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Stability of Rotating Liquid Films

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Abstract

In this paper we consider a modified Young-Laplace-Gauss variational formulation that incorporates a van der Waals potential of adhesion and gives equilibrium configurations of a liquid in contact with a solid and a gas. This model is used to compute the shape of the liquid/gas interface in a coaxial cylindrical container, a configuration like that of the main liquid helium tank in Gravity Probe B and STEP (Satellite Test of Equivalence Principle) space experiments.

We derive the Euler-Lagrange equations and study their rotationally symmetric solutions. Then we compute the second variation and analyze the stability of several configurations. In particular, we prove the existence of a Plateau-Rayleigh instability and curvature-driven instability for very small volume configurations.

We compare our results with previous physical experiments, and we conclude that certain observed instabilities cannot be attributed to the above mechanisms.

1. Introduction

This study of the stability of rotating liquid films is motivated by the STEP (Satellite Test of Equivalence Principle) and Gravity Probe-B Relativity Mission Studies space experiments, whose objectives are to test certain predictions of general relativity, quantitative space curvature, and frame dragging. In these experiments, a cryogenic cooling device, for which a coaxial cylindrical configuration has been considered, is needed to keep the temperature constant. A central rod, which contains the experimental apparatus, is coaxial with a surrounding annular container partly filled with liquid helium [1]. One means for maintaining the center of mass of the helium on the axis of symmetry is to keep

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the configuration rotating at a uniform speed. A cross section of a rotationally symmetric configuration is shown in figure 1. In this situation, a thin film of liquid helium covers the rod and a toroidal bubble of helium gas forms. However this configuration is not stable at slow rotation speed. Therefore one must determine a critical rotation speed ω_c large enough to keep the gas bubble in a stable axisymmetric configuration about the central rod [2], [3]. Another requirement is to find whether this axisymmetric configuration remains stable when gravity is added.

In a one-g physical experiment for exploring the behavior of liquid in a rotating coaxial container, instabilities of the liquid film that coated the central rod were observed [4]. However, in this experiment the liquid was silicon oil rather than liquid helium and some of the container parameters were very different from the ones for the space experiments. For subsequent one-g experiments, described in [5] and [3], no instabilities are mentioned; however the central rod diameter was larger and gravity was directed axially rather than radially as in the earlier experiment, and the liquid was water instead of silicon oil, for which no film formed on the central rod. In [5] and [3] computational results obtained with the Surface Evolver software using a classical Young-Laplace-Gauss (Y-L-G) formulation were compared with the physical experiments; there was a good match between numerical and experimental data, for the materials used. The Surface Evolver is a software program for the study of surfaces subjected to surface tension and other energies, and subject to various constraints. A surface is implemented as a simplicial complex, that is, a union of triangles. The Surface Evolver evolves the surface toward minimal energy by a gradient-descent method [6]. This program was used to compute the tridimensional configurations in figures 2, 5 and 10.

To model the system in which a liquid film coats the solid surface we use a modified Y-L-G variational formulation [7], [8], which incorporates a van der Waals potential of adhesion to give equilibrium configurations of a liquid in contact with a solid and a gas. By considering rotationally symmetric solutions of the Euler-Lagrange equations, we study several solutions, including infinite volume configurations (film-like interface and semi-infinite bubble) and finite volume configurations (one bubble topologically equivalent to a torus).

In [2] the stability of symmetric configurations was studied numerically for the classical case, for which a film-like interface between the central rod and the gas is not present. For the physical experiment in [4], in which film instabilities were observed, it was suggested in [8],[4] that the instabilities could be caused by a Plateau-Rayleigh like phenomenon. To investigate this we study the second variation of the energy in section 5. Based on [8] we present a proof of the existence of Plateau-Rayleigh instability for sufficiently thick film-like solutions. Then we show that for the film thicknesses that arise for our configuration, no instability of this kind is possible. However another kind of instability was found, namely curvature driven instability for very small volume configurations. Our results show that for a fixed rotation speed, infinite volume film-like solutions are stable and for small volume a toroidal configuration is always unstable. This gives theoretical support to the numerical results by Brakke [2] that the critical rotation speed ω_c depends on the volume V of the helium gas. The numerical data (without van der Waals potential) fits the law

$$\omega_c V(\omega_c)^{0.72} = 7.9; \tag{1}$$

however no theoretical explanation for this property has been given. The addition of van der Waals potential does not seem to influence much the value of the critical rotation speed ω_c. For instance, if $V = 50$ and the adhesion coefficient (see later) is $\alpha = 10^{-3}$ then the configuration is stable for $\omega > 0.5$ and unstable for $\omega < 0.5$, whereas the critical value predicted by (1) is $\omega_c = 0.47$. One can define through equation (1) a critical volume V_c for each rotation speed ω. For volumes $V > V_c$ the configuration is stable and for volumes $V < V_c$ the configuration is unstable. The results in section 5 show rigorously that this holds for $V = +\infty$ (film-like solutions) and $V \to 0$.

