., \beta_g$, $\prod_{i=1}^{g} [\alpha_i, \beta_i] = 1$, be canonical generators of the fundamental group of X, and we let $\omega_i \in H^0(X, K)$, $\quad i = 1, ..., g$, be a normalized (dual) basis of holomorphic differentials. By this we mean that $\int_{\alpha_i} \omega_j = \delta_{ij}$ and $\int_{\beta_i} \omega_j = \Pi_{ij}$, where Π_{ij} are the entries of the period matrix, which is symmetric and has a positive definite imaginary part, by the classical Riemann bilinear relations. The results in this section can be found in a slightly different setting in Kra [Kr], Ch. VI, §4, for instance.

The space of one-dimensional linear representations of $\pi_1(X)$, $\mathrm{Hom}(\pi_1(X), GL(1, \mathbb{C})) = \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ is the multiplicative abelian group $(\mathbb{C}^*)^{2g}$, and $\mathrm{Hom}(\Sigma, \mathbb{C}^*) = (\mathbb{C}^*)^g$ is one of its subgroups. Tautologically, all line bundles are stable, so that this notion is irrelevant here, and the moduli space of flat line bundles $\mathcal{M}^{st}(1) = \mathcal{M}(1)$ coincides with the Jacobian of X, $Jac(X)$. In this case, the correspondence between representations of $\pi_1(X)$ and their associated line bundles turns into a well defined holomorphic map between complex manifolds:

$$V : \mathrm{Hom}(\pi_1(X), \mathbb{C}^*) = (\mathbb{C}^*)^{2g} \to Jac(X), \qquad \rho \mapsto V(\rho).$$

We can give a detailed description of the fiber of this map, i.e, find an explicit condition when two representations give the same line bundle.

Lemma 2.1 *Two representations $\rho, \sigma \in Hom(\pi_1(X), \mathbb{C}^*)$ give the same line bundle if and only* *if there exist complex numbers* $c_1, ..., c_g$ *such that, for all j=1,...,g:*

$$\rho(\alpha_j) = e^{2\pi i c_j}\sigma(\alpha_j)$$

$$\rho(\beta_j) = e^{2\pi i \sum_{k=1}^{g} \Pi_{jk} c_k}\sigma(\beta_j)$$

proof: If $V(\rho) = V(\sigma)$ then, by lemma 1.2, there exists $\omega \in H^0(X, K)$ such that the solution of $f^{-1}df = \omega, f(z_0) = 1$ verifies $f(\gamma z) = \rho(\gamma)f(z)\sigma^{-1}(\gamma)$. Using the normalized basis of abelian differentials $\{\omega_i\}_{i=1,...,g}$ and writing $\omega = 2\pi i \sum_{k=1}^{g} c_k\omega_k$, we see that $f(z) = e^{2\pi i \int_{z_0}^{z} \sum_{k=1}^{g} c_k\omega_k}$ is the solution, and substituting α_j and β_j into the equation verified by f, we get the formulas in the lemma. □

Denoting by $\mathcal{U}(1)$ the space of one-dimensional unitary representations of $\pi_1(X)$, $\mathrm{Hom}(\pi_1(X), U(1))$, we have:

Proposition 2.2 *Every line bundle of degree 0 admits a unique unitary representation, i.e, there is an identification* $\mathcal{U}(1) \cong Jac(X)$. *On the other hand, every line bundle of degree 0 admits infinitely many Schottky representations.*

proof: Existence of unitary representations: Let L be a line bundle of degree 0. Since L is flat, $L = V(\sigma)$ for some $\sigma \in \mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$. Let $y_j = \frac{1}{2\pi}\log|\sigma(\alpha_j)|$ and x_k be the solution to the system of g linear equations (which exists since $\mathrm{Im}\Pi_{jk}$ is positive definite, hence invertible)

$$\sum_{k=1}^{g} \mathrm{Im}(\Pi_{jk})x_k + \mathrm{Re}(\Pi_{jk})y_k = \frac{1}{2\pi}\log|\sigma(\beta_j)| \qquad \forall j = 1, ..., g$$

Then, it is an easy computation to see that, for $c_j = x_j + iy_j$, $j = 1, ..., g$ the representation ρ defined by lemma 2.1 is unitary. Uniqueness of unitary representations: let ρ and σ have modulus one and suppose they represent the same line bundle. Then by the previous lemma, we get $1 = |\rho(\alpha_j)/\sigma(\alpha_j)| = |e^{2\pi i c_j}|$, so all the c_j's are real numbers; from the second equation, we get $1 = |\rho(\beta_j)/\sigma(\beta_j)| = |e^{2\pi i \sum_{k=1}^{g} \Pi_{jk}c_k}| = |e^{2\pi \sum_{k=1}^{g} \mathrm{Im}\Pi_{jk}\mathrm{Re}c_k}|$, which implies, since $\mathrm{Im}\Pi_{jk}$ is invertible that all c_k are zero. Hence, $\rho = \sigma$. Existence of Schottky representations: Again, let L be given as $V(\sigma)$, and let $c_j = -\frac{1}{2\pi i}\log(\sigma(\alpha_j))$, for any choice of the logarithm. Then the representation ρ defined by 2.1 is a Schottky representation of L. Finally, the non-uniqueness of Schottky representations follows from the possibility of choosing different branches for the logarithm. A more precise statement is given in the next lemma. □

Lemma 2.3 *The Schottky representations which produce the trivial line bundle over* X *form an abelian group given by the image of the map:*

$$\begin{array}{ccc} \mathbb{Z}^g & \to & Hom(\Sigma, \mathbb{C}^*) \\ (n_1, ..., n_g) & \mapsto & \left\{ \begin{array}{l} \rho_{n_1,...,n_g}(\beta_k) = e^{2\pi i \sum_{k=1}^{g} \Pi_{jk} n_k} \\ \rho_{n_1,...,n_g}(\alpha_k) = 1 \end{array} \right. \end{array}$$

proof: First, for $n_1 = ... = n_g = 0$, $\rho_{0...0}$ is the trivial representation, so $V(\rho_{0...0}) = X \times \mathbb{C}$ is the trivial line bundle. Then, by lemma 2.1 it is clear that all the representations $\rho_{n_1,...,n_g}$ give rise to equivalent line bundles for different $(n_1, ..., n_g) \in \mathbb{Z}^g$, and that if $V(\rho)$ is trivial then $\rho = \rho_{n_1,...,n_g}$ for some integers $n_1, ..., n_g$. $\square$

Remark 2.4 Using this proposition, we can give an alternate description of the Jacobian of X, as the quotient of $\mathrm{Hom}(\Sigma, \mathbb{C}^*)$ by the image of the above map (the subgroup of Schottky representations of the trivial line bundle), and everything fits into the following short exact sequence of group homomorphisms:

$$0 \to \mathbb{Z}^g \to Hom(\Sigma, \mathbb{C}^*) \xrightarrow{W} Jac(X) \to 0.$$

where W denotes the restriction of V to $\mathrm{Hom}(\Sigma, \mathbb{C}^*)$.

From the consideration of line bundles we can already deduce some simple facts about higher rank Schottky vector bundles.

Lemma 2.5 *If E is a rank n Schottky bundle and L is a line bundle, then $E \otimes L$ is also a Schottky bundle.*

proof: Let $E = V(\rho)$ where ρ is Schottky, and by proposition 2.2 we can put $L = V(\sigma)$ for some one dimensional Schottky representation. Hence, $E \otimes L = V(\rho) \otimes V(\sigma) = V(\rho \otimes \sigma)$, and $\rho \otimes \sigma$ is a Schottky representation. $\square$

We can see also that, contrary to the case of unitary representations, Schottky vector bundles do not determine a unique representation:

Proposition 2.6 *Every Schottky vector bundle E has infinitely many non-conjugate Schottky representations associated to it.*

proof: If $E = V(\rho)$ then $E = V(\rho \otimes \rho_{n_1,...,n_g})$, where $\rho_{n_1,...,n_g}$ are the Schottky representations associated with the trivial line bundle, used in lemma 2.3. $\square$

2.2 Schottky vector bundles over an elliptic curve.

Let us now consider vector bundles over the simplest Riemann surfaces. We will see that every flat vector bundle over the Riemann sphere and over a Riemann surface of genus 1, is a Schottky bundle. On the Riemann sphere, Grothendieck showed that every vector bundle is a direct sum of line bundles (see [Gr]). In particular, by Weil's theorem, a flat vector bundle is a direct sum of line bundles of degree 0. Since there is only one isomorphism class of line bundles in each degree, we conclude that any flat bundle over the Riemann sphere is trivial, so this is not an interesting case.

