INSTITUTO SUPERIOR TÉCNICO
Mestrado em Engenharia Física Tecnológica
Ano Lectivo: 2010/2011 Semestre: 1º

MATEMÁTICA COMPUTACIONAL

Resolução do Exame e Testes de 21 de Janeiro de 2011

[1]
(a)[10]

$$x_0 = 0.01234, \qquad u_0 = \sin x_0 = x_0 - \frac{x_0^3}{6} + \ldots$$

$$\tilde{u}_0 = \mathrm{fl}_c(0.01234 - 0.3132 \times 10^{-6} + \ldots) = 0.01233$$

$$z_0 = f(x_0) = x_0 - \sin x_0, \qquad \tilde{z}_0 = 0.1000 \times 10^{-4}$$

$$\delta_{\tilde{z}_0} = 1 - \frac{\tilde{z}_0}{z_0} = -30.93$$

(b)[15]

$$u = \sin x, \qquad z = f(x) = x - u$$

$$\delta_{\tilde{u}}^L = \frac{x \cos x}{\sin x}\,\delta_{\tilde{x}} + \delta_1$$

$$\begin{aligned}
\delta_{\tilde{z}}^L &= \frac{x}{z}\,\delta_{\tilde{x}} - \frac{u}{z}\,\delta_{\tilde{u}}^L + \delta_2 \\
&= \frac{x}{z}\,\delta_{\tilde{x}} - \frac{\sin x}{z}\left(\frac{x\cos x}{\sin x}\,\delta_{\tilde{x}} + \delta_1\right) + \delta_2 \\
&= \frac{x}{f(x)}\,(1 - \cos x)\,\delta_{\tilde{x}} - \frac{\sin x}{f(x)}\,\delta_1 + \delta_2 \\
&=: p_f(x)\,\delta_{\tilde{x}} + q_1(x)\,\delta_1 + \delta_2
\end{aligned}$$

O problema é estável para qualquer $x \in \mathbb{R}$ pois, com $p_f(0) = \lim_{x\to 0} p_f(x) = 3$, $p_f(x) \in [0, 3]$ em todo o domínio de f.

O algoritmo para o cálculo de $f(x)$ é numericamente instável para $x \approx 0$ pois $\lim_{x\to 0} |q_1(x)| = \infty$.

[2]

(a)[10]

$$f : \mathbb{R}^+ \to \mathbb{R}, \qquad f(x) = 2\ln x + \frac{1}{x} + 1 - x$$

$$f'(x) = \frac{2}{x} - \frac{1}{x^2} - 1 = -\frac{1}{x^2}(x-1)^2$$

$$f''(x) = -\frac{2}{x^2} + \frac{2}{x^3} = \frac{2}{x^3}(1-x)$$

$$\lim_{x\to 0} f(x) = +\infty, \qquad \lim_{x\to\infty} f(x) = -\infty, \qquad f \quad \text{é decrescente em} \quad \mathbb{R}^+$$

$\Rightarrow$ f tem um único zero em $\mathbb{R}^+$

(b)[20] $I = [3.9, 4.1]$

$$f(x) = 0 \quad \Leftrightarrow \quad x = g(x), \quad \forall x > 0$$

$$g(x) = x + f(x) = 2\ln x + \frac{1}{x} + 1$$

$$g'(x) = \frac{2}{x} - \frac{1}{x^2} = \frac{2}{x^2}\left(x - \frac{1}{2}\right), \qquad g'(x) > 0, \ \forall x \in I$$

$$g''(x) = -\frac{2}{x^2} + \frac{2}{x^3} = \frac{2}{x^3}(1-x), \qquad g''(x) < 0, \quad \forall x \in I$$

Condições suficientes de convergência do método do ponto fixo com função iteradora g para z, $\forall x_0 \in I$:

(i) $g \in C^1(I)$

(ii) $\max_{x\in I} |g'(x)| = g'(3.9) = 0.447074 < 1$,

pois g' é positiva e decrescente em I.

(iii) $g(I) \subset I$,

pois $g(3.9) = 3.97836 \in I$, $\quad g(4.1) = 4.06588 \in I$,

e g é crescente em I.