At the end of the paper we discuss briefly the time-dependent case with gravity.

2. Young-Laplace-Gauss Formulation

Consider a volume of fluid in contact with a gas and a solid. The classical Y-L-G formulation gives an equilibrium configuration as a stationary value of the energy functional

$$E = \sigma|S| - \sigma\beta|S^*| + \int_X \mathcal{U},$$

subject to a volume constraint. We denote by V the volume of the helium gas. In the case (considered briefly in the end of the paper) in which $\mathcal{U}$ depends on time, we assume the "adiabatic" hypothesis, i.e., the configuration is given by the minimum of energy for each time. Here σ is the interfacial liquid-gas surface tension, S is the liquid-gas free surface, S^* is the wetted area of the solid, $\int_X \mathcal{U}$ is the potential energy and $\beta = \cos\gamma$ is the relative adhesion coefficient with γ the contact angle. The integral is taken over the liquid volume X.

If we assume that the configuration is rotating rigidly at uniform speed ω about the axis of the rod, the potential $\mathcal{U}$ is the rotation potential $\mathcal{K}$

$$\mathcal{K} = -\frac{\rho\omega^2 r^2}{2},$$

where ρ is the difference between the density of the liquid and the density of the gas.

The modified Y-L-G formulation postulates that one can replace the term $-\sigma\beta|\mathcal{S}^*|$ corresponding to the contact energy of the wetted surface, on the rod, with a van der Waals potential

$$\mathcal{W} = -\frac{\alpha}{(r - r_0)^3},$$

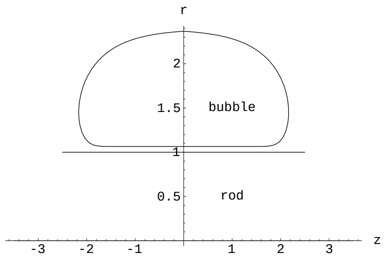

Figure 1: Symmetric configuration with $V = 50$, $\omega = 1$, $\alpha = 10^{-3}$

where $\alpha > 0$ is the adhesion coefficient [8] and r_0 is the radius of the inner cylinder. We assume that the effect of the van der Waals potential arising from parts of the container, other than the central rod, is negligible.

Since our reference system moves with the fluid, any steady transverse gravity components will be time-dependent. Suppose z is the direction of the axis of rotation and x and y are transverse directions, g_z, g_x and g_y the corresponding gravity components, and $g = \sqrt{g_x^2 + g_y^2 + g_z^2}$ the total gravity acceleration. The component g_z of the gravity does not depend on time. Without loss of generality we may assume that the components of gravity along the transverse directions are

$$g_x = g_t \cos \omega t \qquad g_y = g_t \sin \omega t,$$

where $g_t = \sqrt{g_x^2 + g_y^2}$ is the transverse gravity acceleration.

The gravitational potential is

$$\mathcal{G} = \mathcal{G}_t + \mathcal{G}_z = \rho(g_x x + g_y y + g_z z) = \rho(g_t r \cos(\theta - \omega t) + g_z z).$$

$\mathcal{G}_t$ and $\mathcal{G}_z$ are, respectively, the contribuitions to the gravitational potential corresponding to the transverse and z directions.

In the general case the potential energy will be the sum of the three terms

$$\mathcal{U} = \mathcal{K} + \mathcal{W} + \mathcal{G},$$

integrated over the volume X. The equilibrium shapes such as that in figure 1 and 2 represent the balance of these competing influences. We will not consider the case in which the adhesion coefficient α is negative. In this situation the liquid would be non-wetting and there would be no film covering the central rod. Since the van der Waals potential is short-range it would have little influence on the configuration shape or stability. The case $\alpha = 0$ without a film covering the central rod is discussed in [2]. The rotation tends to drive the liquid to the outer wall (figure 3), the van der Waals term pulls it to the central rod (figure 4) and the surface tension decreases the surface area of the bubble. By the scaling properties (see below) increasing surface tension corresponds to decreasing both ω and α.

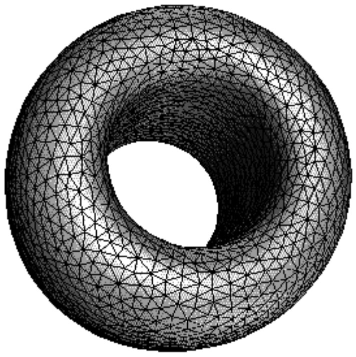

Figure 2: Symmetric configuration solution with $V = 50$, $\omega = 1$, $\alpha = 10^{-3}$, computed with the Surface Evolver.

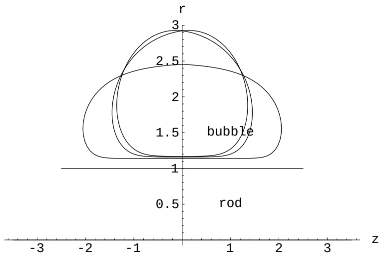

Figure 3: Three different configurations with $V = 50$, $\omega = 0.25, 0.5, 1,$and $\alpha = 10^{-2}$. Increasing values of ω drive the bubble closer to the rod.

