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Schottky uniformization and vector bundles over Riemann surfaces

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Abstract. We study a natural map from representations of a free group of rank g in $GL(n,\mathbb{C})$, to holomorphic vector bundles of degree 0 over a compact Riemann surface X of genus g, associated with a Schottky uniformization of X. Maximally unstable flat bundles are shown to arise in this way. We give a necessary and sufficient condition for this map to be a submersion, when restricted to representations producing stable bundles. Using a generalized version of Riemann's bilinear relations, this condition is shown to be true on the subspace of unitary Schottky representations.

1. Introduction

Let X be a compact Riemann surface of genus g and $\mathcal{G}$ be the sheaf of germs of holomorphic functions from X to $GL(n,\mathbb{C})$. The inclusion $GL(n,\mathbb{C}) \hookrightarrow \mathcal{G}$ (where $GL(n,\mathbb{C})$ is identified with its constant sheaf on X), defines a map:

$$\mathcal{V} : H^1(X, GL(n,\mathbb{C})) \to H^1(X,\mathcal{G}), \tag{1}$$

that sends a flat $GL(n,\mathbb{C})$-bundle into the corresponding holomorphic vector bundle of rank n over X. Two flat bundles are said to be *analytically equivalent* if they have the same image under $\mathcal{V}$. A holomorphic vector bundle E is called *flat* if it lies in the image of $\mathcal{V}$; this happens if and only if every indecomposable component of E have degree 0, by a classical theorem of Weil [W].

There is a well know bijection between the space of flat $GL(n,\mathbb{C})$-bundles over X and the space of representations of the fundamental group of X, $\pi_1 = \pi_1(X)$, modulo overall conjugation $\mathbf{G}_n := \mathrm{Hom}(\pi_1, GL(n,\mathbb{C}))/GL(n,\mathbb{C})$; it is given explicitely by:

$$E_\cdot : \mathbf{G}_n \to H^1(X, GL(n,\mathbb{C})), \qquad \rho \mapsto E_\rho := \tilde{X} \times_\rho \mathbb{C}^n, \tag{2}$$

where π_1 acts diagonally on the trivial rank n vector bundle over the universal cover $\tilde{X}$ of X.

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The subset $\mathcal{M}_n^{st} \subset H^1(X,\mathcal{G})$ of stable holomorphic bundles of rank n and degree 0 is a non-singular quasi-projective algebraic variety of dimension $n^2(g-1)+1$, as shown by Mumford [Mu]. Therefore, if we restrict attention to the subset $\mathbf{G}_n^{st} := \{\rho : E_\rho \text{ is stable}\} \subset \mathbf{G}_n$ which is a (smooth) complex manifold of twice the dimension of $\mathcal{M}_n^{st}$ (see [Gu]), we get a map:

$$V : \mathbf{G}_n^{st} \to \mathcal{M}_n^{st}, \qquad \rho \mapsto V(\rho) := \mathcal{V}(E_\rho) \tag{3}$$

which is surjective, by Weil's theorem (a stable bundle is indecomposable) and is holomorphic (see §3, below) because of the universal property of the (coarse) moduli space $\mathcal{M}_n^{st}$ ([NS1]).

The well known theorem of Narasimhan and Seshadri [NS2], implies that the restriction of V to the subset $\mathbf{U}_n^{st} = \{\rho \in \mathrm{Hom}(\pi_1, U(n))/U(n) : \rho \text{ is irreducible}\}$ is a diffeomorphism.

Now let us fix a canonical basis of $\pi_1 = \pi_1(X)$: elements $a_1, ..., a_g, b_1, ..., b_g$ that generate π_1, subject to the single relation $\prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1} = 1$. Let F_g be a free group on g generators $B_1, ..., B_g$, and $p : \pi_1 \to F_g$ be the homomorphism given by $p(a_i) = 1, p(b_i) = B_i, i = 1, ..., g$. Then we can form the exact sequence of groups:

$$1 \to N \to \pi_1 \xrightarrow{p} F_g \to 1, \tag{4}$$

where N is the smallest normal subgroup of π_1 containing $a_1, ..., a_g$. A Shottky uniformization, representing X as a quotient of a domain of the Riemann sphere $\mathbb{P}^1$ by a free subgroup of $PSL(2,\mathbb{C})$, gives us a prefered representation $\sigma_X \in \mathrm{Hom}(F_g, PSL(2,\mathbb{C}))$ which is unique up to conjugation (see Thm. 4). By analogy, we will say that $\rho \in \mathrm{Hom}(\pi_1, GL(n,\mathbb{C}))$ is a *Schottky representation* if it lies in the image of the inclusion $i := p^* : \mathrm{Hom}(F_g, GL(n,\mathbb{C})) \hookrightarrow \mathrm{Hom}(\pi_1, GL(n,\mathbb{C}))$, induced from sequence (4). Similarly, a holomorphic vector bundle E is called a *Schottky bundle* (of rank n) if it is in the image of the composition

$$\mathbf{S}_n := \mathrm{Hom}(F_g, GL(n,\mathbb{C}))/GL(n,\mathbb{C}) \overset{i}{\hookrightarrow} \mathbf{G}_n \overset{\mathcal{V}\circ E.}{\to} H^1(X,\mathcal{G}).$$

In particular, a Schottky vector bundle is always flat. It is easy to see that $\mathbf{S}_n^{st} := \{\rho : E_\rho \text{ is stable}\} \subset \mathbf{S}_n$ is a complex manifold of the same dimension as $\mathcal{M}_n^{st}$ (Prop. 1).

In section 2, we will give some examples of Schottky vector bundles. Let us say that a vector bundle E of rank 2 and degree 0 is maximally unstable indecomposable, if it is given by a nontrivial extension $0 \to L \to E \to L^{-1} \to 0$, where L is a square root of the canonical line bundle K. We will show that:

Theorem 1. *Every maximally unstable indecomposable vector bundle of rank 2 and degree 0 is Schottky.*

In section 3 we study the spaces of representations $\mathbf{G}_n^{st}$ and $\mathbf{S}_n^{st}$, and consider the restriction

$$W := V|_{\mathbf{S}_n^{st}} : \mathbf{S}_n^{st} \to \mathcal{M}_n^{st}, \qquad \rho \mapsto W(\rho) := V(i(\rho)) \tag{5}$$

of V to $\mathbf{S}_n^{st}$ which is again holomorphic (Prop. 3). One can pose the following problem:

Problem 1. Is the above map W surjective, or at least onto a dense open subset of $\mathcal{M}_n^{st}$?

The answer seems to be unknown at present, to the best of our knowledge, except for the simplest cases: rank 1 or genus one[1], where it is positive (see §6, Appendix). This problem, which can be called the Schottky uniformization for vector bundles, may be of interest for the theory of generalized theta functions, because we can describe them as holomorphic functions on $\mathbf{S}_n^{st}$, by pulling back the determinant line bundle on the moduli space of semistable bundles using W (see Beauville [B]). It can also be useful in describing the Kähler metric on the moduli space of stable bundles (see Takhtajan and Zograf [Ta], [ZT2]) by analogy with the fact that the Fuchsian and Schottky uniformizations of a compact Riemann surface are related through a potential for the Weil-Peterson Kähler metric on Teichmüller space ([ZT1]).

