Viscosity solutions and integrability of Hamiltonian systems

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Hamiltonian:

$$H_\epsilon(p, x) = H_0(p) + \epsilon H_1(p, x),$$

with

- H_0, H_1 smooth,
- $H_0(p)$ strictly convex
- $H_1(p, x)$ bounded with bounded derivatives, and $\mathbb{Z}^n$ periodic in x
- ϵ small parameter.

Dynamics:

$$\dot{p} = D_x H(p, x) \qquad \dot{x} = -D_p H(p, x)$$

Hamilton - Jacobi theory:

- Simplify the dynamics through a change of coordinates:

$$(x, p) \to (X, P)$$

such that

$$\dot{P} = 0 \qquad \dot{X} = -D_P \overline{H}(P).$$

Construction of the change of coordinates:

- Solve the Hamilton-Jacobi equation:

$$H(P + D_x u, x) = \overline{H}(P).$$

- Define, implicitly:

$$p = P + D_x u(x, P) \qquad X = x + D_P u(x, P).$$

Drawbacks: It may not be possible (non-smooth solutions, lack of invertibility, etc)

But: for $\epsilon = 0$ the equation is very simple: $H_0(P + D_x u) = \overline{H}(P)$ has the solution $u = 0$.

And: there are always viscosity solutions.

Plan: Use viscosity solutions to understand when systems close to integrable:

- can be studied by perturbation theory

  OR

- Hamilton-Jacobi integrability fails.

Mather sets

Theorem 1 *Let u be a viscosity solution of*

$$H(P + D_x u, x) = \overline{H}(P).$$

- *For each P there exists a set, the Mather set, invariant under the Hamiltonian dynamics which is contained in the graph*

$$(x, P + D_x u);$$

- *Let $x(t)$ be a trajectory of the Hamiltonian dynamics with initial conditions on this graph. Then*

$$\lim_{t \to \infty} \frac{x(t)}{t} = -D_P \overline{H}(P).$$

I

Stability under small perturbations

Goal:

Given P_0, understand the dependence on ϵ of the viscosity solutions and Mather sets.

Plan:

- Construct approximate solution $\tilde{u}$, $\tilde{H}$
- Choose P_ϵ to keep track of the Mather set by keeping the rotation vector $D_P\overline{H}(P)$ constant.
- Prove estimates for $\tilde{H}$
- Estimate $|\tilde{u} - u|$
- Estimate $|D_x u - D_x \tilde{u}|$

Approximate solution:

$$\begin{aligned}\tilde{u}_N^\epsilon =& \epsilon v_1(x,P_0) + \epsilon(P-P_0)D_P v_1(x,P_0) + \\ &+ \epsilon^2 v_2(x,P_0) + \frac{1}{2}\epsilon(P-P_0)^2 D^2_{PP} v_1(x,P_0) + \\ &+ \epsilon^2(P-P_0)D_P v_2(x,P_0) + \\ &+ \epsilon^3 v_3(x,P_0) + \cdots ,\end{aligned}$$

put in the equation, match powers of ϵ and P and get the expansion.

Each of the terms will satisfy a transport equation:

$$D_pH_0(P_0)D_xw = \lambda + f(\cdots).$$

for some suitable λ, f.

For instance:

$$D_pH_0(P_0)D_xv_1 = \overline{H}_1(P_0) - H_1(P_0, x),$$

$$D_pH_0(P_0)D_x(D_Pv_1) = D_P\overline{H}_1(P_0)$$

$$- D^2_{pp}H_0(P_0)D_xv_1 - D_pH_1(P_0, x),$$

and

$$D_pH_0(P_0)D_xv_2 = \overline{H}_2(P_0)$$

$$- \frac{1}{2}D^2_{pp}H_0(P_0)D_xv_1D_xv$$

$$- D_pH_1(P_0, x)D_xv_1.$$

In general, the equation

$$D_P H_0(P_0) D_x w = \lambda + f(x)$$

will have a periodic solution provided:

- $\lambda = \fint f dx$

- $D_P H_0(P_0)$ satisfies a Diophantine condition:

$$|D_P H_0(P_0) \cdot k| \geq \frac{C}{|k|^s},$$

  for all $k \in \mathbb{Z}^n$, $k \neq 0$.