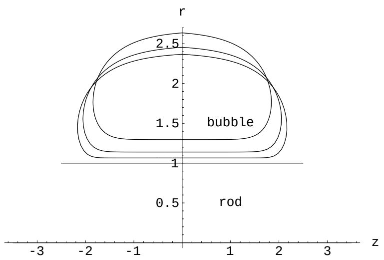

Figure 4: Three different configurations with $V = 50$, $\omega = 1$, and $\alpha = 10^{-3}, 10^{-2}, 10^{-1}$. Increasing values of α drive the bubble away from the rod.

By using the divergence theorem, the energy can be obtained in terms of surface integrals instead of volume integrals. Indeed, since $\mathcal{K}+\mathcal{W}+\mathcal{G}_t(t)$ does not depend on the axial coordinate z, we have

$$\int_X \mathcal{K}+\mathcal{W}+\mathcal{G}_t(t) = \int_{\partial X} \left(\mathcal{K}+\mathcal{W}+\mathcal{G}_t(t)\right) z\vec{e}_z \cdot \vec{n},$$

where $\vec{n}$ is the exterior unit normal to ∂X and $\vec{e}_z$ is a unit vector with axial direction. The energy corresponding to $\mathcal{G}_z$ is

$$\int_X \mathcal{G}_z = \int_{\partial X} \mathcal{G}_z \frac{z}{2}\vec{e}_z \cdot \vec{n}.$$

These formulas will be useful in the following section and in placing the problem in a form suitable for the Surface Evolver program. In this paper most of the analysis will be done with zero gravity. Whenever the effects of gravity are studied we will assume $g_z = 0$ since the existence of a component of the gravity along the z direction would simply cause a drift of the gas towards one end of the container.

We can rescale the problem to a non-dimensional form and take $\rho = r_0 = \sigma = 1$; therefore the only relevant parameters will be V, ω, α and the Bond number B (non-dimensional form of the gravity).

The new variables relate to the originals by the relations:

$$\hat{V} = V/r_0^3 \qquad \hat{\omega} = \omega\sqrt{\frac{\rho r_0^3}{\sigma}} \qquad \hat{\alpha} = \frac{\alpha}{\sigma r_0^2} \qquad B = g\frac{\rho r_0^2}{\sigma}.$$

From now on we will omit the hats for notational simplicity.

In Brakke's paper [2], which studies this problem numerically with the Y-L-G formulation and with gravity (and van der Waals energy absent) the fluid is liquid helium ($\sigma = 0.34\ dyne/cm$, $\rho = 0.15\ g/cm^3$), the radius of the cylinder $r_0 = 13.5\ cm$, and the volume of the bubble is $124000\ cm^3$. The typical rotation speeds are $\omega < 0.6\ rpm$. In the rescaled units we have $V = 50$ and typical angular velocities will be less than 2.

3. Rotationally Symmetric Configurations

We now consider a closed annular container (length l, inner radius $r_0 = 1$, outer boundary of radius R), rotating at a uniform speed ω and partially filled with a liquid rotating rigidly with the container. Assume further that $B = 0$. In general, one or more gas bubbles may appear, but we limit consideration to the case in which there is only a single bubble of gas. Moreover, since we are interested in having a configuration for which the center of mass is on the axis of symmetry, we assume the bubble is topologically equivalent to a torus. At low spinning speed, numerical computations have shown that such a configuration is unstable and that the bubble will evolve to a configuration on one side of the

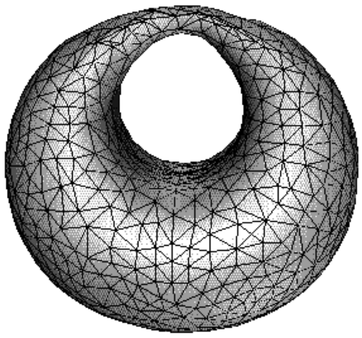

Figure 5: Unstable configuration, $V = 50$, $\omega = 0$, $\alpha = 10^{-3}$, computed with the Surface Evolver.

rod, topologically equivalent to a sphere, which has lower energy (see figure 5). However at higher spinning speed the torus configuration has lower energy than the sphere configuration [2],[3].

By adding a constant to the energy we do not change the equilibria nor their stability. If we choose this constant to be

$$-\int_{X\cup Y} \mathcal{U},$$

where X is the volume of liquid and Y is the volume of gas, we get

$$E = |S| - \int_Y \mathcal{U}.$$

Therefore, by reversing the sign of $\mathcal{U}$, we can perform all integrations over the volume of gas instead of the liquid. From now on we will use this property to compute the energy.