Now, let X be a Riemann surface of genus 1, and let $\mathbb{I}$ denote the trivial line bundle over X. Atiyah proved the following theorem:

Theorem 2.7 (Atiyah [A]) *Let X be a Riemann surface of genus 1, and let $n > 1$. Then:*
a) There is a unique indecomposable vector bundle of rank n and degree 0 over X denoted $\mathbb{F}_n$, such that $dim H^0(X, \mathbb{F}_n) = 1$. Moreover, $\mathbb{F}_n$ is the unique nontrivial extension

$$0 \to \mathbb{I} \to \mathbb{F}_n \to \mathbb{F}_{n-1} \to 0$$

b) Every indecomposable vector bundle E, of rank n and degree 0 over X, is isomorphic to $\mathbb{F}_n \otimes \det E$. $\square$

Lemma 2.8 *For every n, the bundle $\mathbb{F}_n$ is Schottky.*

proof: Let X be represented as the quotient of $\mathbb{C}$ by the lattice generated by $1, \tau \in \mathbb{C}$ where $\mathrm{Im}\tau > 0$, and α, β be the generators of $\pi_1(X)$, acting on $\mathbb{C} = \tilde{X}$ by: $\alpha \cdot z = z + 1$, $\beta \cdot z = z + \tau$. Consider the representation $\rho_n \in \mathrm{Hom}(\pi_1(X), GL(n, \mathbb{C}))$ given by:

$$\rho_n(\beta) = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix} \qquad \rho_n(\alpha) = \mathbf{1}$$

We want to prove, by induction, that $V(\rho_n) = \mathbb{F}_n$. Clearly $\mathbb{F}_1 = \mathbb{I}$. Assume $V(\rho_{n-1}) = \mathbb{F}_{n-1}$. From the way ρ_n is constructed, we see that $V(\rho_n)$ is an extension of $V(\rho_{n-1}) = \mathbb{F}_{n-1}$ by the trivial line bundle. Therefore, $V(\rho_n)$ has sections, because it has a trivial line bundle as sub-bundle. Let us compute the space of sections of $V(\rho_n)$. Any section of $V(\rho_n)$ over $X = \mathbb{C}/ < 1, \tau >$ can be realized as a holomorphic function $s : \mathbb{C} \to \mathbb{C}^n$ satisfying $s(\gamma z) = \rho_n(\gamma) s(z) \quad \forall \gamma \in \pi_1(X)$. This means, in components:

$$s_1(z+1) = s_1(z), \quad ..., \quad s_n(z+1) = s_n(z)$$
$$s_1(z+\tau) = s_1(z) + s_2(z), \quad ..., \quad s_{n-1}(z+\tau) = s_{n-1}(z) + s_n(z),$$
$$s_n(z+\tau) = s_n(z)$$

Then s_n must be a constant, being a holomorphic function on X. Now, if $f : \mathbb{C} \cong \tilde{X} \to \mathbb{C}^n$ is any holomorphic function verifying $f(z+1) = f(z)$ and $f(z+\tau) = f(z) + c$ for some constant $c \in \mathbb{C}$, then $c = 0$, so that f is constant. This is a consequence of the classical fact that an abelian differential with zero "α-periods" (like df, in this case) has to be zero. Applying this to s_{n-1} we get $s_n = 0$, and s_{n-1} constant. Repeating the same argument we find that all s_i, $i = 2, ..., n$ must be zero, and s_1 is a constant. This means that $\dim H^0(X, V(\rho_n)) = 1$, which implies that $V(\rho_n)$ is not the trivial extension (since by hypothesis $V(\rho_{n-1}) = \mathbb{F}_{n-1}$ has sections), and according to Atiyah's theorem, part (a), we conclude that $V(\rho_n) = \mathbb{F}_n$, realizing $\mathbb{F}_n$ as a Schottky bundle. $\square$

Now we immediately obtain:

Theorem 2.9 *Every flat vector bundle over a Riemann surface of genus one is a Schottky vector bundle.*

proof: Every flat bundle E is a direct sum of indecomposable vector bundles of degree 0, so we may assume that E is indecomposable of degree 0. Then by Atiyah's theorem, part (b), $E = \mathbb{F}_n \otimes \det E$. Since $\mathbb{F}_n$ is Schottky, $\det E$ is a line bundle, and tensor products of Schottky representations are Schottky, we conclude that E is also Schottky. □

Chapter 3

Spaces of representations.

Having studied Schottky vector bundles over Riemann surfaces of small genus, we now consider in more detail the relationship between representations of $\pi_1(X)$ and vector bundles over Riemann surfaces, for the case of genus $g \geq 2$. We discuss analytic structures on spaces of conjugacy classes of representations; in particular, for the space of classes of Schottky representations that give rise to *stable* vector bundles $\mathcal{S}^{st}(n)$, we obtain the structure of a complex manifold. Because of a universal property of the moduli space of stable vector bundles $\mathcal{M}^{st}(n)$, the correspondence between representations and vector bundles becomes a holomorphic map between complex manifolds. We will see that the components of the differential of this map are in fact, periods of a certain type of differentials on the Riemann surface which, in case the representation is unitary, satisfy some relations generalizing the classical bilinear relations of Riemann.

3.1 Generalized Schottky spaces.

We will need to consider three spaces of representations: the space of general representations of $\pi_1(X)$, the space of unitary representations, and the space of Schottky representations. Taken modulo conjugation, these spaces are:

$$\widetilde{\mathcal{G}}(n) = \mathrm{Hom}(\pi_1(X), GL(n,\mathbb{C}))/PGL(n,\mathbb{C})$$
$$\widetilde{\mathcal{U}}(n) = \mathrm{Hom}(\pi_1(X), U(n))/PU(n)$$
$$\widetilde{\mathcal{S}}(n) = \mathrm{Hom}(\Sigma(X), GL(n,\mathbb{C}))/PGL(n,\mathbb{C})$$

Here, we consider the conjugation action by $PGL(n,\mathbb{C}) = GL(n,\mathbb{C})/Center$, because it acts freely (at least on a subset of $\mathrm{Hom}(\pi_1(X), GL(n,\mathbb{C}))$, see proposition 3.1 below), and similarly for the case of $PU(n) = U(n)/Center$. Note that Schottky space, the space of equivalence classes of marked Schottky groups (cf. chapter 1), is a subset of $\mathrm{Hom}(\Sigma(X), PSL(2,\mathbb{C}))/PSL(2,\mathbb{C})$, so that $\widetilde{\mathcal{S}}(n)$ can be viewed, in some sense, as a generalization of Schottky space. In general, to obtain nice analytic structures on these spaces we need to restrict the representations allowed. We will work with the subsets of representations that produce *simple* bundles. Let the superscript

* denote these representations and define:

$$\mathcal{G}^{sim}(n) = \text{Hom}^*(\pi_1(X), GL(n,\mathbb{C}))/PGL(n,\mathbb{C})$$

$$\mathcal{U}^{sim}(n) = \text{Hom}^*(\pi_1(X), U(n))/PU(n)$$

$$\mathcal{S}^{sim}(n) = \text{Hom}^*(\Sigma(X), GL(n,\mathbb{C}))/PGL(n,\mathbb{C})$$

We observe that, by proposition 1.6, $\mathcal{U}^{sim}(n)$ is precisely $\mathcal{U}^{st}(n) := \text{Hom}^{irr}(\pi_1(X), U(n))/PU(n)$, the space of *irreducible* unitary representations of $\pi_1(X)$, modulo conjugation. It is known that $\mathcal{U}^{st}(n)$ has the structure of an open complex manifold ([NS1]), and we can give a heuristic computation of its dimension as follows. An element $\rho \in \text{Hom}^{irr}(\pi_1(X), U(n))$ is a $2g$-tuple of unitary $n \times n$ matrices, $(\rho(\alpha_1), ..., \rho(\beta_g))$ subject to one single relation $\prod_{i=1}^{g}[\rho(\alpha_i), \rho(\beta_i)]$, which is a matrix in $SU(n)$. The conjugation action reduces this dimension by $\dim PU(n) = n^2 - 1$. Therefore the real dimension of $\mathcal{U}^{st}(n)$ is $2gn^2 - (n^2-1) - (n^2-1)$, so that its complex dimension is $n^2(g-1)+1$, as is well known. By Narasimhan-Seshadri's theorem, $\mathcal{U}^{st}(n)$ can be identified with the moduli space of *stable* vector bundles of rank n and degree 0 $\mathcal{M}^{st}(n)$, which is known to have the structure of a (*smooth*) quasi-projective algebraic variety, i.e, a Zariski open subset of an algebraic variety (see [Mu] and [NR]).