Choose P_ϵ fixing the rotation vector:

$$D_P H_0(P_0) = D_P \tilde{H}_\epsilon^N(P_\epsilon),$$

that is,

$$\begin{aligned} D_P H_0(P_0) = & D_P \overline{H}_0(P_0) + \epsilon D_P \overline{H}_1(P_0) \\ & + (P_\epsilon - P_0) D^2_{PP} \overline{H}_0(P_0) + \cdots , \end{aligned}$$

Since $H_0 = \overline{H}_0$ is strictly convex

$$\det D^2_{pp} \overline{H}_0 \neq 0$$

the implicit function theorem yields:

$$P_\epsilon = P_0 + \epsilon P_1 + \cdots$$

with

$$P_1 = -\left[D^2_{PP} \overline{H}_0(P_0) \right]^{-1} D_P \overline{H}_1(P_0).$$

Estimates for $\overline{H}$

Proposition 1 (Contreras, et al.)

$$\overline{H}(P) = \inf_{\phi} \sup_{x} H(P + D_x\phi, x),$$

in which the infimum is taken over C^1 periodic functions ϕ.

Then one can prove:

Corollary 1 *We have:*

$$\overline{H}_\epsilon = \tilde{H}_\epsilon^N + O(\epsilon^N).$$

Uniform estimates

Theorem 2 *Suppose*

- $D_P\overline{H}_0(P_0)$ *is Diophantine;*

- *for every* N*, there exists* $P_\epsilon = P_0 + O(\epsilon)$ *such that*

$$D_P\tilde{H}_\epsilon^N(P_\epsilon) = D_P H_0(P_0),$$

Let u^ϵ *be a solution of*

$$H_\epsilon(P_\epsilon + D_x u^\epsilon, x) = \overline{H}_\epsilon(P_\epsilon),$$

and $\tilde{u}_N^\epsilon$ *be the corresponding approximate solution*

Then for every M *there exists* $N(M)$ *such that*

$$\sup_x |u^\epsilon - \tilde{u}_N^\epsilon| = O(\epsilon^M).$$

Proof [sketch]

- Construct an approximate solution $\tilde{u}^{\epsilon}_{N}$ for a sufficiently large but finite N.

- Show that for long (but finite) times there are uniform estimates along trajectories.

- Use Lipschitz estimates for u^{ϵ} and $\tilde{u}^{\epsilon}_{N}$ and ergodization time estimates to extend uniform estimates for all times.

Estimates for the derivative:

Theorem 3 *Let $M > 0$. Suppose*

- $\omega_0 = D_P H_0(P_0)$ *is Diophantine*

- *there exists an approximate solution* $\tilde{u}^\epsilon_N$ *for* N *sufficiently large*

Then

$$\operatorname*{esssup}_x |D_x u^\epsilon - D_x \tilde{u}^\epsilon_N| \leq C\epsilon^{M/2}.$$

Proof [sketch]

- Let G, the **good set**, be defined through:

$$|D_x u^\epsilon - D_x \tilde{u}_N^\epsilon| \le C\epsilon^{M/2},$$

  for some fixed constant C, let B, the **bad set** be its complement.

- Prove an estimate like:

$$\int_0^1 \frac{\gamma}{2} |D_x u^\epsilon(\mathbf{x}(t)) - D_x \tilde{u}_N^\epsilon(\mathbf{x}(t))|^2 dt \le$$
$$\le C\epsilon^N + 2 \sup_x |u^\epsilon - \tilde{u}_N^\epsilon|,$$

- This implies that any point in the **bad set** is mapped into a **not so bad set** in finite time.

- Show that even in the **bad set**

$$|D_x u^\epsilon - D_x \tilde{u}_N^\epsilon| \le C\epsilon^{M/2},$$

  for a slightly larger constant.

II

Small perturbations

Joint work with Enrico Valdinocci.

Without Diophantine hypothesis on the rotation vector, we do not expect Hamilton-Jacobi integrability.

To simplify, we consider the Hamiltonian

$$H(p,x) = \frac{|p|^2}{2} + \epsilon V(x).$$

H is α-**smoothly integrable** at the rotation number Ω if:

(A) There is a X-periodic smooth functions:

$$f(X,\epsilon) = f(X,\epsilon; V,\Omega),$$

and

$$g(X,\epsilon) = P + D_x u(X + f(X,\epsilon),\epsilon),$$

such that

$$(x_0 + \Omega t + f(x_0 + \Omega t,\epsilon), g(x_0 + \Omega t,\epsilon))$$

is a solution to the Hamiltonian dynamics and $\{x + f(x,\epsilon)\,,\ x \in \mathbb{T}^N\} = \mathbb{T}^N$.

(B) f satisfies the bound:

$$|D_X f(X,\epsilon)|, |D^2_{X^2} f(X,\epsilon)| \leq C\epsilon^{\alpha}\,,$$

with $\alpha \in \mathbb{R}$ and $C > 0$, which may depend on Ω and V.