Suppose the configuration is rotationally symmetric and can be parameterized by

$$(x, y, z) = (r(\varphi)\cos\theta, r(\varphi)\sin\theta, z(\varphi)),$$

where $\theta \in [0, 2\pi]$ and $\varphi \in J$, for some (fixed) interval J. ϕ could be thought as a polar angle but we will choose J so that it will be the arclength along a meridian. In the absence of gravity, the energy and volume functionals are

$$E = 2\pi \int_J \mathcal{E} d\varphi \qquad V = 2\pi \int_J \mathcal{V} d\varphi,$$

where $\mathcal{E} = \left(\frac{\omega^2 r^3 z}{2} + \frac{\alpha r z}{(r-1)^3}\right) r' + r\sqrt{r'^2 + z'^2}$ and $\mathcal{V} = zrr'$ ($'$ denotes the derivative with respect to φ) are, respectively, the potential and "volume density" in cylindrical coordinates.

4. Euler-Lagrange Equations

The critical points of the energy functional E with a volume constraint V satisfy the Euler-Lagrange equations for the functional $E + \lambda V$, where λ is a Lagrange multiplier [11]. Since the energy increases with the volume we have $\lambda \leq 0$.

Define $L = \mathcal{E} + \lambda \mathcal{V}$. By performing a change of variables we may assume that the stationary solutions are parameterized by arc length ($r'^2 + z'^2 = 1$). The corresponding Euler-Lagrange equations are $\frac{d}{d\varphi}\frac{\partial L}{\partial z'} - \frac{\partial L}{\partial z} = 0$ and $\frac{d}{d\varphi}\frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = 0$. Note that there are two unknowns, r and z, but three equations, the two Euler-Lagrange equations and the constraint; however these equations are not independent and we can use any two of them to derive the state equations.

The first of the Euler-Lagrange equations can be written as

$$\frac{d}{d\varphi}(rz') = \frac{d}{d\varphi}\left(\frac{\lambda\, r^2}{2} + \frac{\omega^2\, r^4}{8} - \frac{\alpha}{2\,(r-1)^2} - \frac{\alpha}{r-1}\right).$$

This equation shows that

$$I = rz' - \left(\frac{\lambda\, r^2}{2} + \frac{\omega^2\, r^4}{8} - \frac{\alpha}{2\,(r-1)^2} - \frac{\alpha}{r-1}\right)$$

is a conserved quantity. Therefore we can write

$$z' = \frac{h(r;I)}{r}, \tag{2}$$

with

$$h(r;I) = I + \left(\frac{\lambda\, r^2}{2} + \frac{\omega^2\, r^4}{8} - \frac{\alpha}{2\,(r-1)^2} - \frac{\alpha}{r-1}\right).$$

The equation $\frac{d}{d\varphi}\frac{\partial L}{\partial r'} - \frac{\partial L}{\partial r} = 0$ can be written as

$$r'' = g(r;I), \tag{3}$$

where

$$g(r;I) = h(r;I)\left(-\frac{\omega^2 r}{2} - \frac{\lambda}{r} - \frac{\alpha}{r(r-1)^3}\right) + \frac{h(r;I)^2}{r^3}.$$

Now we have to analyze the initial conditions. We may assume $z(0) = 0$, by translation invariance, and, without loss of generality, we may take $r'(0) = 0$. Then $z'(0) = \pm 1$ and its sign is chosen so that the volume is positive.

For each value of λ and $r(0) = r_0$ define $I^{\pm} = I^{\pm}(r_0)$ by

$$I^{\pm}(r_0) = \pm r_0 - \left(\frac{\lambda\, {r_0}^2}{2} + \frac{\omega^2\, {r_0}^4}{8} - \frac{\alpha}{2\,(r_0-1)^2} - \frac{\alpha}{r_0-1}\right),$$

where the $\pm$ sign is the same as the sign of $z'(0)$. The idea is that the constant

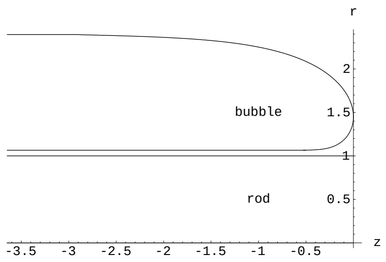

Figure 6: Semi-infinite volume. $\omega = 1$, $\alpha = 10^{-3}$, $\lambda = \lambda_c \simeq -3.279$.

of motion I is defined by both the initial radius r_0 and the sign of $z'(0)$. Suppose $r^{\pm}$ is a solution of the equation

$$g(r; I^{\pm}(r)) = 0,$$

i.e.,

$$\frac{-\omega^2 r^3}{2} \pm 1 - \lambda r - \frac{r\alpha}{(r-1)^3} = 0. \tag{4}$$

The initial condition $r(0) = r^{\pm}$ corresponds to a constant-thickness film-like solution with $r \equiv r^{\pm}$. If $z'(0) = 1$ and α is sufficiently small, we have the asymptotic expansion

$$r^+ = 1 + \left(\frac{\alpha}{-\frac{\omega^2}{2} + 1 - \lambda}\right)^{1/3} + O(\alpha^{2/3}), \tag{5}$$

provided $\lambda < 1 - \frac{\omega^2}{2}$. This is the situation in which a thin film of fluid covers the rod. The solution $r = r^-$ corresponds to the case in which the rod is not covered by liquid but by gas, and there is a cylindrical gas-liquid interface film of radius $r = r^-$.