Now we turn to the determination of the *tangent spaces* of these spaces of representations; these can be given in terms of cohomology of groups with coefficients in group modules. Since we will only need the first cohomology groups, we will work with the following concrete definitions. If π is a group and A is a π-module, we can define the first cohomology group of π with coefficients in the π-module A as

$$H^1(\pi, A) = Z^1(\pi, A)/B^1(\pi, A) \qquad \text{where:}$$

$$Z^1(\pi, A) = \{\phi : \pi \to A : \phi(\gamma_1\gamma_2) = \phi(\gamma_1) + \gamma_1 \cdot \phi(\gamma_2) \quad \forall \gamma_1, \gamma_2 \in \pi\}$$

$$B^1(\pi, A) = \{\phi : \pi \to A : \exists a \in A \, \text{with} \; \phi(\gamma) = a - \gamma \cdot a \quad \forall \gamma \in \pi\}$$

Given a Riemann surface X, and a representation ρ of $\pi_1(X)$ in $GL(n,\mathbb{C})$, the Lie algebra of $GL(n,\mathbb{C})$ (the space of all $n \times n$ complex matrices, $Mat_{n\times n}(\mathbb{C})$) becomes a $\pi_1(X)$-module via the adjoint representation:

$$\gamma \cdot B = \rho(\gamma) B \rho(\gamma)^{-1} \qquad \forall \gamma \in \pi_1(X), B \in Mat_{n\times n}(\mathbb{C})$$

We will denote the first cohomology group of $\pi_1(X)$ with coefficients in this module $Mat_{n\times n}(\mathbb{C})$, by $H^1(\pi_1(X), \text{Ad}_\rho)$.

If now ρ is a Schottky representation, $\rho \in \text{Hom}(\Sigma, GL(n,\mathbb{C}))$, then $Mat_{n\times n}(\mathbb{C})$ becomes also a Σ-module, with the same adjoint representation, and we denote the first cohomology group of Σ with coefficients in this Σ-module by $H^1(\Sigma, \text{Ad}_\rho)$. Its dimension can be computed as follows. Since Σ is a free group, there is no cocycle condition on $Z^1(\Sigma, \text{Ad}_\rho)$ and an element $\phi \in Z^1(\Sigma, \text{Ad}_\rho)$ can be viewed as a g-tuple of $n \times n$ complex matrices of the form $(\phi(\beta_1), ..., \phi(\beta_g))$. The dimension of $Z^1(\Sigma, \text{Ad}_\rho)$ is therefore gn^2. Now, the subspace $B^1(\Sigma, \text{Ad}_\rho) \subset Z^1(\Sigma, \text{Ad}_\rho)$ consists of those g-tuples of the form:

$$(B - \rho(\beta_1) B \rho(\beta_1)^{-1}, \; ..., \; B - \rho(\beta_g) B \rho(\beta_g)^{-1})$$

for some $B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$. Hence its dimension equals n^2 minus the dimension of the kernel of the linear map which sends $B \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ to the cocycle with the form given above. This kernel is precisely $Z(\rho)$, the space of matrices which commute with all matrices $\rho(\gamma),\ \gamma \in \pi_1(X)$. Considering only representations which give rise to simple bundles we have $\dim Z(\rho) = 1$ so that

$$\begin{aligned}\dim_{\mathbb{C}} H^1(\Sigma, \mathrm{Ad}_\rho) &= \dim_{\mathbb{C}} Z^1(\Sigma, \mathrm{Ad}_\rho) - \dim_{\mathbb{C}} B^1(\Sigma, \mathrm{Ad}_\rho)\\ &= gn^2 - (n^2-1) = n^2(g-1)+1\end{aligned}$$

and coincides with the dimension of the moduli space of flat stable bundles. It is easy to verify that $Z^1(\Sigma, \mathrm{Ad}_\rho)$ is contained in $Z^1(\pi_1(X), \mathrm{Ad}_\rho)$ (the inclusion being the map which sends a cocycle $(B_1, ..., B_g) \in Z^1(\Sigma, \mathrm{Ad}_\rho)$ to the cocycle $(0, ..., 0, B_1, ..., B_g) \in Z^1(\pi_1(X), \mathrm{Ad}_\rho))$, and that this induces an inclusion:

$$H^1(\Sigma, \mathrm{Ad}_\rho) \hookrightarrow H^1(\pi_1(X), \mathrm{Ad}_\rho).$$

We can now establish the following:

Proposition 3.1 *$\mathcal{G}^{sim}(n)$ and $\mathcal{S}^{sim}(n)$ are complex manifolds of complex dimensions $2[n^2(g-1)+1]$ and $n^2(g-1)+1$, respectively, and whose tangent spaces at the equivalence class of the representation ρ are $H^1(\pi_1(X), Ad_\rho)$ and $H^1(\Sigma, Ad_\rho)$, respectively. Moreover, $\mathcal{S}^{sim}(n)$ is a complex analytic sub-manifold of $\mathcal{G}^{sim}(n)$ and the corresponding inclusion of tangent spaces is the inclusion of cohomology groups: $H^1(\Sigma, Ad_\rho) \hookrightarrow H^1(\pi_1(X), Ad_\rho)$.*

proof: For the proof that $\mathcal{G}^{sim}(n)$ is a complex manifold of the required dimension, see [Gu]. For $\mathcal{S}^{sim}(n)$, the semicontinuity theorem (see theorem 4.3 below) implies that $\mathrm{Hom}^*(\Sigma, GL(n,\mathbb{C}))$ is an open dense subset of the complex manifold $\mathrm{Hom}(\Sigma, GL(n,\mathbb{C})) = GL(n,\mathbb{C})^g$, in the usual topology. Therefore, we just need to prove that the action of $PGL(n,\mathbb{C})$ is free at a every representation $\rho \in \mathrm{Hom}^*(\Sigma, GL(n,\mathbb{C}))$. By definition, this means that, for a matrix $M \in GL(n,\mathbb{C})$, the condition

$$M\rho(\beta_i)M^{-1} = \rho(\beta_i) \quad \forall i = 1, ..., g$$

implies that M is the identity in $PGL(n,\mathbb{C})$, i.e, that M is a scalar matrix. This is equivalent to saying that ρ has only scalar commutants, and this is the case since $V(\rho)$ is simple (cf. proposition 1.4). So the action is free. The realization of the tangent spaces as cohomology groups can be found in [Gu] (the Schottky case being an easy consequence of the formulas in §9). Since the inclusion $H^1(\Sigma, \mathrm{Ad}_\rho) \hookrightarrow H^1(\pi, \mathrm{Ad}_\rho)$ is complex linear when ρ is Schottky, $\mathcal{S}^{sim}(n)$ becomes a complex analytic submanifold of $\mathcal{G}^{sim}(n)$, with the obvious inclusion map. □

Because of the fact that when two unitary matrices are conjugate then they are conjugate via a unitary matrix, we can see that the space $\mathcal{U}^{sim}(n) = \mathcal{U}^{st}(n)$ is also contained in $\mathcal{G}^{sim}(n)$. When ρ is a unitary representation of $\pi_1(X)$, the Lie algebra of $U(n)$ (the space of skew-hermitian matrices) becomes a $\pi_1(X)$-module via the adjoint representation of ρ. To distinguish it from the previous cohomology groups, let us denote the first cohomology group of $\pi_1(X)$ with coefficients in this $\pi_1(X)$-module by $H^1(\pi_1(X), \mathrm{Ad}_{U(n)}(\rho))$. This is a real vector space whose complexification is $H^1(\pi_1(X), \mathrm{Ad}_\rho)$. If we forget the complex structure on $\mathcal{U}^{st}(n)$, we have the following fact proved, for instance in [NS1]: *the real tangent space of* $\mathcal{U}^{st}(n)$ *at the equivalence class of the representation* ρ *can be identified with* $H^1(\pi_1(X), Ad_{U(n)}(\rho))$. Therefore we see that $\mathcal{U}^{st}(n)$ sits inside $\mathcal{G}^{sim}(n)$ as a *real analytic* submanifold. Contrary to the case of $\mathcal{S}^{sim}(n)$, this inclusion is *not* complex analytic.

Most Schottky bundles are stable or, equivalently, associated to irreducible unitary representations. This is a consequence of another result of Narasimhan-Seshadri. Let M be a complex manifold. We say that $\mathcal{E}$ is a **holomorphic family of vector bundles over** X **parametrized by** M if $\mathcal{E}$ is a holomorphic vector bundle over $M \times X$. If m is a point in M, we denote by $\mathcal{E}_m$ the vector bundle over X corresponding to the point m. More precisely, if $i_m : X \to M \times X, \quad x \mapsto (m, x)$ is the inclusion, we have: $\mathcal{E}_m := i_m^*\mathcal{E}$.