In section 4, in order to compute the differential of W, we consider the following period map:

$$P_{Ad_\rho} : H^0(X, End E_\rho \otimes K) \to H^1(\pi_1, Ad_\rho), \qquad P_{Ad_\rho}(\phi) := [\gamma \mapsto \textstyle\int_\gamma \phi]$$

where $\rho \in \mathbf{G}_n$, Ad_ρ denotes the π_1-module of $n \times n$ matrices M with the action $\gamma \cdot M = \rho(\gamma) M \rho(\gamma)^{-1}$, $H^1(\pi_1, Ad_\rho)$ the first cohomology of π_1 with values in this module, and K is the canonical line bundle on X. Then we have:

Theorem 2. *Let X be a compact Riemann surface of genus $g \geq 2$. Then $dW_\rho : T_\rho \mathbf{S}_n^{st} \to T_{E_\rho} \mathcal{M}_n^{st}$ is surjective if and only if $H^1(F_g, Ad_\rho) \cap Im(P_{Ad_\rho}) = 0$.*

In §5, we consider the case of representations that are both Schottky and unitary. This space is also important, since it is one of the Bohr-Sommerfeld orbits of the real polarization of the pre-quantum system of flat unitary connections on X (see Tyurin [Ty]). We use a generalization of Riemann's bilinear relations to prove:

Theorem 3. *If $\rho \in \mathbf{S}_n^{st} \cap \mathbf{U}_n^{st}$ then dW_ρ is surjective. In particular, The image $W(\mathbf{S}_n^{st})$ contains a nonempty open set $U \subset \mathcal{M}_n^{st}$ (in the complex topology) such that $W(\mathbf{S}_n^{st} \cap \mathbf{U}_n^{st}) \subset U$.*

In section 6, we study Schottky bundles in the easy case $g = 1$, and in the appendix, we consider the case of line bundles, which have been considered before from another viewpoint (compare [K], Ch. VI, §4).

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[1] in a slightly modified version, to take into account semistable bundles.

2. Schottky uniformization and unstable vector bundles.

We begin by recalling the classical Schottky uniformization of compact Riemann surfaces. For details we refer to [C, AS, Ms, Bs1, Bs2]. Schottky groups are an important class of Kleinian groups: groups of Möbius transformations acting properly discontinuously on some domain of the Riemann sphere $\mathbb{P}^1$. A *marked Schottky group of genus* g is a strictly loxodromic (this includes hyperbolic) finitely generated free Kleinian group Σ of rank g [Ms], together with a choice of g generators $T_1, ..., T_g \in PSL(2, \mathbb{C})$ of Σ. Two marked Schottky groups $(\Sigma, T_1, ..., T_g)$ and $(\Sigma', T_1', ..., T_g')$ are said to be *equivalent* if there exists a Möbius transformation M such that $T_i' = MT_iM^{-1}$ for all $i = 1, ..., g$. Thus, the set of equivalence classes of marked Schottky groups of genus g is a subset of $\mathrm{Hom}(F_g, PSL(2, \mathbb{C}))/PSL(2, \mathbb{C})$ (where F_g is a free group on g generators, and $PSL(2, \mathbb{C})$ acts by overall conjugation), called *Schottky space of genus* g. It is an open set in $\mathbb{C}^{3g-3}$ [C]. Let us denote by $\Omega_\Sigma \subset \mathbb{P}^1$ the domain of discontinuity of Σ (for $g > 1$, its complement is a Cantor set [C],[AS]).

A Schottky group Σ gives rise to a compact Riemann surface $X := \Omega_\Sigma/\Sigma$. Every marked Schottky group $(\Sigma, T_1, ..., T_g)$ has a (non unique) standard fundamental domain (see [C]), which is a region $D \subset \mathbb{P}^1$ bounded by smooth closed curves $C_1, ..., C_g, C_1', ..., C_g'$, each lying on the outside of all the others, such that $T_i(C_i) = C_i'$. If we orient each C_i clockwise and each C_i' counterclockwise, the canonical holomorphic map $\Omega_\Sigma \to X$, sends the boundary curves of D onto smooth non-intersecting simple oriented closed curves $\alpha_1, ..., \alpha_g$ on X.

In this way, we see that a marked Schottky group plus the choice of a standard fundamental domain determines a Riemann surface with a distinguished set of curves $\{\alpha_1, ..., \alpha_g\}$. Conversely, the classical retrosection theorem of Koebe (see [Bs1, AS]), states that every compact Riemann surface arises this way: *For every compact Riemann surface of genus g with a choice of g smooth simple non-intersecting, homologically independent, oriented closed curves $\alpha_1, ..., \alpha_g$, there exists a marked Schottky group of genus g, $(\Sigma, T_1, ..., T_g)$ and a fundamental domain for Σ with $2g$ boundary curves $C_1, ..., C_g, C_1', ..., C_g'$, such that $X = \Omega_\Sigma/\Sigma$ and the map $\Omega_\Sigma \to X$ sends both C_i and C_i' to α_i, preserving orientations as above. The marked Schottky group $(\Sigma, T_1, ..., T_g)$ satisfying these conditions is uniquely determined by (X, α_i) up to equivalence.*

We can restate this classical theorem in the following way, more convenient for our purposes:

Theorem 4. *Let X be a Riemann surface of genus $g \geq 1$ with a canonical basis for π_1, and F_g be as in sequence (4). Then there is a unique (up to conjugation) representation $\sigma_X \in Hom(F_g, PSL(2, \mathbb{C}))/PSL(2, \mathbb{C})$ such that $X = \Omega_\Sigma/\Sigma$ where $\Sigma = Im(\sigma_X)$.*

To avoid heavier notaion, we will denote representations and their equivalence classes under conjugation by the same symbols; hopefuly, this should cause no confusion.

Definition 1. *A representation ρ of π_1 in a group G will be called a* Schottky representation, *if it lies in the image of the inclusion induced by sequence (4) $i : Hom(F_g, G) \hookrightarrow Hom(\pi_1, G)$. A holomorphic vector bundle E over X is called a* Schottky vector bundle over X *if $E = V(\rho)$, for some Schottky representation $\rho \in Hom(\pi_1, GL(n, \mathbb{C}))$, where $V = \mathcal{V} \circ E.$ is the map (3).*

It is clear that ρ is Schottky if and only if $\rho(a_i) = \mathbf{1}$, for all $i = 1, ..., g$. Another easy consequence of the definition is that the tensor product and direct sum of Schottky vector bundles is also Schottky. Later, we will see that every line bundle of degree 0 is Schottky (see §A), so the property of being Schottky is preserved under tensoring by a line bundle of degree 0.

We now give some interesting examples of Schottky vector bundles of rank 2, which are not semistable[2]. Let E be an extension $0 \to L \to E \to L^{-1} \to 0$, where L is a line bundle of degree $g-1$. For E to be indecomposable, the space that classifies these extensions $H^1(X, L^2) = H^0(X, KL^{-2})^*$, has to be non zero, so L is a square root of the canonical bundle K. In this case E is unique, and called a *maximally unstable indecomposable vector bundle* of rank 2 and degree 0, and denoted here by E_L. We now prove that all these E_L are Schottky.