Define λ_c by the equation $I^+(r^+) = I^-(r^-)$. In this case the two equilibrium points r^+ and r^- may be connected by an heteroclinic trajectory. This corresponds to a semi-infinite volume, figure 6. In this case, the initial condition is $z'(-\infty) = 0$ instead of $z'(0) = 0$.

To compute solutions corresponding to finite volume configurations we use the following algorithm. First we choose some $\lambda < \lambda_c$. Then for fixed $r(0)$ (close to r^+ since we expect a film-like region to develop) we solve equation (2) and (3) with initial conditions $r(0)$, $z(0) = 0$, $r'(0) = 0$ and $z'(0) = 1$. Then we integrate until $\varphi = \varphi_1$ so that $r'(\varphi)$ first becomes negative. Continue to solve, and stop in the first iteration corresponding to $\varphi_2 > \varphi_1$ such that $r'(\varphi_2) > 0$. This will yield a value $z(\varphi_2; r(0))$, which, in general, is not zero, and we have to solve numerically the equation $z(\varphi_2; r(0)) = 0$ to obtain an appropriate value of $r(0)$. To do this we use the bisection method. Thus, for each λ we get a solution with some volume $V(\lambda)$. Then we solve for λ as a function of V. In figure 1 we

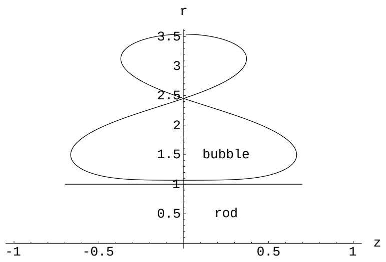

Figure 7: Self-intersecting solution with $\omega = 1$, $\alpha = 10^{-3}$, $\lambda = -3$.

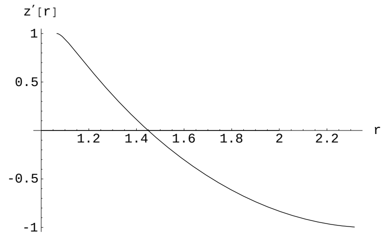

Figure 8: $z'(r)$ for $\lambda = -3.29 < \lambda_c$, $\omega = 1$, $\alpha = 10^{-3}$

can see one of these solutions and in figure 2 the corresponding tridimensional configuration computed by the Surface Evolver. In order to make the film visible in the pictures we took $\alpha = 10^{-3}$, which is much larger than physical values which can be as small as 10^{-15}.

If $\lambda \to \lambda_c$ the corresponding volumes V_λ increase to infinity. These configurations converge to the one connecting the two equilibrium points $r = r^+$ and $r = r^-$. Thus, for large volumes, these configurations will be almost cylindrical both in the part close to the rod (inner film) and the outer surface ("outer film").

Not all solutions of the Euler-Lagrange equations are physically meaningful; for instance, they may have self-intersections (see figure 7). This property can be easily explained. Indeed, we know that $z'^2 + r'^2 = 1$, or equivalently,

$$\frac{h(r;I)^2}{r^2} + r'^2 = 1.$$

The periodic solutions of the Euler-Lagrange equations have r between two values $r_1 \leq r \leq r_2$, where $\frac{h(r_i,I)^2}{r^2} = 1$, since $r' = 0$ at maxima or minima. In our range of parameters, for $\lambda < \lambda_c$ and $r_1 < r < r_2$, the graph of $z' = \frac{h(r;I)}{r}$ is decreasing taking values from -1 to 1 (see figure 8). However, for $\lambda > \lambda_c$ this function is not increasing (see figure 9); instead it decreases up to a certain value $z' > -1$ and then increases again until $z' = 1$. This causes the solution to self intersect.

Now we consider the problem of very small volumes $V = O(\epsilon^2)$ for $\epsilon > 0$

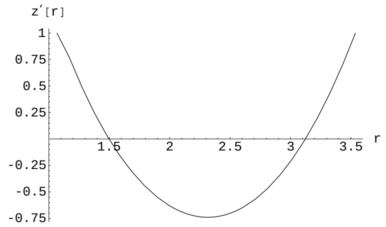

Figure 9: $z'(r)$ for $\lambda = -3 > \lambda_c$, $\omega = 1$, $\alpha = 10^{-3}$

sufficiently small. Assume that we can write

$$r = r_c + \epsilon r_1\left(\frac{\varphi}{\epsilon}\right) \qquad z = \epsilon z_1\left(\frac{\varphi}{\epsilon}\right),$$

for some suitable r_c. A full justification of these expansions requires bifurcation theory analysis; we will not discuss the details here but give only the main ideas.