Proposition 3.2 (Narasimhan-Seshadri [NS2], Thm 2(B)) *Let* $\mathcal{W}$ *be a holomorphic family of vector bundles parametrized by a complex manifold* T, *and let* T^s *be the set of* $t \in T$ *such that* $\mathcal{W}_t$ *is stable. Then* $T \smallsetminus T^s$ *is an analytic subset of* T. □

This means that in any parameter space, stable bundles form an open subset which is dense, if nonempty. This allows us to prove:

Proposition 3.3 *Let* X *be a compact Riemann surface of genus* $g \geq 2$, *and let* $n \geq 1$. *There are open dense sets* $\mathcal{S}^{st}(n) \subset \mathcal{S}^{sim}(n)$ *and* $\mathcal{G}^{st}(n) \subset \mathcal{G}^{sim}(n)$, *such that for every* $\rho \in \mathcal{S}^{st}(n), \mathcal{G}^{st}(n)$, *the bundle* $V(\rho)$ *is stable.*

proof: Consider the trivial rank n vector bundle over $\mathcal{G}^{sim}(n) \times \tilde{X}$, with the $\pi_1(X)$-action:

$$\gamma \cdot (\rho, z, v) = (\rho, \gamma z, \rho(\gamma)v) \qquad \forall \quad \gamma \in \pi_1(X),\ \rho \in \mathcal{G}^{sim}(n),\ z \in \tilde{X},\ v \in \mathbb{C}^n$$

As in chapter 1, this action defines a holomorphic bundle over $\mathcal{G}^{sim}(n) \times X$, i.e, a holomorphic family of vector bundles $\mathcal{F}$ parametrized by the complex manifold $\mathcal{G}^{sim}(n)$. Moreover, it is clear that, for every conjugacy class of representations $\rho \in \mathcal{G}^{st}(n)$, we have $\mathcal{F}_\rho = i_\rho^*\mathcal{F} = V(\rho)$ (see section 1.1). Then, defining

$$\mathcal{G}^{st}(n) = \{\rho \in \mathcal{G}^{sim}(n) : V(\rho) \text{ is stable}\},$$

$\mathcal{G}^{sim}(n) \smallsetminus \mathcal{G}^{st}(n)$ is an analytic subset, by proposition 3.2, and since $\mathcal{G}^{st}(n)$ is not empty (there exist stable vector bundles, given by irreducible unitary representations) $\mathcal{G}^{st}(n)$ is open and dense in $\mathcal{G}^{sim}(n)$. The same arguments apply to $\mathcal{S}^{st}(n)$, provided stable Schottky vector bundles exist, which is proved in the following proposition. □

Proposition 3.4 *For every* $n \geq 1$, $g \geq 2$, *there exist unitary, Schottky* n*-dimensional irreducible representations of* $\pi_1(X)$. *Therefore, there exist stable Schottky vector bundles of any given rank.*

proof: We may suppose that $n \geq 2$. Let $\lambda_1, ..., \lambda_n$ be n distinct complex numbers of modulus one. Let B_1 be the unitary diagonal matrix $diag(\lambda_1, ..., \lambda_n)$, and B_2 be the permutation matrix $e_1 \mapsto e_2, ..., e_n \mapsto e_1$, for a canonical basis $e_1, ..., e_n \in \mathbb{C}^n$. It is easy to see that these two unitary $n \times n$ matrices B_1, B_2 form an irreducible set of matrices (i.e, there is no subspace of $\mathbb{C}^n$ preserved by both). Hence, the representation of $\pi_1(X)$ given by $\rho(\alpha_i) = \mathbf{1}, \quad \forall i = 1, ...g, \quad \rho(\beta_1) = B_1, \quad \rho(\beta_2) = B_2, \quad \rho(\beta_i) = \mathbf{1} \quad \forall i = 3, ..., g$, is unitary, Schottky and irreducible. The second statement is immediate from Narasimhan-Seshadri's theorem (cf. theorem 1.5), since for a representation ρ as above, $V(\rho)$ is stable. $\square$

3.2 The moduli map.

Since the representation spaces we considered and the moduli space of stable vector bundles, $\mathcal{M}^{st}(n)$, are complex manifolds, we can hope that the association of a vector bundle to a representation as in §1.1 defines a holomorphic map. This is in fact the case, because of a universal property of the moduli space $\mathcal{M}^{st}(n)$, which is best formulated using the general theory of moduli problems. Since the discussion of this theory would lead us too far and will not be important in the sequel, we will state the relevant property of $\mathcal{M}^{st}(n)$ only for this particular case and refer to Newstead [N] for an introduction to moduli problems and for a proof of the theorem.

Theorem 3.5 *The moduli space of flat stable holomorphic vector bundles $\mathcal{M}^{st}(n)$, has the following universal property: for any holomorphic family $\mathcal{E}$ of flat stable vector bundles over X, parametrized by a complex manifold S, the map sending $s \in S$ to the vector bundle $\mathcal{E}_s$ in the moduli space $\mathcal{M}^{st}(n)$ is a holomorphic map.* $\square$

By applying this fact to suitable families of holomorphic vector bundles we get:

Proposition 3.6 *The maps*

$$V : \mathcal{G}^{st}(n) \to \mathcal{M}^{st}(n) \qquad\qquad W = V|_{\mathcal{S}^{st}(n)} : \mathcal{S}^{st}(n) \to \mathcal{M}^{st}(n)$$

which send a representation ρ to the vector bundle $V(\rho)$, are holomorphic.

proof: Consider again the holomorphic family $\mathcal{F}$ of vector bundles over X parametrized by the complex manifold $\mathcal{G}^{st}(n)$, given in the proof of 3.3. Since $\mathcal{F}_\rho = V(\rho)$, the previous theorem says that the map $\rho \mapsto V(\rho)$ is holomorphic. For the case of $\mathcal{S}^{st}(n)$ exactly the same construction applies. $\square$

Because these maps are related to universal properties of moduli spaces, we will call the map V the **general moduli map**, and W the **Schottky moduli map**.

We are interested in identifying the fibers of the map $V : \mathcal{G}^{st}(n) \to \mathcal{M}^{st}(n)$ over some bundle E. This fiber consists of those representations ρ which give rise to E (i.e, those with $V(\rho) = E$), and recalling proposition 1.2, we see that it is convenient to introduce, for a given $\rho \in \mathcal{G}^{st}(n)$, the holomorphic function ψ_ρ, defined by:

$$\begin{array}{cccc} \psi_\rho : & H^0(X, \mathrm{End}V(\rho)\otimes K) & \to & \mathcal{G}^{st}(n) \\ & \omega & \mapsto & \sigma(\gamma) := f_\omega(\gamma z)\rho(\gamma)f_\omega(z)^{-1} \end{array}$$

where f_ω is the solution of the differential equation $f^{-1}df = \omega, f(z_0) = I$. Note that the function ψ_ρ depends on the basepoint z_0, and that it is holomorphic due to the analytic dependence of f_ω on $\omega \in H^0(X, \mathrm{End}V(\rho)\otimes K)$.

For every conjugacy class of representations ρ, we can compose the differential of ψ_ρ at zero, with the differential of V at ρ, to get the sequence:

$$H^0(X, \mathrm{End}V(\rho)\otimes K) \stackrel{d(\psi_\rho)_0}{\to} H^1(\pi_1(X), \mathrm{Ad}_\rho) \stackrel{dV_\rho}{\to} T_{V(\rho)}\mathcal{M}^{st}(n).$$

which turns out to be exact:

Lemma 3.7 *The image of ψ_ρ coincides with the fiber of the map V lying above $V(\rho)$, in other words, $\psi_\rho(H^0(EndV(\rho)\otimes K)) = V^{-1}(V(\rho))$. For every class of representations $\rho \in \mathcal{G}^{st}(n)$, Im $d(\psi_\rho)_0 =$ Ker dV_ρ.*

proof: σ is in the image of ψ_ρ if and only if condition 3) in lemma 1.2 is verified. This is then equivalent to $V(\rho) = V(\sigma)$ or $\sigma \in V^{-1}(V(\rho))$. For any $\eta \in H^0(X, \mathrm{End}V(\rho)\otimes K)$ and $t \in \mathbb{C}$, we have $V(\psi_\rho(t\eta)) \cong V(\psi_\rho(0))$ (again by lemma 1.2). Letting $t \to 0$, we see that the differential of the composition $V \circ \psi_\rho$, at the origin, is zero, i.e, $d(V\circ\psi_\rho)_0(\eta) = dV_\rho \circ d(\psi_\rho)_0(\eta) = 0$. Hence $\mathrm{Im}d(\psi_\rho)_0 \subset \mathrm{Ker}dV_\rho$. Conversely, if $\phi \in \mathrm{Ker}dV_\rho \subset H^1(\pi_1(X), \mathrm{Ad}_\rho)$, then ϕ is tangent to the fiber of the map V at ρ, which means tangent to the image of ψ_ρ at 0, so that there is an $\eta \in H^0(X, \mathrm{End}V(\rho)\otimes K)$, such that $\phi = d(\psi_\rho)_0(\eta)$. □