Método do ponto fixo:

$$x_m = g(x_{m-1}), \quad m \in \mathbb{N}_1, \qquad x_0 \in I$$

$$|z - x_m| \le \frac{L}{1-L}|x_m - x_{m-1}| =: B_m, \qquad m \ge 1$$

$$L = \max_{x\in I} |g'(x)| = 0.447074$$

m	x_m	B_m
0	4.0	
1	4.02259	0.183×10^{-1}
2	4.03245	0.797×10^{-2}

$z = 4.03245 + \Delta, \qquad |\Delta| < B_2$

[3]

(a)[15]

O método de Jacobi convergirá para a solução do sistema $Ax = b$ se e só se o raio espectral da matriz iteradora do método for inferior à unidade, $r_\sigma(C_J) < 1$.

$$A = D + L + U, \qquad M_J = D = 2I, \qquad N_J = L + U = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$C_J = -M_J^{-1} N_J = -\frac{1}{2} N$$

$$\det(C_J - \lambda I) = \begin{vmatrix} -\lambda & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\lambda & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\lambda \end{vmatrix} = -\lambda \left(\lambda^2 - \frac{1}{2} \right)$$

$$\sigma(C_J) = \left\{ 0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\}$$

$$r_\sigma(C_J) = \frac{1}{\sqrt{2}} < 1$$

(b)[20]

Método de Jacobi:

$$x^{(m+1)} = D^{-1} \left(b - (L + U) x^{(m)} \right), \quad m \geq 0$$

Estimativa de erro do método de Jacobi:

$$\|z - x^{(m)}\|_2 \leq \frac{c^m}{1 - c} \|x^{(1)} - x^{(0)}\|_2 < \varepsilon, \qquad c = \|C_J\|_2$$

$$m > \frac{\log \dfrac{(1 - c)\varepsilon}{\|x^{(1)} - x^{(0)}\|_2}}{\log c}$$

$$c = \|C_J\|_2 = \sqrt{r_\sigma(C_J^T C_J)} = \sqrt{r_\sigma(C_J^2)} = r_\sigma(C_J) = \frac{1}{\sqrt{2}}$$

$$x^{(1)} = \frac{1}{2}\left(\begin{bmatrix}4\\5\\4\end{bmatrix} - \begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \frac{3}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$x^{(1)} - x^{(0)} = \frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}, \qquad \|x^{(1)} - x^{(0)}\|_2 = \frac{\sqrt{3}}{2}$$

$\varepsilon = 10^{-6}: \quad m > 42.9912$

$m = 43$

[4][10]

Norma matricial associada à norma da soma em $\mathbb{R}^n$:

$$\|A\|_1 = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|_1}{\|x\|_1}$$

Seja $x \in \mathbb{R}^n \setminus \{0\}$. Então:

$$\|Ax\|_1 = \sum_{i=1}^n |(Ax)_i| = \sum_{i=1}^n \left|\sum_{j=1}^n a_{ij}x_j\right| \leq \sum_{j=1}^n |x_j| \sum_{i=1}^n |a_{ij}|$$

Seja c tal que $c = \max_{1\leq j\leq n} \sum_{i=1}^n |a_{ij}|$

Então: $\|Ax\|_1 \leq c\|x\|_1, \quad \forall x \in \mathbb{R}^n \setminus \{0\}$

Seja k tal que $c = \sum_{i=1}^n |a_{ik}|$.

Seja $y = e_k$, o vector com componentes $y_j = \delta_{jk} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$.

$$\|Ay\|_1 = \sum_{i=1}^n |(Ay)_i| = \sum_{i=1}^n \left|\sum_{j=1}^n a_{ij}\delta_{jk}\right| = \sum_{i=1}^n |a_{ik}| = c = c\|y\|_1$$

Conclui-se pois que $\|A\|_1 = c$.

[5][20]

Método da Newton generalizado:

$$\begin{cases} x^{(m+1)} = x^{(m)} + \Delta x^{(m)} \\ J_f\left(x^{(m)}\right)\Delta x^{(m)} = -f\left(x^{(m)}\right), \quad m \geq 0 \end{cases}$$

$$J_f(x) = \begin{bmatrix} -6 & 2x_2 & 2x_3 \\ 1 & -5 & 3x_3^2 \\ 2x_1 & -3 & 6 \end{bmatrix}$$

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 & 0 & 0 \\ 1 & -5 & 0 \\ 0 & -3 & 6 \end{bmatrix} \Delta x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Leftrightarrow \quad \Delta x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \end{bmatrix}$$

$$x^{(1)} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \end{bmatrix}$$

$$\begin{bmatrix} -6 & 0 & \frac{1}{3} \\ 1 & -5 & \frac{1}{12} \\ 0 & -3 & 6 \end{bmatrix} \Delta x^{(1)} = -\begin{bmatrix} \frac{1}{36} \\ \frac{1}{216} \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \Delta \tilde{x}^{(1)} = \begin{bmatrix} \frac{359}{76680} \\ \frac{2}{1065} \\ \frac{1}{1065} \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} \frac{359}{76680} \\ \frac{2}{1065} \\ \frac{119}{710} \end{bmatrix} = \begin{bmatrix} 0.00468179 \\ 0.00187793 \\ 0.167606 \end{bmatrix}$$