We can take $r_1(0) = 0$ and $z_1'(0) = 0$. Heuristic considerations lead us to define r_c to be the solution of the equation

$$\mathcal{U}'(r_c) = -\frac{1}{\epsilon r_c},$$

where $\mathcal{U}(r) = \frac{\omega^2 r^2}{2} + \frac{\alpha}{(r-1)^3}$ is the potential and $\lambda = -\frac{\omega^2 r_c^2}{2} - \frac{\alpha}{(r_c-1)^3} - \frac{1}{\epsilon}$. Note that r_c has an asymptotic expansion $r_c = 1 + (3\alpha\epsilon)^{1/4} + O(\epsilon^{1/2})$ and, if r_1 is uniformly bounded as $\epsilon \to 0$ then $r > 1$ for small ϵ. Moreover we can expand the equilibrium equations in Taylor series about $r = r_c$. Note that

$$I = -\left(\frac{\lambda r_c^2}{2} + \frac{\omega^2 r_c^4}{8} - \frac{\alpha}{2(r_c-1)^2} - \frac{\alpha}{r_c - 1}\right),$$

and so

$$\frac{h(r; I)}{r} = -r_1 + O(\epsilon^{7/4} r_1^3).$$

Thus

$$z_1' = -r_1 + O(\epsilon^{7/4}), \qquad r_1'' = -r_1 + O(\epsilon^{3/4}).$$

Therefore a solution of the Euler-Lagrange equations will have asymptotic expansion

$$r = r_c + \epsilon \sin\frac{\varphi}{\epsilon}, \qquad z = \epsilon \cos\frac{\varphi}{\epsilon},$$

up to higher order terms, with $\varphi \in [0, 2\pi\epsilon]$.

In the general case, an equilibrium configuration may not be symmetric. If we assume that the surface can be parameterized by

$$(x, y, z) = (r(\varphi,\theta)\cos\theta, r(\varphi,\theta)\sin\theta, z(\varphi,\theta)),$$

again with $\theta \in [0, 2\pi]$ and $\varphi \in I$, for some interval $I = [0, M]$ with $M > 0$, the Lagrangian is

$$L = \left(\frac{\omega^2 r^3 z}{2} + \frac{\alpha r z}{(r-1)^3}\right) r' + \sqrt{r^2 r'^2 + r^2 z'^2 + (z_\theta r' - z' r_\theta)^2} + \lambda z r r',$$

denoting by r_θ and z_θ the derivatives with respect to θ. The corresponding Euler-Lagrange equations are

$$\frac{d}{d\varphi}\frac{\partial L}{\partial r'} + \frac{d}{d\theta}\frac{\partial L}{\partial r_\theta} = \frac{\partial L}{\partial r}, \qquad \frac{d}{d\varphi}\frac{\partial L}{\partial z'} + \frac{d}{d\theta}\frac{\partial L}{\partial z_\theta} = \frac{\partial L}{\partial z}.$$

Since there is no explicit dependence on θ, any solution of the rotation invariant problem will satisfy these Euler-Lagrange equations.

5. Stability Analysis

In this section we discuss the stability of critical points of the energy by analyzing the second variation. The new results in this section are the proof that a Plateau-Rayleigh type instability (i.e. instability of the film covering the rod) does not develop, at least for small α, and the existence of asymmetric instabilities driven by large curvature that can appear for a sufficiently small volume in a toroidal configuration. Thus infinite-volume configurations corresponding to constant-thickness film-like solutions are stable, whereas for a fixed rotation speed, sufficiently small volume configurations are unstable. This is supports the numerical results by Brakke on the existence of a critical volume for each rotation speed ω.

The second variation of the energy can be written in the form

$$\delta^2(E + \lambda V)[\phi] = \int (T\phi)\phi,$$

in which T, the second variation operator, is a linear differential operator acting on the variation ϕ along the normal component. If T has a negative eigenvalue that satisfies the volume constraint $\delta V[\phi] = 0$, the equilibrium is unstable. Otherwise if all eigenvalues (that satisfy the volume constraint) are positive the equilibrium is stable. In our case the infinitesimal volume constraint condition is $\int r\phi = 0$.

A general form for $T\phi$ was calculated in [9], and is

$$T\phi = -\Delta\phi - (k_1^2 + k_2^2)\phi + \nabla\mathcal{U} \cdot \vec{n}\phi,$$

where Δ is the intrinsic Laplacian [10] on the free surface $\mathcal{S}$, k_1 and k_2 are the principal curvatures of the surface and $\vec{n}$ is the outward normal.