Proof. (of theorem 1): Let L be a square root of K (i.e, $L^2 = K$). First, note that $E_{L\otimes L_0} = E_L \otimes L_0$, for any line bundle L_0 such that $L_0^2 = \mathcal{O}$; moreover if E_L is Schottky, so is $E_L \otimes L_0$, by the remarks after definition 1. Therefore, if the result is true for some square root of K, then it is also true for any other square root. If $g = 1$, and $\mathbb{I}$ is the trivial line bundle, $E_{\mathbb{I}}$ is Atiyah's bundle $\mathbb{F}_2$ (see section 6), so the result follows from lemma 5 below. So, assuming $g \geq 2$, consider the following diagram, induced by the quotient homomorphism $\nu : SL(2, \mathbb{C}) \to PSL(2, \mathbb{C})$, whose commutativity is easily established, and whose vertical sequences are principal fibrations:

$$\begin{array}{ccccc}
\mathrm{Hom}(F_g, \mathbb{Z}_2) & \stackrel{i}{\to} & \mathrm{Hom}(\pi_1, \mathbb{Z}_2) & \stackrel{E.}{\to} & H^1(X, \mathbb{Z}_2) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}(F_g, SL(2,\mathbb{C})) & \stackrel{i}{\to} & \mathrm{Hom}(\pi_1, SL(2,\mathbb{C})) & \stackrel{E.}{\to} & H^1(X, SL(2,\mathbb{C})) \\
\downarrow \nu & & \downarrow \nu & & \downarrow \mu \\
\mathrm{Hom}(F_g, PSL(2,\mathbb{C})) & \stackrel{i}{\to} & \mathrm{Hom}(\pi_1, PSL(2,\mathbb{C}))^+ & \stackrel{P.}{\to} & H^1(X, PSL(2,\mathbb{C}))^+
\end{array} \qquad (6)$$

Here $\mathrm{Hom}(\pi_1, PSL(2, \mathbb{C}))^+$ denotes the connected component which contains the image of the inclusion $i : \mathrm{Hom}(F_g, PSL(2, \mathbb{C})) \hookrightarrow \mathrm{Hom}(\pi_1, PSL(2, \mathbb{C}))$, and similarly for $H^1(X, PSL(2, \mathbb{C}))^+$. Using the techniques of Gunning ([Gu1]), we see that the marked Schottky group $\sigma_X \in \mathrm{Hom}(F_g, PSL(2, \mathbb{C}))$, of theorem 4, gives rise to a flat projective Schottky bundle $P_{i(\sigma_X)} \in H^1(X, PSL(2, \mathbb{C}))$, which by construction is in the image of the natural map from the space of all projective structures on X to $H^1(X, PSL(2, \mathbb{C}))$. Then, By [Gu1], th. 2, there is a flat vector bundle E_1 such that $\mu(E_1) = P_{i(\sigma_X)}$ and E_1 has divisor order $g-1$, so that it can be given as an extension $0 \to L_1 \to$

[2] I thank I. Biswas for suggesting this example

$E_1 \to L_1^{-1} \to 0$, with $\deg(L_1) = g-1$. E_1 is indecomposable because of Weil's theorem, thus $L_1^2 = K$, and by uniqueness, $E_1 = E_{L_1}$. On the other hand, if $\rho \in \mathrm{Hom}(F_g, SL(2,\mathbb{C}))$ is such that $\nu(\rho) = \sigma_X$, then $E_{i(\rho)}$ is Schottky by definition, and the commutativity of the diagram implies $\mu(E_{i(\rho)}) = P_{i(\sigma_X)} = \mu(E_1)$. Thus, by exactness, E_{L_1} and $E_{i(\rho)}$, differ by tensoring with a $\mathbb{Z}_2$-bundle $L_2 \in H^1(X, \mathbb{Z}_2)$ ($L_2^2 = \mathcal{O}$), so E_{L_1} is Schottky as required.

Considering vector bundles of the form $E_L \oplus \mathbb{I}^n$, we get Schottky vector bundles which are clearly not semistable, for any $g \geq 2$ and any rank ≥ 2.

3. Spaces of Schottky representations.

To study the maps V, W of (3),(5), we start by studying the spaces of representations $\mathbf{G}_n$ and $\mathbf{S}_n$. Let $\mathrm{Hom}(\pi_1, GL(n,\mathbb{C}))^o$ and $\mathrm{Hom}(F_g, GL(n,\mathbb{C}))^o$ denote the subsets consisting of representations ρ having only scalar commutants (i.e, if a matrix M commutes with all matrices in the image of ρ, then M is a scalar) and let

$$\mathbf{G}_n^o := \mathrm{Hom}(\pi_1, GL(n,\mathbb{C}))^o / GL(n,\mathbb{C}) \qquad \mathbf{S}_n^o := \mathrm{Hom}(F_g, GL(n,\mathbb{C}))^o / GL(n,\mathbb{C})$$

be their quotients under simultaneous conjugation. We remark that, in terms of the initial map $\mathcal{V}$ (1), this constitutes no restriction, since by [Gu],[Gu2] every flat holomorphic bundle admits a representation with only scalar commutants. Their *tangent spaces* can be given in terms of cohomology of groups with coefficients in group modules. Denote by $H^k(\Gamma, M)$ the k-th cohomology group of Γ with coefficients in the Γ-module M. The adjoint representation, together with a representation ρ of π_1 (resp. F_g) in $GL(n,\mathbb{C})$, endows the space of all $n \times n$ matrices, with a π_1- (resp. F_g-) module structure ($\gamma \cdot M := \rho(\gamma) M \rho(\gamma)^{-1}$ for a matrix M), denoted by Ad_ρ. When ρ has only scalar commutants, it is not difficult to see that $\dim_{\mathbb{C}} H^1(F_g, \mathrm{Ad}_\rho) = n^2(g-1)+1$, which is also the dimension of $\mathcal{M}_n^{st}$. Note that the inclusion $Z^1(F_g, \mathrm{Ad}_\rho) \hookrightarrow Z^1(\pi_1, \mathrm{Ad}_\rho) : (B_1, ..., B_g) \mapsto (0, ..., 0, B_1, ..., B_g)$ induces an inclusion $H^1(F_g, \mathrm{Ad}_\rho) \hookrightarrow H^1(\pi_1, \mathrm{Ad}_\rho)$, which is complex linear.

In [Gu, Gu2] it is shown that $\mathbf{G}_n^o$ has the structure of a complex analytic manifold of dimension $2(n^2(g-1)+1)$ such that the natural projection $\mathrm{Hom}(\pi_1, GL(n,\mathbb{C}))^o \to \mathbf{G}_n^o$ is a complex analytic principal $PGL(n,\mathbb{C})$-bundle, and whose tangent space at the equivalence class of ρ is $H^1(\pi_1, \mathrm{Ad}_\rho)$. $\mathbf{G}_n^o$ has also the structure of a complex symplectic manifold, [3] whose symplectic form ω is defined by cup product, an invariant bilinear pairing B on the Lie algebra of $GL(n,\mathbb{C})$, and evaluation on the fundamental homology class $c \in H_2(\pi_1, \mathbb{C})$, $\omega : H^1(\pi_1, \mathrm{Ad}_\rho) \times H^1(\pi_1, \mathrm{Ad}_\rho) \stackrel{\cup}{\to} H^2(\pi_1, \mathrm{Ad}_\rho \otimes \mathrm{Ad}_\rho) \stackrel{B}{\to} H^2(\pi_1, \mathbb{C}) \stackrel{\cdot c}{\to} \mathbb{C}$ (see [Go]). For the case of $\mathbf{S}_n^o$ we find:

[3] $\mathbf{G}_n^o$ has also the structure of a hyperKähler manifold but we will not need this extra structure.

Proposition 1. $\mathbf{S}_n^o$ *is a complex analytic manifold of dimension* $n^2(g-1)+1$ *whose tangent space at the equivalence class of the representation* ρ *is* $H^1(F_g, Ad_\rho)$*. Moreover,* $\mathbf{S}_n^o$ *is a Lagrangian submanifold of* $\mathbf{G}_n^o$*.*

Proof. Since $\rho \in \mathrm{Hom}(F_g, GL(n,\mathbb{C}))$ has only scalar commutants, the action of $PGL(n,\mathbb{C})$ by conjugation is free and given $\rho, \rho' \in \mathrm{Hom}(F_g, GL(n,\mathbb{C}))^o$ in the same orbit, there is a unique $T \in PGL(n,\mathbb{C})$ such that $\rho' = T\rho T^{-1}$. The same arguments used in [Gu], §9, prove that $\mathbf{S}_n^o = \mathrm{Hom}(F_g, GL(n,\mathbb{C}))^o/GL(n,\mathbb{C}) = \mathrm{Hom}(F_g, GL(n,\mathbb{C}))^o/PGL(n,\mathbb{C})$ (the center acts trivially) is a complex manifold, and that the tangent spaces agree with the required cohomology groups. Moreover, $\mathbf{S}_n^o$ is an analytic submanifold of $\mathbf{G}_n^o$, since the inclusion $H^1(F_g, \mathrm{Ad}_\rho) \hookrightarrow H^1(\pi_1, \mathrm{Ad}_\rho)$ is complex linear when ρ is Schottky. Since F_g is a free group, $H^2(F_g, M) = 0$, for any F_g-module M. Therefore, the symplectic form vanishes on any two tangent vectors to $\mathbf{S}_n^o$.