To compute the kernel of dV, we can now consider the differential of this map ψ_ρ at the origin. The computations yield:

Proposition 3.8 *The differential at the origin, of the map ψ_ρ defined above, assigns to $\eta \in H^0(EndV(\rho)\otimes K)$ the cocycle $\phi_{\rho,\eta} \in H^1(\pi_1(X), Ad_\rho)$ whose γ-component is:*

$$\phi_{\rho,\eta}(\gamma) = \int_{z_0}^{\gamma z_0} \eta$$

proof: Consider a family of matrix valued differentials of the form $\omega_t = t\eta$, $\eta \in H^0(\mathrm{End}V(\rho)\otimes K)$, where $t \in \mathbb{C}$. We can write the solution of $f^{-1}df = t\eta, f(z_0) = I$ as

$$f_{t\eta}(z) = P\exp(\int_{z_0}^{z} t\eta),$$

where $P\exp$ denotes the path ordered exponential. For t small we can expand $f_t := f_{t\eta}$ and f_t^{-1} as

$$f_t(z) = I + t\int_{z_0}^{z}\eta + O(t^2) \qquad f_t^{-1}(z) = I - t\int_{z_0}^{z}\eta + O(t^2)$$

Let ρ_t denote the representation $\psi_\rho(t\eta)$. Discarding second order terms we find:

$$\rho_t(\gamma) = f_t(\gamma z)\rho(\gamma)f_t(z)^{-1} = \left(I + t\int_{z_0}^{\gamma z}\eta\right)\rho(\gamma)(I - t\int_{z_0}^{z}\eta) + O(t^2) =$$

$$= \rho(\gamma) + t\left[((\int_{z_0}^{\gamma z}\eta)\rho(\gamma) - \rho(\gamma)\int_{z_0}^{z}\eta)\right] + O(t^2)$$

The derivative of the curve of representations ρ_t is then given by $\dot\rho_t\rho_t^{-1}$, where for every $\gamma \in \pi_1(X)$ the dot derivative is just $\dot\rho_t(\gamma) = \lim_{s\to t}\frac{\psi_\rho(s\eta)-\psi_\rho(t\eta)}{s-t}$. (since for a general curve g_t in a Lie group the derivative is represented by the vector $\dot g_t g_t^{-1}$, in the Lie algebra). So the differential at $\omega = 0$ in the η direction is finally, given by:

$$\phi_{\rho,\eta}(\gamma) := d(\psi_\rho)_0(\eta)(\gamma) = \dot\rho_t\rho_t^{-1}\big|_{t=0} = \lim_{t\to 0}\frac{\psi_\rho(t\eta)(\gamma) - \psi_\rho(0)(\gamma)}{t}\rho(\gamma)^{-1} =$$

$$= \int_{z_0}^{\gamma z}\eta - \rho(\gamma)\left(\int_{z_0}^{z}\eta\right)\rho(\gamma)^{-1} = \int_{z_0}^{\gamma z}\eta - \int_{z_0}^{z}(\eta\circ\gamma)\gamma' = \int_{z_0}^{\gamma z_0}\eta$$

Where we have used the fact that η is a matrix valued holomorphic differential: $\eta(\gamma z)\gamma'(z) = \rho(\gamma)\eta(z)\rho(\gamma)^{-1}$. The fact that this is a cocycle in $Z^1(\pi_1(X), \mathrm{Ad}_\rho)$, can be also verified directly:

$$\int_{z_0}^{\gamma_1\gamma_2 z_0}\eta = \int_{z_0}^{\gamma_1 z_0}\eta + \int_{\gamma_1 z_0}^{\gamma_1\gamma_2 z_0}\eta = \int_{z_0}^{\gamma_1 z_0}\eta + \int_{z_0}^{\gamma_2 z_0}(\eta\circ\gamma_1)\gamma_1' =$$

$$= \int_{z_0}^{\gamma_1 z_0}\eta + \rho(\gamma_1)(\int_{z_0}^{\gamma_2}\eta)\rho(\gamma_1)^{-1} \qquad \square$$

3.3 Periods of differentials.

We now define a general "period map" for representations of $\pi_1(X)$, which turns out, in a special case, to be the differential of the general moduli map computed above. Let ρ be a representation of $\pi_1(X)$ in a vector space $\mathbb{V}$ and X a Riemann surface of genus $g \geq 2$. We want to consider sections of the vector bundle $V(\rho)\otimes K$, where K is the canonical line bundle on X, and $V(\rho) = \mathbb{H}\times_\rho \mathbb{V}$. These sections will be called "ρ-twisted differentials" because, since X can be given as $X = \mathbb{H}/\Gamma$, where $\Gamma \in PSL(2,\mathbb{R})$ is a Fuchsian group isomorphic to $\pi_1(X)$, they can be viewed as a closed, holomorphic differential 1-forms $\omega = h(z)dz$, (with $h : \mathbb{H} \to \mathbb{C}^n$ holomorphic) on the upper half plane, satisfying:

$$h(\gamma z)\gamma'(z) = \rho(\gamma)h(z) \qquad \forall\ \gamma \in \pi_1(X) \cong \Gamma \subset PSL(2,\mathbb{R}).$$

Now, choose a basepoint $z_0 \in \mathbb{H}$ and let

$$P^\gamma_{z_0,\rho}(\omega) = \int_{z_0}^{\gamma z_0}\omega, \qquad \gamma \in \pi_1(X)$$

Taking γ to be any one of the generators of $\pi_1(X)$ we get $2g$ integrals which can be put together to form the "period map" for differentials twisted by ρ:

$$\begin{array}{rccc} P_{z_0,\rho}: & H^0(X, K\otimes V(\rho)) & \to & (\mathbb{C}^n)^{2g} \\ & \omega & \mapsto & (\int_{z_0}^{\alpha_1 z_0}\omega, ..., \int_{z_0}^{\alpha_g z_0}\omega, \int_{z_0}^{\beta_1 z_0}\omega, ..., \int_{z_0}^{\beta_g z_0}\omega) \end{array}$$

Let us see the effect of changing the basepoint from z_0 to z_1:

$$\begin{aligned} P^{\gamma}_{z_1}(\omega) &= \int_{z_1}^{z_0} \omega + \int_{z_0}^{\gamma z_0} \omega + \int_{\gamma z_0}^{\gamma z_1} \omega \\ &= \int_{z_1}^{z_0} \omega + \int_{z_0}^{z_1} (\omega \circ \gamma)\gamma' + P^{\gamma}_{z_0}(\omega) = (\rho(\gamma) - 1) \int_{z_0}^{z_1} \omega + P^{\gamma}_{z_0}(\omega) \end{aligned}$$

And consider also the period of a composed loop:

$$\begin{aligned} &P^{\gamma_1\gamma_2}_{z_0}(\omega) = P^{\gamma_2}_{z_0}(\omega) + P^{\gamma_1}_{\gamma_2 z_0}(\omega) = \\ &= P^{\gamma_2}_{z_0}(\omega) + (\rho(\gamma_1) - 1) \int_{z_0}^{\gamma_2 z_0} \omega + P^{\gamma_1}_{z_0}(\omega) = \rho(\gamma_1)P^{\gamma_2}_{z_0}(\omega) + P^{\gamma_1}_{z_0}(\omega) \end{aligned}$$

This means that for every ω, $P_{z_0,\rho}(\omega)$ is a cocycle in $Z^1(\pi_1(X), \rho)$ and, since changing the basepoint adds a coboundary, by the computations above, we can drop the subscript z_0 and define $P_\rho(\omega)$ as an element of $H^1(\pi_1(X), \rho)$. So, we have:

Definition 3.9 *Given a representation $\rho \in Hom(\pi_1(X), GL(n, \mathbb{C}))$, the map*

$$P_\rho : H^0(X, V(\rho) \otimes K) \to H^1(\pi_1(X), \rho)$$

defined above is called the period map *associated to ρ.*

When ρ is a unitary representation, this period map can be used in a generalization of the classical Riemann bilinear relations to ρ-twisted differentials, at least in two important cases: when ρ is a one-dimensional representation (and hence $V(\rho)$ is a line bundle), and the case of the adjoint representation, where we can see that $V(\mathrm{Ad}_\rho) = V(\rho)^* \otimes V(\rho) = \mathrm{End}V(\rho)$.