The case of a configuration with two parallel plates at a distance l, and between

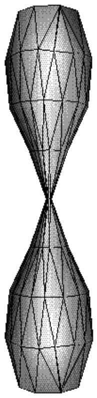

Figure 10: Plateau-Rayleigh instability for $\frac{l}{r} = 4 > \pi$, computed with the Surface Evolver.

them a cylindrical film of outer radius $r = r^+ > 1$ covering a rod of radius $r_0 = 1$, corresponds to

$$T\phi = -\phi'' - \frac{1}{r^2}\phi_{\theta\theta} - \left(\omega^2 r - \frac{3\alpha}{(r-1)^4} + \frac{1}{r^2}\right)\phi.$$

This case was analyzed in [8]. To recover these results we assume periodic boundary conditions in θ and fixed boundary when $\varphi = 0, l$.

Then the eigenvalues are

$$\mu_{k,m} = -\omega^2 r + \frac{3\alpha}{(r-1)^4} + \frac{m^2\pi^2}{l^2} + \frac{k^2-1}{r^2},$$

where k is any integer; m is any positive integer if $k \neq 0$, m is any positive integer greater than 1 if $k = 0$, and $r = r^+$. If we assume that $\mathcal{U} = 0$, i.e., $\alpha = \omega = 0$ and we do not fix the radius of the film at both ends, then for $l > \pi r$ the configuration is unstable. This is the classical Plateau-Rayleigh instability, and if we apply a small perturbation to the film, the configuration evolves to a shape like the one in figure 10. If, in addition to the volume constraint, the radius of the film at both ends is kept fixed, the Plateau-Rayleigh instability appears for $l > 2\pi r$ instead of for $l > \pi r$. We now assume this additional constraint, as done in [8].

In the presence of rotation and van der Waals potential energy the stability criteria are more complex. We now quote some results from [8] that follow from the analysis of the sign of the eigenvalues. If $\mathcal{U} \neq 0$ then there is a value r_1 given by

$$\frac{\partial \mathcal{U}}{\partial r}(r_1) = \frac{1}{r_1^2} \tag{6}$$

$(r_1 > 1)$, such that if the radius r satisfy $1 \leq r \leq r_1$ then the film is stable. An axisymmetric instability sets up whenever $r > r_1$ and l satisfies

$$l \geq l_{0,2} \equiv \frac{2\pi}{\sqrt{\frac{1}{r^2} - \frac{\partial \mathcal{U}}{\partial r}}} = \frac{2\,\pi}{\sqrt{\frac{1}{r^2} - \frac{3\,\alpha}{(r-1)^4} + r\,\omega^2}}.$$

This instability is axially symmetric. In the presence of rotation there is also a non-axisymmetric instability. There is a value $r_2 > r_1$ such that if $r > r_2$ and

$$l \geq l_{1,1} \equiv \frac{\pi}{\sqrt{-\frac{\partial \mathcal{U}}{\partial r}}} = \frac{\pi}{\sqrt{\frac{-3\,\alpha}{(r-1)^4} + r\,\omega^2}}$$

then the configuration is unstable.

In the case of rotationally symmetric configurations with finite volume, like the ones considered in the previous section, a film-like region is formed. If the thickness of the bubble-rod interface film were sufficiently large, it would be reasonable to suppose that a Plateau-Rayleigh type instability would develop. However, in our numerical experiments with Surface Evolver, we never observed this type of instability. This phenomenon can be explained easily. From the previous results we know that a Plateau-Rayleigh type instability can exist only if the radius r of the film is larger than some critical radius r_1 defined by equation (6). This equation yields the asymptotic formula

$$r_1 = 1 + \left(\frac{3\alpha}{1+\omega^2}\right)^{1/4} + O(\alpha^{1/2}),$$

as $\alpha \to 0$. The film thickness $\Delta \equiv r^+ - 1$ is, by expression (5), of the order $O(\alpha^{1/3})$. Since $\alpha^{1/4} >> \alpha^{1/3}$ for small values of α, we do not expect $r > r_1$ for small values of α and correspondingly no Plateau-Rayleigh type instability could be expected to develop.

Now suppose the rod is not covered by liquid but by gas, and there is a cylindrical gas-liquid interface film of radius $r = r^-$. This models the "outer film" described in the previous section. The second variation operator is given by

$$T\phi = -\phi'' - \frac{1}{r^2}\phi_{\theta\theta} - \left(-\omega^2 r + \frac{3\alpha}{(r-1)^4} + \frac{1}{r^2}\right)\phi.$$

The eigenvalues of T, satisfying the same boundary condition and the volume constraint are

$$\mu_{k,m} = \omega^2 r - \frac{3\alpha}{(r-1)^4} + \frac{m^2\pi^2}{l^2} + \frac{k^2-1}{r^2},$$

where k is any non-negative integer, for $k = 0$, m is any positive integer greater than 1, for $k > 0$, m is any positive integer, and $r = r^-$.