Observe that $\mathbf{U}_n^o = \mathrm{Hom}(\pi_1, U(n))^o/U(n)$ is also a subset of $\mathbf{G}_n^o$, diffeomorphic to $\mathcal{M}_n^{st}$ by the theorem of Narasimhan-Seshadri (since a unitary representation with only scalar commutants is irreducible). When ρ is a unitary representation of π_1, the Lie algebra of $U(n)$ is again a π_1-module (denoted $\mathrm{Ad}_\rho^{\mathbb{R}}$) via the adjoint representation composed with ρ . The tangent space of $\mathbf{U}_n^{st}$ at the equivalence class of the representation ρ can be identified with $H^1(\pi_1, \mathrm{Ad}_\rho^{\mathbb{R}})$ [NS1], but the inclusion $H^1(\pi_1, \mathrm{Ad}_\rho^{\mathbb{R}}) \hookrightarrow H^1(\pi_1, \mathrm{Ad}_\rho)$ is *not* complex linear. Thus, $\mathbf{U}_n^{st}$ sits inside $\mathbf{G}_n^o$ as a *real analytic* submanifold but *not* complex analytic, in contrast to $\mathbf{S}_n^o$.

Lemma 1. *For every* $n \geq 1$, $g \geq 2$, *there are unitary, Schottky* n*-dimensional irreducible representations of* $\pi_1(X)$*. Therefore, there are stable Schottky vector bundles of any given rank.*

Proof. We may suppose that $n \geq 2$ (see §A). Let $\lambda_1, ..., \lambda_n$ be n distinct complex numbers of modulus 1. Let $B_1 = diag(\lambda_1, ..., \lambda_n)$, and B_2 be the permutation matrix $e_1 \mapsto e_2, ..., e_n \mapsto e_1$, for a canonical basis $e_1, ..., e_n \in \mathbb{C}^n$. It is easy to see that B_1, B_2 form an irreducible set of unitary matrices (i.e, no subspace of $\mathbb{C}^n$ is preserved by both). Hence, the representation of π_1 given by $\rho(a_i) = \mathbf{1}, \quad i = 1, ..., g; \quad \rho(b_i) = B_i, \quad i = 1, 2; \quad \rho(b_i) = \mathbf{1}, \quad i = 3, ..., g$, is unitary, Schottky and irreducible. Stability is clear from Narasimhan-Seshadri's theorem.

Consider now the subsets $\mathbf{S}_n^{st}$ and $\mathbf{G}_n^{st}$, of $\mathbf{S}_n^o$ and $\mathbf{G}_n^o$, respectively, consisting of conjugacy classes of representations ρ such that E_ρ is stable (since a stable bundle is simple, it has only scalar commutants), and let $\mathcal{E}$ be the holomorphic family of vector bundles over X parametrized by the complex manifold $\mathbf{G}_n^o$, whose fiber over $X \times \{\rho\}$ is E_ρ. It is given by the following π_1-action on $\mathbf{G}_n^o \times \tilde{X} \times \mathbb{C}^n$:

$$\gamma \cdot (\rho, z, v) = (\rho, \gamma \cdot z, \rho(\gamma) v) \qquad \forall \gamma \in \pi_1, (z, v) \in \tilde{X} \times \mathbb{C}^n.$$

Proposition 2. *Let X be a compact Riemann surface of genus $g \geq 2$, and let $n \geq 1$. $\mathbf{S}_n^{st}$ and $\mathbf{G}_n^{st}$ are dense open complex submanifolds of $\mathbf{S}_n^o$ and $\mathbf{G}_n^o$, respectively.*

Proof. In any holomorphic parameter space, stable bundles form the complement of an analytic subset [NS2], Thm 2(B). Since $\mathbf{G}_n^{st}$ is non-empty (irreducible unitary representations give stable bundles, for $g \geq 2$), the existence of the family $\mathcal{E}$ implies that $\mathbf{G}_n^{st}$ is open and dense in $\mathbf{G}_n^o$. The same applies to $\mathbf{S}_n^{st}$, since stable Schottky bundles exist, by lemma 1.

Proposition 3. *$V : \mathbf{G}_n^{st} \to \mathcal{M}_n^{st}$ and $W : \mathbf{S}_n^{st} \to \mathcal{M}_n^{st}$ are holomorphic maps.*

Proof. This follows from [NS1],§2, where it is shown that the space of (isomorphism classes of) simple vector bundles with degree 0 and rank n, $\mathcal{M}_n^{sim}$, verifies the universal property of a coarse moduli space in the holomorphic category. In other words, for every holomorphic family $\mathcal{A}$ of simple vector bundles over X, parametrized by a complex manifold S, the "universal map" $S \to \mathcal{M}_n^{st}$ sending $s \in S$ to the equivalence class of $\mathcal{A}_s$ is holomorphic. Since our maps V and W are actually the "universal maps" for the holomorphic families $\mathcal{E}|_{\mathbf{G}_n^{st}}$ and $\mathcal{E}|_{\mathbf{S}_n^{st}}$ constructed above (consisting of stable, hence simple, bundles), and $\mathcal{M}_n^{st}$ is an open subset of $\mathcal{M}_n^{sim}$, we have the coarse moduli space universal property for $\mathcal{M}_n^{st}$ as well.

We end this section by observing that $\mathbf{S}_n^o$ can be viewed as a natural generalization of Schottky space. Recall the canonical map $\nu : \mathrm{Hom}(F_g, SL(2,\mathbb{C})) \to \mathrm{Hom}(F_g, PSL(2,\mathbb{C}))$.

Proposition 4. *Schottky space of genus $g \geq 2$ is contained in $\nu(\mathbf{S}_2^o)$. More concretely, if the image of a representation $\sigma : F_g \to PSL(2,\mathbb{C})$ is a Schottky group Σ' of genus g, then $\nu^{-1}(\sigma)$ has only scalar commutants.*

Proof. By definition of Schottky group, every $T \in \Sigma' = Im(\sigma)$ is loxodromic, and any two generators of Σ' have distinct fixed points. Since two non trivial Möbius transformations commute if and only if they have the same fixed points, only the identity commutes with all elements in Σ'. Therefore, every representation $\rho \in \nu^{-1}(\sigma)$ has only scalar commutants.

4. The period map.

To compute the differential of W, we now consider a certain period map, and prove theorem 2. Let $\mathbb{V}_\sigma$ denote the (left) π_1-module defined on $\mathbb{C}^n$ by a representation $\sigma : \pi_1 \to GL(n,\mathbb{C})$. A global holomorphic section ϕ of $E_\sigma \otimes K$ will be called an *E_σ-differential.* In terms of a local coordinate $z \in \tilde{X}$, an E_σ-differential can be viewed as a closed, holomorphic differential 1-form $\phi = \phi(z)dz$, satisfying $\phi(\gamma z)\gamma'(z) = \sigma(\gamma)\phi(z)$, for all $\gamma \in \pi_1$. For a fixed basepoint $z_0 \in \tilde{X}$ and fixed ϕ, simple computations show that the map $\Phi_{z_0} : \pi_1 \to \mathbb{V}_\sigma$ defined by $\gamma \mapsto \int_{z_0}^{\gamma z_0} \phi$, is a cocycle in $Z^1(\pi_1, \mathbb{V}_\sigma)$, (since $\Phi_{z_0}(\gamma_1\gamma_2) = \Phi_{z_0}(\gamma_1) + \gamma_1 \cdot \Phi_{z_0}(\gamma_2)$) whose equivalence class $[\Phi_{z_0}] \in H^1(\pi_1, \mathbb{V}_\sigma)$ does not depend on the basepoint z_0 (because $\Phi_{z_1}(\gamma) = \int_{z_1}^{\gamma z_1} \phi = (\sigma(\gamma) - 1)\int_{z_0}^{z_1} \phi + \Phi_{z_0}(\gamma)$).