Let ρ be a one-dimensional unitary representation; we can define a (positive definite) hermitian inner product on the space of ρ-twisted differentials:

$$\begin{aligned} &(,) : H^0(X, K \otimes V(\rho))^{\otimes 2} \to \mathbb{C} \\ &(\omega_1, \omega_2) := \int_X \omega_1 \wedge {}^*\omega_2 = i \int_D h_1(z)\overline{h_2(z)}\ dz \wedge \overline{dz} \end{aligned}$$

Where D denotes a fundamental domain for $X = \mathbb{H}/\Gamma$, and the star is the Hodge star which gives ${}^*\omega = i\overline{\omega}$, for holomorphic forms. Note that $dz \wedge \overline{dz} = -2idx \wedge dy$, so that $|\omega|^2 = (\omega, \omega) \geq 0$. We have the following lemma:

Lemma 3.10 (Bilinear relations for twisted abelian differentials). *Let X be a compact Riemann surface, ρ a $U(1)$ representation of $\pi_1(X)$ and $P = P_\rho$ its associated period map. Then for all $\omega_1, \omega_2 \in H^0(X, V(\rho) \otimes K)$ there are points $z_i, w_i, i = 1, ..., g$ in $\mathbb{H}$, such that:*

$$\int_X \omega_1 \wedge {}^*\omega_2 = -i \sum_{j=1}^{g} \rho(\beta_j)^{-1} P^{\beta_j}_{z_0}(\omega_1)\overline{P^{\alpha_j}_{z_j}(\omega_2)} + \rho(\alpha_j)^{-1} P^{\alpha_j}_{z_0}(\omega_1)\overline{P^{\beta_j}_{w_j}(\omega_2)}$$

proof: Let X be realized as the quotient $\mathbb{H}/\Gamma$, for a Fuchsian group Γ, and let $f(z) := \int_{z_0}^{z} \omega_1$, so that $\omega_1 = df$. By the properties of the periods, $f(\gamma z) = \rho(\gamma)f(z) + P^{\gamma}_{z_0}(\omega_1)$, for all $\gamma \in \pi_1(X)$.

Using Stokes' theorem in a fundamental domain $D \subset \mathbb{H}$ for X, we compute:

$$(\omega_1, \omega_2) = \int_D \omega_1 \wedge {}^*\omega_2 = \int_{\partial D} f^*\omega_2 =$$
$$= \sum_{i=1}^{g} (\int_{\alpha_i} f^*\omega_2 + \int_{\beta_i} f^*\omega_2 + \int_{\alpha_i^{-1}} f^*\omega_2 + \int_{\beta_i^{-1}} f^*\omega_2)$$

But now, if α denotes any of the α_i, and writing $\omega_2 = h(z)dz$, we get:

Figure 3.1:

$$\int_\alpha f^*\omega_2 + \int_{\alpha^{-1}} f^*\omega_2 =$$
$$= i \int_w^{\alpha w} \left[f(z)\overline{h(z)}^t - f(\beta z)\overline{h(\beta z)}\ \overline{\beta'(z)} \right] \overline{dz} =$$
$$= i \int_w^{\alpha w} \left[f(z)\overline{h(z)} - (\rho(\beta) f(z) + P_{z_0}^{\beta}(\omega_1))\overline{\rho(\beta) h(z)} \right] \overline{dz} =$$
$$= -i \left[P_{z_0}^{\beta}(\omega_1)\overline{\rho(\beta)} \int_w^{\alpha w} \overline{h(z)}\ \overline{dz} \right] =$$
$$= -i \left[P_{z_0}^{\beta}(\omega_1)\rho(\beta)^{-1}\overline{P_w^{\alpha}(\omega_2)} \right]$$

To find the integrals along the "beta" cycles we perform similar computations. □

Corollary 3.11 *Let ρ be a Schottky and unitary one-dimensional representation. If, for some $z_0 \in \mathbb{H}$, $P_{z_0,\rho}^{\alpha_i}(\omega) = 0$ for all $i = 1, ..., g$, then $\omega = 0$.*

proof: Since ρ is Schottky $\rho(\alpha_i) = 1$, for all $i = 1, ..., g$, so $P_{z_0}^{\gamma}(\omega) = 0$, implies $P_{z_1}^{\gamma}(\omega) = 0$ for any other basepoint z_1. Now the lemma says $|\omega|^2 = (\omega, \omega) = 0$ so that $\omega = 0$. □

Now, when ρ is an n-dimensional unitary representation, we can define as before a (positive definite) hermitian inner product on the space of "matrix-valued" differentials, using the trace:

$$(\omega_1, \omega_2) := \int_X \mathrm{tr}(\omega_1 \wedge {}^*\omega_2) = -\frac{i}{2} \int_D \mathrm{tr}(h_1(z)\overline{h_2(z)}^t)\ dz \wedge \overline{dz}$$

Where D is again a fundamental domain for $X = \mathbb{H}/\Gamma$.

Proposition 3.12 (Bilinear relations for endomorphism-valued differentials).
Let X be a compact Riemann surface, $\rho \in Hom(\pi_1(X), U(n))$, and let $P = P_{Ad_\rho}$ be the period

map associated to the adjoint representation. Then, for any $\omega_1, \omega_2 \in H^0(X, EndV(\rho) \otimes K)$, *there are points* $z_i, w_i, i = 1, ...g$ *in* $\mathbb{H}$ *such that:*

$$(\omega_1, \omega_2) = -i \sum_{j=1}^{g} tr[P_{z_0}^{\beta_j}(\omega_1) Ad_{\rho(\beta_j)} \overline{P_{z_j}^{\alpha_j}(\omega_2)}] + tr[P_{z_0}^{\alpha_j}(\omega_1) Ad_{\rho(\alpha_j)} \overline{P_{w_j}^{\beta_j}(\omega_2)}]$$

proof: The proof is very similar to the abelian case. Again, let $X = \mathbb{H}/\Gamma$, for a Fuchsian group Γ and let $f(z) := \int_{z_0}^{z} \omega_1$, so that $\omega_1 = df$ and in this case, $f(\gamma z) = \sigma(\gamma)\rho(\gamma)f(z)\rho(\gamma)^{-1} + P_{z_0}^{\gamma}(\omega_1)$, for all $\gamma \in \pi_1(X)$. Using Stokes' theorem, we compute:

$$\begin{aligned}(\omega_1, \omega_2) &= \int_D \mathrm{tr}(\omega_1 \wedge {}^*\omega_2) = \int_{\partial D} \mathrm{tr}(f^* \omega_2) = \\ &= \sum_{i=1}^{g} \left(\int_{\alpha_i} \mathrm{tr} f^* \omega_2 + \int_{\beta_i} \mathrm{tr} f^* \omega_2 + \int_{\alpha_i^{-1}} \mathrm{tr} f^* \omega_2 + \int_{\beta_i^{-1}} \mathrm{tr} f^* \omega_2 \right)\end{aligned}$$

But now, if α denotes any of the α_i, and writing $\omega_2 = h(z)dz$, we get in this case:

$$\begin{aligned}&\int_{\alpha} \mathrm{tr} f^* \omega_2 + \int_{\alpha^{-1}} \mathrm{tr} f^* \omega_2 = \\ &= i \int_{w}^{\alpha w} \mathrm{tr} \left[f(z)\overline{h(z)}^t - f(\beta z)\overline{h(\beta z)}^t \, \overline{\beta'(z)} \right] \overline{dz} = \\ &= i \int_{w}^{\alpha w} \mathrm{tr} \left[f(z)\overline{h(z)}^t - \rho(\beta) f(z) \rho(\beta)^{-1} + P_{z_0}^{\beta}(\omega_1)) \rho(\beta) \overline{h(z)}^t \rho(\beta)^{-1} \right] \overline{dz} = \\ &= -i \quad \mathrm{tr} \left[P_{z_0}^{\beta}(\omega_1) \rho(\beta) \int_{w}^{\alpha w} \overline{h(z)}^t \, \overline{dz} \rho(\beta)^{-1} \right] = \\ &= -i \quad \mathrm{tr} \left[P_{z_0}^{\beta}(\omega_1) \rho(\beta) \overline{P_{w}^{\alpha}(\omega_2)}^t \rho(\beta)^{-1} \right]\end{aligned}$$

To find the integrals along the "beta" cycles we perform similar computations. □

As before, putting $\omega_1 = \omega_2$ and using the positivity of the inner product in $H^0(X, \mathrm{End}E \otimes K)$, we get:

Corollary 3.13 *Let* ρ *be a Schottky and unitary representation. If* $P_{z_0, Ad_\rho}^{\alpha_i}(\omega) = 0$ *for all* $i = 1, ..., g$ *and some* $z_0 \in \mathbb{H}$, *then* $\omega = 0$.

proof: The same as corollary 3.11. □

Chapter 4

Schottky bundles and unitary bundles.