If $\omega^2 r$ were sufficiently small there could be a symmetric instability. For small α and ω we have

$$r^- = \left(\frac{2}{\omega}\right)^{2/3} + O(\alpha + \omega^{-1/3}).$$

Thus

$$\omega^2 r - \frac{3\alpha}{(r-1)^4} - \frac{1}{r^2} = 2^{2/3}\omega^{4/3} - \left(\frac{\omega}{2}\right)^{4/3} + O(\alpha + \omega^{5/3}) > 0$$

and so the outer film is always stable, for sufficiently small rotation speed ω and very large volumes.

If the term corresponding to the gradient of $\mathcal{U}$ is small and $k_1^2 + k_2^2$ is sufficiently large, the operator T may have negative eigenvalues and so the configuration can be unstable. This situation happens in the case of very small volumes. As the volume $V \to 0$, $k_1^2 + k_2^2 \to \infty$ since the computations in the previous section imply

$$k_1^2 + k_2^2 = \frac{1}{\epsilon^2} + O(1),$$

when $V = 2\pi^2\epsilon^2$. The intrinsic Laplacian for a torus parameterized by $r = r_c + \epsilon \sin\varphi$ and $z = \epsilon \cos\varphi$ with $\varphi \in [0, 2\pi]$ (we reparameterized φ) is

$$\Delta\phi = \frac{\frac{\partial}{\partial\varphi}\left[\frac{r_c + \epsilon\sin\varphi}{\epsilon}\frac{\partial\phi}{\partial\varphi}\right]}{\epsilon(r_c + \epsilon\sin\varphi)} + \frac{\frac{\partial^2\phi}{\partial\theta^2}}{(r_c + \epsilon\sin\varphi)^2} = \frac{1}{\epsilon^2}\frac{\partial^2\phi}{\partial\varphi^2} + O(\frac{1}{\epsilon}).$$

Finally note that on this torus $|\nabla\mathcal{U}| = O(\frac{1}{\epsilon})$. Thus

$$T\phi = -\frac{1}{\epsilon^2}\frac{\partial^2\phi}{\partial\varphi^2} - \frac{\phi}{\epsilon^2} + O(\frac{1}{\epsilon}).$$

If we take $\phi = \cos\theta$ (note that this function satisfies the infinitesimal volume constraint) we have

$$\int \phi T\phi = -\frac{1}{\epsilon^2} + O(\frac{1}{\epsilon})$$

which implies that, for sufficiently small ϵ, T has negative eigenvalues and so the configuration is unstable.

6. Conclusions

The present results suggest that for each rotation speed ω there exists a critical volume $V_c(\omega)$, decreasing with ω, such that for $V > V_c(\omega)$ there exists a symmetric configuration that is a local minimizer and therefore stable. If $V < V_c$ then an asymmetric instability will develop. This was confirmed by numerical computations using the Surface Evolver.

With a (transverse) gravity field we can write the energy of a configuration as

$$E = E_0 + BG(t),$$

where E_0 is the part of the energy in the absence of gravity and $BG(t)$ is energy corresponding to the transverse gravity field (B is the Bond number). Suppose Ω_0 is a stationary configuration in the absence of gravity. Under what conditions can we guarantee the existence of a configuration $\Omega_B(t)$, close to Ω_0 such that $\Omega_B(t)$ is a critical point of E for each time t? More precisely, the problem consists in finding a function $\phi(t, B)$, satisfying the volume constraint $\delta V[\phi] = 0$ such that for any ϕ satisfying the same condition $\delta V[\psi] = 0$, we have $\delta E|_{\Omega_0 + B\phi}[\psi] = 0$. Consider the expression for the energy

$$E|_{\Omega_0 + B\phi + \epsilon\psi} = E_0|_{\Omega_0} + \delta E_0|_{\Omega_0}[B\phi + \epsilon\psi] + \delta^2 E_0|_{\Omega_0}[B\phi + \epsilon\psi, B\phi + \epsilon\psi] +$$

$$+BG|_{\Omega_0} + B\delta G|_{\Omega_0}[B\phi + \epsilon\psi] + O(B^2 + \epsilon^2).$$

If we differentiate in ϵ and set $\epsilon = 0$, we get

$$B\left(2\delta^2 E_0|_{\Omega_0}[\phi, \psi] + \delta G|_{\Omega_0}[\psi]\right) + O(B^2)$$

If $\delta^2 E_0|_{\Omega_0}$ is positive definite, we can apply the Lax-Milgram theorem [11] to find ϕ. Thus the stability of the configuration without gravity implies existence of a stable configuration with sufficiently small gravity.

A related experiment was carried out by McCuan et al. (see [4]). For this case $r_0 = 0.1\ cm$ and $V = 10\ cm^3$. Thus $\hat{V} = 10^4$, if $\omega < 1000\ rpm$ we have $\hat{\omega} < 0.35$ and for $g = 9.8\ m/s^2$ we will have $B = 4.2$. Since B is not small, the adiabatic hypothesis is no longer valid. Experimentally several instabilities were observed. It would be important to carry out experiments in low gravity to confirm our predictions and verify if other possible sources of instability are negligible. Such sources might include convection, viscosity effects, and surface tension variations due to temperature gradients.

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