Definition 2. *Given a representation σ, the map:*

$$P_\sigma : H^0(X, E_\sigma \otimes K) \to H^1(\pi_1, \mathbb{V}_\sigma)$$

defined by $P_\sigma(\phi) := [\Phi_{z_0}]$ is called the period map *associated to σ.*

If $\rho_1, \rho_2 \in \mathrm{Hom}(\pi_1, GL(n, \mathbb{C}))$, one can ask when are E_{ρ_1} and E_{ρ_2} analytically equivalent. Let $z_0 \in \tilde{X}$, and denote by f_ϕ the unique solution of the differential equation $f^{-1}df = p^*\phi, f(z_0) = I$, for a given $\phi \in H^0(X, \mathrm{End}E_{\rho_1} \otimes K)$.

Lemma 2. *The following are equivalent: (1) $E_{\rho_1} \cong E_{\rho_2}$, (2) there is a holomorphic function $f : \tilde{X} \to GL(n, \mathbb{C})$ such that $f(\gamma z) = \rho_2(\gamma) f(z) \rho_1(\gamma)^{-1}$ for all $\gamma \in \pi_1, z \in \tilde{X}$, and (3) There exists $\omega \in H^0(X, EndE_{\rho_1} \otimes K)$ and $C \in GL(n, \mathbb{C})$ such that $\rho_2(\gamma) = C f_\omega(\gamma z) \rho_1(\gamma) f_\omega(z)^{-1} C^{-1}$, for all $\gamma \in \pi_1, z \in \tilde{X}$.*

Proof. ($1 \Leftrightarrow 2$): An isomorphism between E_{ρ_1} and E_{ρ_2} is a holomorphic global section of $E^*_{\rho_1} \otimes E_{\rho_2} = E_{{}^t\rho_1^{-1} \otimes \rho_2}$, consisting of invertible matrices. So it corresponds to a holomorphic $f : \tilde{X} \to GL(n, \mathbb{C})$ such that $f(\gamma \cdot z) = ({}^t\rho_1^{-1} \otimes \rho_2)(\gamma) f(z) = \rho_2(\gamma) f(z) \rho_1(\gamma)^{-1}$. ($2 \Leftrightarrow 3$): If we have f as in 2), then put $h(z) := C^{-1} f(z)$ where $C := f(z_0)^{-1}$. We obtain $h(z_0) = I$, $\rho_2(\gamma) = C h(\gamma z) \rho_1(\gamma) h(z)^{-1} C^{-1}$ and $\omega := h^{-1}dh$ belongs to $H^0(X, \mathrm{End}E_{\rho_1} \otimes K)$, since $d(C^{-1}\rho_2(\gamma)C) = 0$ is equivalent to $((h^{-1}dh) \circ \gamma)\gamma' = \rho_1(\gamma)(h^{-1}dh)\rho_1(\gamma)^{-1}$. Conversely, if we have 3), then clearly $f = Ch$ verifies 2).

Now define for a fixed $\rho \in \mathrm{Hom}(\pi_1, GL(n, \mathbb{C}))$ the following map which, because of the previous lemma, does not depend on z_0, or on the choice of representative ρ in its conjugacy class:

$$\begin{array}{rcl} Q_\rho : H^0(X, \mathrm{End}E_\rho \otimes K) & \to & \mathbf{G}_n \\ \omega & \mapsto & [f_\omega(\gamma z) \rho(\gamma) f_\omega(z)^{-1}] \end{array}$$

Q_ρ is holomorphic, due to the analytic dependence of f_ϕ on $\phi \in H^0(X, \mathrm{End}E_\rho \otimes K)$. Lemma 2 is easily seen to be equivalent to:

Lemma 3. *Two bundles E_{ρ_1} and E_{ρ_2} are analytically equivalent if and only if there is $\omega \in H^0(EndE_{\rho_1} \otimes K)$ such that $Q_{\rho_1}(\omega) = \rho_2$. In fact, $Q_\rho(H^0(EndE_\rho \otimes K)) = V^{-1}(E_\rho)$.*

Lemma 4. *(a) For any representation ρ, Ker $dV_\rho =$ Im $d(Q_\rho)_0$.*
(b) $d(Q_\rho)_0 : H^0(X, EndE_\rho \otimes K) \to H^1(\pi_1, Ad_\rho)$ coincides with P_{Ad_ρ}.

Proof. (a) For any $\eta \in H^0(X, \mathrm{End}E_\rho \otimes K)$ and $t \in \mathbb{C}$, we have $V(Q_\rho(t\eta)) \cong V(Q_\rho(0))$ (again by lemma 2). Letting $t \to 0$, we see that the differential of $V \circ Q_\rho$, at the origin is zero: $d(V \circ Q_\rho)_0(\eta) = dV_\rho \circ d(Q_\rho)_0(\eta) = 0$. Hence $\mathrm{Im}d(Q_\rho)_0 \subset \mathrm{Ker}dV_\rho$. Conversely, if $\phi \in \mathrm{Ker}dV_\rho \subset H^1(\pi_1, \mathrm{Ad}_\rho)$, then ϕ is tangent to the fiber of the map V at ρ, which means tangent to the image of Q_ρ at 0, so that there is an $\eta \in H^0(X, \mathrm{End}E_\rho \otimes K)$, such that $\phi = d(Q_\rho)_0(\eta)$.

(b) Let $\eta \in H^0(\mathrm{End}E_\rho \otimes K)$ and $t \in \mathbb{C}$. For small t, we can expand $f_{t\eta}$ as: $f_{t\eta}(z) = I + t\int_{z_0}^{z} \eta + O(t^2)$, Let ρ_t denote $Q_\rho(t\eta)$, for brevity. Discarding second order terms, and computing at $z = z_0$, we find:

$$\rho_t(\gamma) = f_{t\eta}(\gamma z_0)\rho(\gamma) f_{t\eta}(z_0)^{-1} = f_{t\eta}(\gamma z_0)\rho(\gamma) = \rho(\gamma) + t(\int_{z_0}^{\gamma z} \eta)\rho(\gamma) + O(t^2).$$

The derivative of the curve of representations ρ_t is given by $\dot{\rho}_t \rho_t^{-1}$, so the differential at $\phi = 0$ in the η direction is finally given by:

$$[d(Q_\rho)_0(\eta)](\gamma) = \dot{\rho}_t\rho_t^{-1}\big|_{t=0}(\gamma) = \lim_{t\to 0} \frac{Q_\rho(t\eta)(\gamma) - Q_\rho(0)(\gamma)}{t}\rho(\gamma)^{-1} = \int_{z_0}^{\gamma z} \eta$$