[6]
(a)[20]

Fórmula de Newton às diferenças divididas:

i	x_i	$f[x_i]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	-1.0	-1.0			
			1.9093		
1	0.0	0.9093		-1.28765	
			-0.6660		0.7865
2	1.0	0.2433		1.07185	
			1.4777		
3	2.0	1.721			

$$p_3(x) = f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$+f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2)$$

$$p_3(x) = -1.0 + 1.9093(x+1.0) - 1.28765(x+1.0)x + 0.7865(x+1.0)x(x-1.0)$$

$$p_3(x) = 0.9093 - 0.16485x - 1.28765x^2 + 0.7865x^3$$

- Alternativa. Fórmula interpoladora de Lagrange:

$$p_3(x) = \sum_{j=0}^{3} f(x_j) l_j(x), \qquad l_j(x) = \prod_{i=0, i\neq j}^{3} \frac{x - x_i}{x_j - x_i}$$

$$p_3(x) = -l_0(x) + 0.9093\, l_1(x) + 0.2433\, l_2(x) + 1.721\, l_3(x)$$

$$l_0(x) = \frac{(x-0.0)(x-1.0)(x-2.0)}{(-1.0-0.0)(-1.0-1.0)(-1.0-2.0)} = -\frac{1}{6}\, x(x-1.0)(x-2.0)$$

$$l_1(x) = \frac{(x+1.0)(x-1.0)(x-2.0)}{(0.0+1.0)(0.0-1.0)(0.0-2.0)} = \frac{1}{2}\,(x+1.0)(x-1.0)(x-2.0)$$

$$l_2(x) = \frac{(x+1.0)(x-0.0)(x-2.0)}{(1.0+1.0)(1.0-0.0)(1.0-2.0)} = -\frac{1}{2}\,(x+1.0)x(x-2.0)$$

$$l_3(x) = \frac{(x+1.0)(x-0.0)(x-1.0)}{(2.0+1.0)(2.0-0.0)(2.0-1.0)} = \frac{1}{6}\,(x+1.0)x(x-1.0)$$

$$p_3(x) = 0.9093 - 0.16485x - 1.28765x^2 + 0.7865x^3 \quad \bullet$$

Erro de interpolação:

$$e_3(x) = f(x) - p_3(x) = \frac{f^{(4)}(\xi)}{4!}\, W_4(x), \qquad x \in [-1.0, 2.0]$$

$$W_4(x) = (x+1)x(x-1)(x-2), \qquad \xi \in [-1; 2; x] \subset [-1, 2]$$

$$|f(x) - p_3(x)| \leq \frac{1}{24} \max_{x\in[-1,2]} |f^{(4)}(x)| \max_{x\in[-1,2]} |W_4(x)|, \quad \forall x \in [-1, 2]$$

$$f(x) = x + \sin(2x+2), \qquad f^{(4)}(x) = 16\sin(2x+2)$$

$$\max_{x\in[-1,2]} |f^{(4)}(x)| = 16$$

$$W_4(x) = x^4 - 2x^3 - x^2 + 2x$$

$$W_4'(x) = 4x^3 - 6x^2 - 2x + 2 = 4(x-z_1)(x-z_2)(x-z_3)$$

$$z_1 = \frac{1-\sqrt{5}}{2}, \qquad z_2 = \frac{1}{2}, \qquad z_3 = \frac{1+\sqrt{5}}{2}$$

$$\max_{x\in[-1,2]} |W_4(x)| = \max\{|W_4(z_1)|, |W_4(z_2)|, |W_4(z_3)|\} = \max\left\{1, \frac{9}{16}, 1\right\} = 1$$

$$|f(x) - p_3(x)| \leq \frac{16}{24} = \frac{2}{3}, \quad \forall x \in [-1, 2]$$

$$\max_{x\in[-1,2]} |f(x) - p_3(x)| \leq \frac{2}{3}$$

(b)[15]

Erro de interpolação:

$$e_3(x) = f(x) - q_3(x) = \frac{f^{(4)}(\xi)}{4!}\, \tilde{W}_4(x), \qquad x \in [-1.0, 2.0]$$

$$\tilde{W}_4(x) = \prod_{i=0}^{3}(x - x_i), \qquad \xi \in [-1; 2; x] \subset [-1, 2]$$

$$\max_{x\in[-1,2]} |f(x) - q_3(x)| \leq \frac{2}{3} \max_{x\in[-1,2]} |\tilde{W}_4(x)|$$