Main result

If the rotation number Ω is exponentially close to a resonance, that is, there exists $T_n \to \infty$ and

$$\omega_n \in \frac{1}{T_n}\mathbb{Z}^N$$

such that

$$|\Omega - \omega_n| \le e^{-bT_n},$$

then there are **analytic** potentials V for which smooth integrability fails.

Related results: J. Mather, U. Bessi.

Key ideas

- Observe that smooth integrability implies that the trajectories $(\Omega t + f(\cdots), g(\cdots))$ are minimizers.

- Build periodic orbits with rotation number ω_n and show that they have different action.

- Estimate the difference between their action and minimizing orbits at rotation number Ω.

- Build a potential V for which this implies a contradiction.

Estimates

- Difference in action between periodic orbits with rotation vector ω_n is estimated by Melnikov gap:

$$\epsilon \int [V(x_0 + \omega_n t) - V(x_1 + \omega_n t)]\, dt$$

- Difference in action between ω_n orbits and Ω_n orbits:

$$|\omega_n - \Omega_n| T_n$$

- Choose

$$V(x) = \sum_{n=1}^{\infty} e^{-cT_n} \cos(2\pi k_n \cdot x),$$

with $k_n \cdot \omega_n = 0$ and extract a contradiction.

III

Large perturbations

Joint work with Adam Oberman.

Problem

Given $H(p,x)$ and some point (p,x) decide whether there is P and u solving

$$H(P + D_x u, x) = \overline{H}(P)$$

such that $(p,x) = (P + D_x u, x)$.

Goal

Get a criteria that is based on **computable** quantities such as integrals on **finite** intervals of solutions of the Hamiltonian dynamics.

Heuristics

If $(x,p)=(x,P+D_xu)$ then

$$\frac{x(T)}{T}\to Q, \qquad \frac{1}{T}\int_0^T L(x,\dot{x})dt\to \overline{L}(Q),$$

as $T\to\infty$.

If (x,p) is in the non-integrable region we expect

$$\liminf_{T\to\infty}\frac{1}{T}\int_0^T L(x,\dot{x})dt > \overline{L}(Q)+\epsilon$$

for some $\epsilon>0$.

Since $\overline{L}(Q)-PQ-\overline{H}(P)=0$,

$$\int_0^T L(x,\dot{x})+P\dot{x}+\overline{H}(P)ds > T\epsilon\to\infty,$$

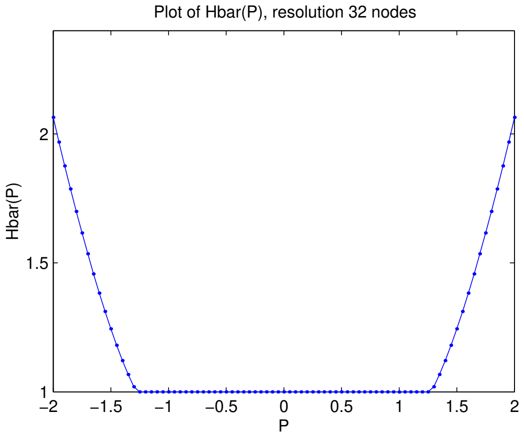
as $T\to\infty$.

Numerical computation of $\overline{H}$:

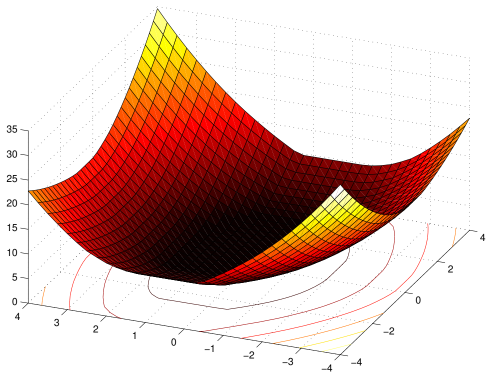
Discretize the variational formula

$$\overline{H}(P) = \inf_{\phi} \sup_{x} H(P + D_x\phi, x).$$

Results



$\bar{H}(P)$ for the one-dimensional pendulum.



$\bar{H}(P)$ for the double pendulum

Cartoon problem

The Hamiltonian for a pendulum is

$$H(p,x) = \frac{p^2}{2} - \cos 2\pi x.$$

and the Lagrangian is

$$L(x,v) = \frac{v^2}{2} + \cos 2\pi x.$$

- The point $(x,p) = (0,0)$ is a fixed point for the dynamics.

- One can show that $\overline{H}(P) \geq 1$.

The action S is

$$S \geq \int_0^T 2dt = 2T,$$

which is unbounded.