In this chapter we compare unitary and Schottky vector bundles. We give examples of Schottky vector bundles E, of rank $n > 1$, which do not admit unitary representations (and hence are not stable). The period map, studied in chapter 3, is used to show that the Schottky moduli map is a local diffeomorphism at some representations, and conclude that, in the moduli space of flat stable vector bundles, the Schottky ones form an open subset. This techniques, being of local nature, are insufficient for a global study of this open subset.

4.1 Non-unitary Schottky vector bundles.

Let $\sigma_i \in \text{Hom}(\pi_1(X), \mathbb{C}^*)$, $i = 1, ..., n$, be one-dimensional representations, so that $F_i = V(\sigma_i)$ are line bundles of degree 0, and let $\rho \in \text{Hom}(\pi_1(X), GL(n, \mathbb{C}))$ be a representation of the form:

$$\rho(\gamma) = \begin{pmatrix} \sigma_1(\gamma) & * & \dots & * \\ 0 & \sigma_2(\gamma) & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & \sigma_n(\gamma) \end{pmatrix} \in GL(n, \mathbb{C}) \qquad \forall \gamma \in \pi_1(X)$$

Then the vector bundle $E = V(\rho)$ has a strictly increasing filtration $0 \subset E_1 \subset E_2 \subset ... \subset E_n = E$, of flat sub-vector bundles each of which is an extension (with $E_1 = F_1$):

$$0 \to F_1 \to E_2 \to F_2 \to 0$$
$$0 \to E_2 \to E_3 \to F_3 \to 0$$
$$\vdots$$
$$0 \to E_{n-1} \to E \to F_n \to 0$$

Since all bundles in this filtration have degree 0 and line bundles are semistable, we see that E is a semistable bundle which is not stable, by repeated application of the following lemma:

Lemma 4.1 *Consider an extension $0 \to V \to E \to W \to 0$, where all bundles have the same slope μ, and V, W are semistable. Then E is semistable but not stable.*

proof: Since $V \subset E$ and $\mu(V) = \mu(E) = \mu$, E is not stable. Suppose that E is not semistable. Then there is a stable sub-bundle $F \subset E$ with $\mu(F) > \mu(E)$ (see [NS2]). Consider the following long exact sequence:

$$0 \to H^0(X, \mathrm{Hom}(F,V)) \to H^0(X, \mathrm{Hom}(F,E)) \to H^0(X, \mathrm{Hom}(F,W)) \to ...$$

Since V and W are semistable and $\mu(F) > \mu(V) = \mu(W) = \mu$, by proposition 1.4, $H^0(X, \mathrm{Hom}(F,V)) = H^0(X, \mathrm{Hom}(F,W)) = 0$ and so $H^0(X, \mathrm{Hom}(F,E)) = 0$, which contradicts $F \subset E$. □

In particular, we can take all the σ_i and ρ to be Schottky, so that:

$$\rho(\alpha_1) = ... = \rho(\alpha_g) = \mathbf{1}$$

$$\rho(\beta_1) = \begin{pmatrix} a_1 & * & \dots & * \\ 0 & b_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & z_1 \end{pmatrix}, ..., \qquad \rho(\beta_g) = \begin{pmatrix} a_g & * & \dots & * \\ 0 & b_g & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & z_g \end{pmatrix},$$

and $E = V(\rho)$ will be a Schottky semistable and non-stable vector bundle. Bundles like these form a holomorphic family parametrized by a vector space. More precisely, let $M = \mathbb{C}^{g\frac{n(n+1)}{2}}$, be the space of g-tuples of $n \times n$ complex upper triangular matrices, and for $m = (m_1, ..., m_g) \in M$, denote by ρ_m its corresponding representation $(\rho(\alpha_i) = 1, \rho(\beta_i) = m_i)$. Let $\mathcal{F} = M \times \tilde{X} \times \mathbb{C}^n$, and consider the action of $\pi_1(X)$ on $\mathcal{F}$ given by:

$$\gamma \cdot (m, z, v) = (m, \gamma z, \rho_m(\gamma) v) \qquad \forall\, \gamma \in \pi_1(X),\ m \in M,\ z \in \tilde{X},\ v \in \mathbb{C}^n$$

Then $\mathcal{F}$ is a vector bundle over $M \times X$ (i.e, by definition, it is a family of vector bundles over X, parametrized by M), and we see that $\mathcal{F}_m := i_m^* \mathcal{F}$ (where $i_m : X \to M \times X$ is the inclusion $x \mapsto (m, x)$) is isomorphic to $V(\rho_m)$. Now we claim that almost all bundles $\mathcal{F}_m$ in this family are simple. Consequently, since they are not stable, they do not admit unitary representations (cf. proposition 1.6).

Theorem 4.2 *Let X be a Riemann surface of genus $g \geq 2$. For every $n \geq 2$ there is a (nonempty) dense open subset $M^o \subset M = \mathbb{C}^{g\frac{n(n+1)}{2}}$ such that for every $m \in M^o$, the vector bundle $V(\rho_m)$ is Schottky, simple but not stable. Consequently, there are Schottky bundles in any rank $n > 1$, which do not admit a unitary representation.*

Remark: Although these bundles are not unitary, they are *S-equivalent* to one which is unitary. The notion of S-equivalence is defined as follows. Every semistable vector bundle over a Riemann surface admits a filtration

$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_n = E$$

where the quotients E_{i+1}/E_i are stable, $i = 1, ..., n-1$ (see Seshadri [S]). Two semistable bundles are called S-equivalent if all these quotients are isomorphic. It is then clear that two *stable* bundles are S-equivalent if and only if they are isomorphic. Thus, any semistable bundle E with a filtration as above is S-equivalent to $E_2/E_1 \oplus \ldots \oplus E_n/E_{n-1}$. Since the latter is a direct sum of stable bundles, it is a unitary vector bundle, by Narasimhan-Seshadri's theorem. The notion of S-equivalence was introduced by Seshadri [S] because the space of S-equivalence classes of semistable vector bundles admits the structure of a projective variety.

To prove the theorem, we will use the following special case of the *upper semicontinuity theorem for cohomology* (see [H] for a proof).

Theorem 4.3 *Let $\mathcal{F}$ be a holomorphic family of vector bundles over X, parametrized by a complex manifold (resp. algebraic variety) T. Then, for any $k \in \mathbb{N}$, the set:*

$$T_k = \{m \in T : dim H^0(X, \mathcal{F}_m) \geq k\}$$

is an analytic (resp. algebraic) subset of T. Hence its complement $T \setminus T_k$, if non empty, is a dense (resp. Zariski) open set in T. □

Using as $\mathcal{F}$ a suitable holomorphic family, we get a density result for simple vector bundles, which was already used in proposition 3.1.

Proposition 4.4 *: Let M be a complex manifold parametrizing a holomorphic family of vector bundles $\{V(\rho_m)\}_{m\in M}$, and M^{sim} be the set of points $m \in M$ such that $V(\rho_m)$ is simple. Then, if M^{sim} is non empty, it is a dense open subset of M.*

proof: Consider the following action of $\pi_1(X)$ on $\mathcal{E} = M \times \tilde{X} \times \mathbb{C}^{n^2}$:

$$\gamma \cdot (m, z, b) = (m, \gamma z, \rho_m(\gamma) b \rho_m(\gamma)^{-1}) \quad \forall\, \gamma \in \pi_1(X), m \in M, z \in \tilde{X}, b \in \mathbb{C}^{n^2}$$

Under this action, $\mathcal{E}$ becomes an analytic family of vector bundles over X parametrized by M and such that $\mathcal{E}_m \cong \mathrm{End}V(\rho_m)$. By theorem 4.3 we know that, for any integer k, $M_k := \{m \in M : \dim H^0(X, \mathcal{E}_m) \geq k\}$ is an analytic subset of M. In particular, for $k = 2$, the complement $M \setminus M_2 = \{m \in M : \dim H^0(X, \mathrm{End}V(\rho_m)) = 1\}$, is precisely M^{sim} (because the dimension of $H^0(X, \mathrm{End}V(\rho_m))$ is always at least 1) and therefore M^{sim}, if non-empty, is a dense open subset of M. □

Hence, one example of a representation ρ_m, $m \in M$ whose associated bundle $V(\rho_m)$ is simple, will suffice to prove the theorem. To find one such bundle, we will use the lemma.