Proof. (of theorem 2): Since $\mathbf{S}_n^{st}$ and $\mathcal{M}_n^{st}$ have the same dimension, $n^2(g-1)+1$ (§1), we just need to show that dW_ρ has trivial kernel. Since W is the composition $\mathbf{S}_n^{st} \xrightarrow{i} \mathbf{G}_n^{st} \xrightarrow{V} \mathcal{M}_n^{st}$, the differential is the composition: $H^1(F_g, \mathrm{Ad}_\rho) \xrightarrow{di} H^1(\pi_1, \mathrm{Ad}_\rho) \xrightarrow{dV_\rho} T_{E_\rho}\mathcal{M}_n^{st}$, and so, (using lemma 4):

$$\mathrm{Ker}dW_\rho = H^1(F_g, \mathrm{Ad}_\rho) \cap \mathrm{Ker}dV_\rho = H^1(F_g, \mathrm{Ad}_\rho) \cap \mathrm{Im}(P_{Ad_\rho}).$$

5. Unitary Schottky vector bundles.

In this section we compute, for unitary Schottky bundles, the period map using a generalized version of Riemann's bilinear relations, and prove theorem 3. If σ is a unitary representation of π_1 in the vector space $\mathbb{V}_\sigma$, there is a hermitian inner product $\langle , \rangle$ on $\mathbb{V}_\sigma$, which is invariant under the π_1 action: $\langle \gamma \cdot v_1, \gamma \cdot v_2 \rangle = \langle v_1, v_2 \rangle \quad \forall \gamma \in \pi_1;\ v_1, v_2 \in \mathbb{V}_\sigma$ and so, we define also a hermitian inner product on the space of E_σ- differentials, as follows:

$$(\phi_1, \phi_2) := i \int_D \langle h_1(z), h_2(z) \rangle \; dz \wedge \overline{dz},$$

where $D \in \tilde{X}$ is a fundamental domain for $X = \tilde{X}/\pi_1$, and $\phi_i = h_i(z)dz$ for $z \in \tilde{X}$. (positive definiteness follows from $\frac{i}{2} dz \wedge \overline{dz} = dx \wedge dy$). From the Hermitian pairing $\langle , \rangle$ on $\mathbb{V}_\sigma$, we can also form a pairing in $H^1(\pi_1, \mathbb{V}_\sigma)$, as follows. First allow our cocycles $\Phi \in H^1(\pi_1, \mathbb{V}_\sigma)$ to be defined by linearity on the group ring $\mathbb{Z}[\pi_1]$. Let $R_k = \prod_{j=1}^k a_j b_j a_j^{-1} b_j^{-1}$ $(k = 1, ..., g)$, and put $R = R_g$. Using the notation of the Fox calculus, we set: $\frac{\partial R}{\partial a_k} = R_{k-1} - R_k b_k$, $\quad \frac{\partial R}{\partial b_k} = R_{k-1} a_k - R_k$. There is a natural involution # in $\mathbb{Z}[\pi_1]$ given by $\# : \sum n_j\gamma_j \mapsto \sum n_j\gamma_j^{-1}$, so that, e.g: $\#\frac{\partial R}{\partial a_k} = R_{k-1}^{-1} - b_k^{-1}R_k^{-1}, \quad \#\frac{\partial R}{\partial b_k} = a_k^{-1}R_{k-1}^{-1} - R_k^{-1}$.

In this notation, the fundamental 2-cycle $c \in H_2(\pi_1, \mathbb{Z})$, corresponding to $[X]$, under the isomorphism $H_2(X,\mathbb{Z}) \cong H_2(\pi_1,\mathbb{Z})$, is given by $c := \sum_{k=1}^g [\frac{\partial R}{\partial a_k}|a_k] + [\frac{\partial R}{\partial b_k}|b_k]$, where the bar notation $[a|b]$ $(a, b \in \pi_1)$ denotes the equivalence class of the 2-cycle $(1, a, ab)$ in group homology (see [Br], ch. II). Finally, for $\Phi, \Psi \in Z^1(\pi_1, \mathbb{V}_\sigma)$ define:

$$\Phi \Cup \Psi := \sum_{k=1}^g (-\langle \Phi(\#\tfrac{\partial R}{\partial a_k}), \Psi(a_k)\rangle + \langle \Phi(\#\tfrac{\partial R}{\partial b_k}), \Psi(b_k)\rangle)$$

This pairing is well defined in cohomology (it depends only on the cohomology classes $[\Phi], [\Psi] \in H^1(\pi_1, \mathbb{V}_\sigma)$) and, by an easy calculation, it is the composition of the cup product, followed by contraction in $\mathbb{V}_\sigma$ using $\langle , \rangle$, and by evaluation on the fundamental 2-cycle $c \in H_2(\pi_1, \mathbb{Z})$ (compare [Go], [Gu]):

$$\Cup : H^1(\pi_1, \mathbb{V}_\sigma) \times H^1(\pi_1, \mathbb{V}_\sigma) \overset{\cup}{\to} H^2(\pi_1, \mathbb{V}_\sigma \otimes \mathbb{V}_\sigma) \overset{\langle,\rangle}{\to} H^2(\pi_1, \mathbb{C}) \overset{\cdot c}{\to} \mathbb{C}.$$

Proposition 5. (Bilinear relations for E_σ-differentials).
Let $(\mathbb{V}_\sigma, \langle , \rangle)$ be a unitary representation. Then, for all $\phi, \psi \in H^0(X, E_\sigma \otimes K)$, we have:

$$(\phi, \psi) = i\ \{P_\sigma(\phi) \Cup P_\sigma(\psi)\}.$$

Proof. Let $\Phi(\gamma) = \int_{z_0}^{\gamma z_0} \phi$ be a cocycle representative of $P_\sigma(\phi)$ and similarly for ψ. Let $f(z) := \int_{z_0}^z \phi$, so that $\phi = df$ and $f(\gamma z) = \gamma \cdot f(z) + \Phi(\gamma)$, for all $\gamma \in \pi_1$. By Stokes' theorem, we have:

$$(\phi, \psi) = i \int_{\partial D} \langle f(z), \psi(z)\rangle \overline{dz} = i(\sum_{k=1}^{4g} \int_{\gamma_k} \langle f(z), \psi(z)\rangle \overline{dz}),$$

where the curves γ_k are the $4g$ sides of the boundary of the polygon $D \subset \tilde{X}$, whose vertices can be ordered as $\{z_0, a_1z_0, a_1b_1z_0, R_1a_1b_1a_1^{-1}z_0 = R_1b_1z_0, R_1z_0, ..., R_gz_0 = z_0\}$. Half of the $4g$ sides give (using the notation $f^\gamma = f \circ \gamma$):

$$\int_{R_{k-1}z_0}^{R_{k-1}a_kz_0} \langle f, \psi\rangle \overline{dz} + \int_{R_{k-1}a_kb_kz_0}^{R_kb_kz_0} \langle f, \psi\rangle \overline{dz} =$$
$$= \int_{z_0}^{a_kz_0} \langle f^{R_{k-1}}, \psi^{R_{k-1}}\rangle \overline{R_{k-1}{}'(z)}\overline{dz} - \int_{z_0}^{a_kz_0} \langle f^{R_kb_k}, \psi^{R_kb_k}\rangle\ \overline{(R_kb_k)'(z)}\overline{dz} =$$
$$= \int_{z_0}^{a_kz_0} [\langle R_{k-1}\cdot f + \Phi(R_{k-1}), R_{k-1}\cdot\psi\rangle - \langle R_kb_k \cdot f + \Phi(R_kb_k), R_kb_k\cdot\psi\rangle]\,\overline{dz} =$$
$$= \int_{z_0}^{a_kz_0} [\langle \Phi(R_{k-1}), R_{k-1}\cdot\psi\rangle - \langle \Phi(R_kb_k), R_kb_k\cdot\psi\rangle]\,\overline{dz} =$$
$$= \langle \Phi(R_{k-1}), R_{k-1}\cdot\Psi(a_k)\rangle - \langle \Phi(R_kb_k), R_kb_k\cdot\Psi(a_k)\rangle =$$
$$= -\langle \Phi(R_{k-1}^{-1}), \Psi(a_k)\rangle + \langle \Phi(b_k^{-1}R_k^{-1}), \Psi(a_k)\rangle = -\langle \Phi(\#\frac{\partial R}{\partial a_k}), \Psi(a_k)\rangle.$$

A similar computation for the remaining $2g$ sides gives the desired formula.