Lemma 4.5 *Let X be a compact Riemann surface uniformized as before, $X = \mathbb{H}/\Gamma$, and let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function verifying:*

$$(*) \quad \begin{cases} f(\alpha_i z) = f(z) \\ f(\beta_i z) = \lambda_i f(z) + c_i, \end{cases}$$

for some constants $c_i \in \mathbb{C}$, and $\lambda_i \in U(1)$. Then $c_i = d(1-\lambda_i)$ for some $d \in \mathbb{C}$, and in particular, $f = \frac{c_i}{1-\lambda_i} = d$ is a constant.

proof: Let f satisfy $(*)$. Then the differential of f, $\omega = df$ verifies:

$$(**) \qquad \begin{cases} \omega(\alpha_i z)\alpha_i'(z) = \omega(z) \\ \omega(\beta_i z)\beta_i'(z) = \lambda_i \omega(z), \end{cases}$$

so, ω is a ρ-twisted differential, where in this case, ρ is the $U(1)$ representation of $\pi_1(X)$ given by $\rho(\alpha_1) = ... = \rho(\alpha_g) = 1, \quad \rho(\beta_1) = \lambda_1, \quad ..., \quad \rho(\beta_g) = \lambda_g$. Therefore, $\int_{z_0}^{\alpha_i z_0} \omega = f(\alpha_i z_0) - f(z_0) = 0$. Hence, $P_w^{\alpha_i}(\omega)$ vanishes for any $i = 1, ..., g$, and any basepoint w. Then, by corollary 3.11 we must have $\omega = 0$ and so f is constant as asserted. □

Now we can construct an example of simple, Schottky, non stable bundle in rank $n = 2$.

Proposition 4.6 *Let $X = \mathbb{H}/\Gamma$ be a compact Riemann surface of genus $g \geq 2$, as above. Let ρ be the representation given by:*

$$\rho(\alpha_1) = ... = \rho(\alpha_g) = \mathbf{1}$$

$$\rho(\beta_1) = \begin{pmatrix} a_1 & c_1 \\ 0 & b_1 \end{pmatrix} ... \quad \rho(\beta_g) = \begin{pmatrix} a_g & c_g \\ 0 & b_g \end{pmatrix}$$

If $a_i/b_i \in S^1$, $a_i \neq b_i$ and $\frac{c_i}{b_i - a_i} \neq \frac{c_j}{b_j - a_j}$ when $i \neq j$, then $V(\rho)$ is a simple vector bundle over X.

proof: We need to verify that the only functions in $H^0(X, \mathrm{End}V(\rho))$ are the constant multiples of the identity. An element $F \in H^0(X, \mathrm{End}V(\rho))$, can be considered as a function

$$F = \begin{pmatrix} f & g \\ h & k \end{pmatrix} : \mathbb{H} \to Mat_{2\times 2}(\mathbb{C})$$

which verifies: $F(\gamma z) = \rho(\gamma) F(z) \rho(\gamma)^{-1}, \quad \forall \gamma \in \pi_1(X)$. Explicitly, this means:

$$\begin{cases} h \circ \alpha_i = h \\ k \circ \alpha_i = k \\ f \circ \alpha_i = f \\ g \circ \alpha_i = g \end{cases} \qquad \begin{cases} h \circ \beta_i = \frac{b_i}{a_i} h \\ k \circ \beta_i = k - h\frac{c_i}{a_i} \\ f \circ \beta_i = f + h\frac{c_i}{a_i} \\ g \circ \beta_i = \frac{a_i}{b_i} g + \frac{c_i}{b_i}(k - f) \end{cases} \qquad \forall i = 1, ..., g.$$

By hypothesis, $1 \neq \frac{a_i}{b_i} \in S^1$, so we conclude that $h = 0$, f and k are constants, and g satisfies the hypothesis of lemma 4.5. So, $g = C^{te} = \frac{c_i}{b_i - a_i}(k - f)$. Since by hypothesis $\frac{c_i}{b_i - a_i} \neq \frac{c_j}{b_j - a_j}$, for all $i \neq j$, we see that $k = f$ and $g = 0$. So F is a constant multiple of the identity. □

If we perform the same computations, but now letting b_i be an $(n-1)\times(n-1)$ matrix like the ones in the last proposition, and c_i be an $(n-1)$-vector, we get, by induction, a corresponding result for vector bundles of rank n, and the proof of theorem 4.2 is complete.

Proposition 4.7 *Let X be a compact Riemann surface of genus $g \geq 2$ and let $n \geq 2$. If $m = (m^1, ..., m^g)$ is a g-tuple of $n \times n$ complex upper triangular matrices (with entries $(m^i_{ab}), i = 1, ..., g; 1 \leq a \leq b \leq n$) satisfying the conditions:*
1) for every $i = 1, ..., g$ we have $\quad m^i_{aa}/m^i_{bb} \in U(1) \setminus \{1\}$ for all $a \neq b$, and
2) for every $i \neq j$, $\quad \frac{m^i_{ab}}{m^i_{bb} - m^i_{aa}} \neq \frac{m^j_{ab}}{m^j_{bb} - m^j_{aa}}$.
Then the flat vector bundle $V(\rho_m)$ is simple. □

4.2 Schottky vector bundles in the moduli space.

We are now ready to prove that there are stable bundles F such that every bundle E, in some neighborhood of F (in the moduli space of stable flat vector bundles), has a Schottky representative.

Proposition 4.8 *The Schottky moduli map* $W : \mathcal{S}^{st}(n) \to \mathcal{M}^{st}(n)$ *of proposition 3.6, has maximal rank at the points of* $\mathcal{S}^{st}(n) \cap \mathcal{U}^{st}(n)$.

proof: Since $\mathcal{S}^{st}(n)$ and $\mathcal{U}^{st}(n)$ are both complex manifolds of the same dimension, $n^2(g-1)+1$ (cf. proposition 3.1), we just need to show that the differential of W, dW_ρ has trivial kernel when $\rho \in \mathrm{Hom}(\Sigma, U(n))$. Since $W = V \circ i$ is the composition

$$\mathcal{S}^{st}(n) \stackrel{i}{\to} \mathcal{G}^{st}(n) \stackrel{V}{\to} \mathcal{M}^{st}(n)$$

the differential is the composition:

$$H^1(\Sigma, \mathrm{Ad}_\rho) \stackrel{di}{\to} H^1(\pi, \mathrm{Ad}_\rho) \stackrel{dV_\rho}{\to} T_{V(\rho)}\mathcal{M}^{st}(n),$$

and so, by lemma 3.7,

$$\mathrm{Ker} dW_\rho = \mathrm{Im}(H^1(\Sigma, \mathrm{Ad}_\rho)) \cap \mathrm{Ker} dV_\rho = \mathrm{Im}(H^1(\Sigma, \mathrm{Ad}_\rho)) \cap \mathrm{Im} d(\psi_\rho)_0.$$

This means that $B = (B_1, ..., B_g) \in \mathrm{Ker} dW_\rho$ is equivalent to the existence of a $\eta \in H^0(\mathrm{End} V(\rho) \otimes K)$ such that $d(\psi_\rho)_0(\eta) \in H^1(\Sigma, \mathrm{Ad}_\rho)$, that is, $d(\psi_\rho)_0(\eta)$ is of the form $[(0, ..., 0, B_1, ..., B_g)]$. Since $d(\psi_\rho)_0(\eta)^\gamma = \int_{z_0}^{\gamma z_0} \eta$, this means that $P^\alpha_{z_0}(\eta) = 0 \quad \forall i = 1, ..., g$ and so, by corollary 3.13, $\eta = 0$. Hence, $d(\psi_\rho)_0(\eta)^\gamma = 0$, from which it follows that $B = 0$, and so, $\mathrm{Ker} dW_\rho = 0$. □

Corollary 4.9 *the map* $W : \mathcal{S}^{st}(n) \to \mathcal{M}^{st}(n)$ *is a local diffeomorphism except in an analytic subset* R. *In particular,* $W : \mathcal{S}^{st}(n) \setminus R \to \mathcal{M}^{st}(n)$ *is a covering map onto its image.*

proof: We recall that, by proposition 3.6 W is a holomorphic map, so that the singular set $R = \{\rho \in \mathcal{S}^{st}(n) : det(dW_\rho) = 0\}$ is a closed analytic subset of $\mathcal{S}^{st}(n)$. □

Recall that the fiber of the map W contains infinitely many representations, since there are infinitely many Schottky representations of the trivial line bundle. Finally, we get:

Theorem 4.10 *There is an open set* $\mathcal{M}'(n) \subset \mathcal{M}^{st}(n)$ *in the moduli space of stable flat vector bundles of rank* n*, such that the map* $W : \mathcal{S}^{st}(n) \to \mathcal{M}^{st}(n)$ *has maximal rank onto points of* $\mathcal{M}'(n)$. *In particular, every bundle in* $\mathcal{M}'(n)$ *admits a Schottky representation.*

proof: We can just take $\mathcal{M}'(n) = W(\mathcal{S}^{st}(n) \setminus R)$. □

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