We observe that, in the special case $\mathbb{V}_\sigma = \mathbb{C}$ with the trivial action of π_1, and the usual inner product $\langle z_1, z_2 \rangle = z_1 \overline{z_2}$, $z_1, z_2 \in \mathbb{C}$, this proposition reduces to the classical Riemann bilinear relations, because it says that for any $\phi, \psi \in H^0(X, K)$, we have (here all R_k are trivial):

$$\int_X \phi\overline{\psi}\, dz \wedge \overline{dz} = -i(\phi, \psi) = P(\Phi) \Cup P(\Psi) = \sum_{k=1}^{g} (\Phi(a_k)\overline{\Psi(b_k)} - \Phi(b_k)\overline{\Psi(a_k)}).$$

Let us now consider the important case of a representation which is both Schottky and unitary. Then proposition 5 has an immediate consequence, for the period map associated to Ad_ρ, $P_{Ad_\rho} : H^0(X, End E_\rho \otimes K) \to H^1(\pi_1, \mathrm{Ad}_\rho)$.

Proposition 6. *Let $\rho \in Hom(\pi_1, GL(n, \mathbb{C}))$ be a Schottky and unitary representation. If $\phi \in H^0(X, End E_\rho \otimes K)$ is such that $P_{Ad_\rho}(\phi) \in H^1(F_g, Ad_\rho)$, then $\phi = 0$.*

Proof. Since F_g is a free group, $H^2(F_g, M) = 0$, for any F_g-module M. Therefore, if $P_{Ad_\rho}(\phi) \in H^1(F_g, Ad_\rho)$, then $P_{Ad_\rho}(\phi) \Cup P_{Ad_\rho}(\phi) = 0$ and by proposition 5, $(\phi, \phi) = 0$. Therefore, $\phi = 0$.

We can now prove theorem 3:

Proof. (of theorem 3): By Prop. 6 and Thm. 2, dW_ρ is a submersion when $\rho \in \mathbf{S}_n^{st} \cap \mathbf{U}_n^{st}$. $R := \{\rho \in \mathbf{S}_n^{st} : det(dW_\rho) = 0\}$ is a closed analytic subset of $\mathbf{S}_n^{st}$ which is not the whole set. Since W is a holomorphic map between the complex manifolds $\mathbf{S}_n^{st}$ and $\mathcal{M}_n^{st}$, on the complement $\mathbf{S}_n^{st} \setminus R$, W is a local diffeomorphism, and therefore, it is an open map. So, we can take $U = W(\mathbf{S}_n^{st} \setminus R)$.

6. Schottky bundles over an elliptic curve.

Let X be a Riemann surface of genus 1. Atiyah [A] proved the following:

Theorem 5. *(a) For any $n \geq 1$, there is a unique indecomposable vector bundle of rank n and degree 0 over X denoted $\mathbb{F}_n$, such that $dim H^0(X, \mathbb{F}_n) = 1$. Moreover, (for $n > 1$) $\mathbb{F}_n$ is the unique nontrivial extension $0 \to \mathbb{I} \to \mathbb{F}_n \to \mathbb{F}_{n-1} \to 0$. (b) Every indecomposable vector bundle E, of rank n and degree 0 over X, is isomorphic to $\mathbb{F}_n \otimes \det E$.*

Lemma 5. *For every $n \geq 1$, the bundle $\mathbb{F}_n$ is Schottky.*

Proof. Write $X = \mathbb{C}/\langle a, b |\ ab = ba \rangle$, where a, b act by $a \cdot z = z + 1$, $b \cdot z = z + \tau$ and $\mathrm{Im}\tau > 0$. Consider the Schottky representation $\rho_n \in Hom(F_g, GL(n, \mathbb{C}))$ given by assigning to b the matrix $N_{ij} = 1$ if $j = i$ or $j = i + 1$, and $N_{ij} = 0$ otherwise. (i.e, the only nonzero entries of N are ones on the principal diagonal and on the diagonal above it). Clearly $\mathbb{F}_1 = E_{\rho_1} = \mathbb{I}$. Assume, by induction, that $E_{\rho_{n-1}} = \mathbb{F}_{n-1}$. Since E_{ρ_n} is an extension of $E_{\rho_{n-1}} = \mathbb{F}_{n-1}$ by the trivial line bundle: $0 \to \mathbb{I} \to E_{\rho_n} \to \mathbb{F}_{n-1} \to 0$, by Thm. 5(a), we just need to prove that $\dim H^0(X, E_{\rho_n}) = 1$. Any section of E_{ρ_n} over X corresponds to a holomorphic function $s : \mathbb{C} \to \mathbb{C}$ satisfying $s(\gamma z) = \rho_n(\gamma)s(z) \quad \forall\gamma \in \pi_1$. This means that s_n is a constant (being a holomorphic doubly periodic function on $\mathbb{C}$), and that the other components of s have 1 as period, and verify $s_i(z+\tau) = s_i(z) + s_{i+1}(z) \quad \forall i = 1, ..., n-1$. Since an abelian differential with zero "a-periods" (like ds_{n-1}) has to be zero, we get $s_n = 0$, and s_{n-1} constant. Repeating this argument we get $s_i = 0$, for all $i = 2, ..., n$, and s_1 is a constant and so, $\dim H^0(X, E_{\rho_n}) = 1$.

Theorem 6. *Every flat vector bundle over a Riemann surface of genus one is Schottky.*

Proof. By Weil's theorem, we may assume that E is indecomposable of degree 0. Then by Thm. 5(b), $E = \mathbb{F}_n \otimes \det E$. Since $\mathbb{F}_n$ is Schottky and $\det E$ is a line bundle, we conclude that E is also Schottky (see the remarks after definition 1).

Over a Riemann surface of genus one, there are no *stable* vector bundles of degree 0 and rank $n > 1$ (see Tu, [Tu]), so the map V does not make sense in this case, but we can construct an analog, considering the moduli space of *semistable* vector bundles of degree 0, which is isomorphic to the n-th symmetric product of the Jacobian: $\mathcal{M}_n^{ss} = Sym^n(Jac(X))$ ([Tu]). Similarly, we can consider semisimple representations:

$$\mathbf{G}_n^{ss} = \{\rho \in \mathrm{Hom}(\pi_1, GL(n,\mathbb{C})) : \rho(\gamma) \text{ is diagonalizable for all } \gamma \in \pi_1\}/GL(n,\mathbb{C}),$$

Since π_1 is commutative, all matrices $\rho(\gamma)$ can be simultaneously diagonalized, and so, $\mathbf{G}_n^{ss}$ is the n-th symmetric product of one dimensional representations, $\mathbf{G}_n^{ss} = Sym^n(\mathrm{Hom}(\pi_1, \mathbb{C}^*))$. It is then easy to verify that $V_n : \mathbf{G}_n^{ss} \to \mathcal{M}_n^{ss}$ (sending ρ to E_ρ) is the n-th symmetric power of $V_1 : \mathrm{Hom}(\pi_1, \mathbb{C}^*) \to \mathcal{M}_1$ (§A). The space of Schottky representations in $\mathbf{G}_n^{ss}$, will be $\mathbf{S}_n^{ss} = Sym^n(\mathrm{Hom}(F_g, \mathbb{C}^*))$, and we can describe the map $W_1 : \mathbf{S}_n^{ss} \to \mathcal{M}_n^{ss}$ using the lattice $\Lambda := \{2\pi i n\omega\} \subset H^0(X, K)$ (where ω is a normalized differential), as follows:

Proposition 7. *For a Riemann surface of genus 1, the map $W : \mathbf{S}_n^{ss} \to \mathcal{M}_n^{ss}$ is the n-th symmetric power of the Λ-bundle $W_1 : Hom(F_g, \mathbb{C}^*) \to \mathcal{M}_1$, where Λ acts on $Hom(F_g, \mathbb{C}^*)$ as in Prop. 8.*

A. Schottky line bundles.

It is known that every line bundle of degree 0 is Schottky ([K], Ch. VI, §4), so we will comment briefly this case in our setting. The moduli space of degree 0 holomorphic line bundles $\mathcal{M}_1$ is the Jacobian variety of X, $Jac(X)$, which is a group under tensor product. $H^1(X; \mathbb{C}^*) = Hom(\pi_1, \mathbb{C}^*)$ is also an abelian group (under tensor product of representations) isomorphic to $(\mathbb{C}^*)^{2g}$ upon the choice of generators of π_1; and the map $V = V_1 : \mathbf{G}_1 = \mathrm{Hom}(\pi_1, \mathbb{C}^*) \to$

$Jac(X)$ becomes a (non-algebraic) surjective homomorphism (a degree 0 line bundle is flat). The holomorphic map Q_ρ gives now an explicit action of $H^0(X,K)$ on $Hom(\pi_1,\mathbb{C}^*)$:

$$\omega\cdot\rho := Q_\rho(\omega)(\gamma) = e^{\int_\gamma \omega}\rho(\gamma),$$

which represents the Jacobian as $Hom(\pi_1,\mathbb{C}^*)/H^0(X,K)$. Let $\rho_1 \in Hom(\pi_1,\mathbb{C}^*)$ represent a line bundle L; finding a Schottky representation ρ_2 of the same L amounts, by lemma 3 to finding a holomorphic differential ω with $\rho_2(a_j) = e^{\int_\gamma \omega}\rho_1(a_j) = 1$ for all j. Since this equation is always solved with $\omega = -\sum_{j=1}^{g}\log(\rho_1(a_j))\,\omega_j$ (for any choice of branches of log, and where $\omega_1,...,\omega_g$ is a normalized basis of $H^0(X,K)$, i.e, $\int_{a_i}\omega_j = \delta_{ij}$ and $\int_{b_i}\omega_j = \Pi_{ij}$ $(i,j=1,...,g)$; Π_{ij} is the period matrix, symmetric with positive definite imaginary part), we see that *every degree 0 line bundle L, admits a Schottky representation.*

Let Λ denote the lattice $\Lambda := \{2\pi i(n_1\omega_1 + ... + n_g\omega_g) : n_1,...,n_g \in \mathbb{Z}\}$ inside $H^0(X,K)$. Then, *two Schottky representations ρ_1 and ρ_2 produce the same holomorphic line bundle if and only if there exists $\omega \in \Lambda$ such that $Q_{\rho_1}(\omega) = \rho_2$.* To see this, let $E_{\rho_1} = E_{\rho_2}$ and $\omega \in H^0(X,K)$ be the form such that $Q_{\rho_1}(\omega) = \rho_2$; then $e^{\int_{a_j}\omega} = 1$ for all j. Writing $\omega = \sum c_i\omega_i$, this implies $c_j \in 2\pi i\mathbb{Z}$, which means $\omega \in \Lambda$. The converse is immediate, and we conclude that:

Proposition 8. *The map $W = W_1 : \mathbf{S}_1 = Hom(F_g,\mathbb{C}^*) \to \mathcal{M}_1 = Jac(X)$ is a holomorphic principal Λ-bundle, under the action $\omega\cdot\rho := Q_\rho(\omega)$ of Λ on $\mathbf{S}_1$.*

Let $\mathbf{1}$ denote the trivial representation, and $\mathbb{I}$ the trivial line bundle over X. Contrary to the case of stable bundles, Schottky bundles do not determine a unique representation:

Corollary 1. *If E is a Schottky vector bundle, then there are infinitely many non-conjugate Schottky representations that give rise to E.*

Proof. If $E = E_\rho$ for some $\rho \in Hom(F_g, GL(n,\mathbb{C}))$, then $\rho\otimes(\omega\cdot\mathbf{1})$ is a non-conjugate Schottky representation, for all $\omega \in \Lambda\backslash\{0\}$. Moreover $E_{\rho\otimes(\omega\cdot\mathbf{1})} = E_\rho \otimes E_{\omega\cdot\mathbf{1}} = E_\rho\otimes\mathbb{I} = E$, by Prop. 8.

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References

[A] Atiyah, M., *Vector bundles over an elliptic curve*, Proc. London Math. Society **VII** (1957), 414-52

[AS] Ahlfors, L. and Sario, L.,*Riemann Surfaces*, Princeton University Press, 1960.

[B] Beauville, A., *Vector bundles over curves and generalized theta functions: recent results and open problems*, Preprint (alg-geom 9404001).

[Bs1] Bers, L., *Automorphic forms for Schottky groups*, Advances in Math. **16**, 332-361 (1975)

[Bs2] Bers, L., *Uniformization on bounded domains*, Comm. Pure and Applied Math. **14** (1961)

[Br] Brown, K., Cohomology of Groups, GTM 87, Springer-Verlag, 1982.

[C] Chuckrow, V., *On Schottky groups with applications to Kleinian groups*, Annals of Math. **88** (1968), 47-61.

[Go] Goldman, W., *The symplectic nature of fundamental groups of surfaces*, Advances in Math. **54**, 200-225 (1984).

[Gu] Gunning, R., *Lectures on vector bundles over Riemann surfaces*, Princeton Academic Press, 1967.

[Gu1] Gunning, R., *Special coordinate coverings of Riemann surfaces*, Math. Annalen 170, 67-86 (1967).

[Gu2] Gunning, R., *Analytic structures on the space of flat vector bundles over a compact Riemann surfaces*, LNM 185, 47-62. Springer-Verlag (1971).

[K] Kra, I., *Automorphic forms and Kleinian groups.*, Reading, Mass., W. A. Benjamin, 1972.

[Ms] Maskit, B., *A characterization of Schottky groups*, Journal d'Analyse Math. **19** (1967), 227-230.

[Mu] Mumford, D., *Projective invariants of projective structures*, Proceedings of the Int. Math. Congress, Stockholm, 526-530 (1962).

[NS1] Narasimhan, M. and Seshadri, C., *Holomorphic vector bundles on a compact Riemann surface*, Math. Annalen **155** (1964) 69-80.

[NS2] Narasimhan, M. and Seshadri, C., *Stable and unitary vector bundles on a compact Riemann surface*, Annals of Mathematics **82** (1965) 540-67.

[Ta] Takhtajan, L., *Uniformization, local index theorem and geometry of the moduli spaces of Riemann surfaces and vector bundles*, Proc. Symp. Pure Math. 49 (1989), part 1.

[Tu] Tu, L., *Semistable bundles over an elliptic curve*, Adv. in Math. 98 (1993), 1-26.

[Ty] Tyurin, A., *Quantization and theta functions*, Jussieu Math. Preprint.

[W] Weil, A., *Généralisation des fonctions abéliennes*, J. Math. Pures et Appliqués 17, 47 (1938).

[ZT1] Zograf, P. and Takhtajan, L., *On uniformization of Riemann surfaces and the Weil-Peterson metric on Teichmüller and Schottky spaces*, Math. Sbornik **132** (1987), 303-320.

[ZT2] Zograf, P. and Takhtajan, L., *On the geometry of moduli spaces of vector bundles over a Riemann surface*, Math. USSR, Izvestiya, vol **35** (1990) No. 1.