Sabe-se que a quantidade

$$\max_{t\in[-1,1]} \left|\prod_{i=0}^{3}(t - t_i)\right|$$

toma o menor valor possível (1/8) quando os nós t_0, t_1, t_2, t_3 são os zeros do polinómio de Chebyshev de grau 4, T_4. Usando a definição destes polinómios obtemos:

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1, \qquad T_3(x) = x(4x^2 - 3)$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

Zeros de T_4: $\quad -t_0 = t_3 = \dfrac{\sqrt{2+\sqrt{2}}}{2}, \qquad -t_1 = t_2 = \dfrac{\sqrt{2-\sqrt{2}}}{2}$

A quantidade $\max_{x\in[-1,2]} |\tilde{W}_4(x)|$ tomará então o menor valor possível (1/8) quando os pontos x_0, x_1, x_2, x_3 forem dados por

$$x_i = X(t_i), \quad i = 0, 1, 2, 3$$

$$X : [-1, 1] \to [-1, 2], \qquad X(t) = -1 + \frac{3}{2}\,(t + 1)$$

Isto é:

$$x_i = -1 + \frac{3}{2}\,(t_i + 1), \quad i = 0, 1, 2, 3$$

Resulta então:

$$\max_{x\in[-1,2]} |f(x) - q_3(x)| \leq \frac{1}{12}$$

[7][20]

Fórmula dos trapézios composta ($f \in C([a,b])$):

$$I(f) = \int_a^b f(x)\,dx \quad \approx \quad I_1^{(M)}(f) = \frac{h_M}{2}\left[f(x_0) + f(x_M) + 2\sum_{j=1}^{M-1} f(x_j)\right]$$

$$h_M = \frac{b-a}{M}, \qquad x_j = a + jh_M, \quad j = 0, 1, \ldots, M$$

$$\begin{cases} a = 0, \quad b = 2, \qquad M = 4, \qquad h_M = \dfrac{1}{2} \\ x_j = 0.5j, \quad j = 0, 1, 2, 3, 4 \end{cases}$$

$$I_1^{(4)}(f) = \frac{1}{4}\left[f(0.0) + f(2.0) + 2\Big(f(0.5) + f(1.0) + f(1.5)\Big)\right] = 4.19385$$

$$\Big(\; I(f) = 4.23653\Big)$$

Erro da fórmula dos trapézios composta:

$$E_1^{(M)}(f) = -\frac{b-a}{12}\,h_M^2\,f''(\xi), \qquad \xi \in]a,b[$$

$$\left|E_1^{(4)}(f)\right| \leq \frac{1}{24}\max_{x\in[0,2]} |f''(x)|$$

$$f(x) = e^{\sin x}, \qquad f'(x) = f(x)\cos x, \qquad f''(x) = f(x)(\cos^2 x - \sin x)$$

$$f'''(x) = f(x)\cos x(\cos^2 x - 3\sin x - 1) = -\frac{1}{2}f(x)\sin(2x)(3 + \sin x)$$

$$f'''(x) = 0 \quad \Leftrightarrow \quad x = 0 \quad \vee \quad x = \frac{\pi}{2}, \qquad x \in [0,2]$$

$$\max_{x\in[0,2]} |f''(x)| = \max\left\{|f''(0)|, \left|f''\left(\frac{\pi}{2}\right)\right|, |f''(2)|\right\} = \max\{1, e, 1.82747\} = e$$

$$\left|E_1^{(4)}(f)\right| \leq \frac{e}{24}$$

[8]

(a)[15] Método de Runge-Kutta clássico de 2ª ordem:

$$y_1 = y_0 + \frac{h}{4}\left[f(x_0, y_0) + 3f\left(x_0 + \frac{2h}{3}, y_0 + \frac{2h}{3}\,f(x_0, y_0)\right)\right]$$

$$f(x,y) = x(1-y^2) - y, \qquad x_0 = 1, \quad y_0 = 2$$

$$y_1 = 2 + \frac{h}{4}\left[f(1,2) + 3f\left(1+\frac{2h}{3}, 2+\frac{2h}{3}f(1,2)\right)\right]$$

$$f(1,2) = -5$$

$$f\left(1+\frac{2h}{3}, 2-\frac{10h}{3}\right) = -5 + \frac{44h}{3} - \frac{20h^2}{9} - \frac{200h^3}{27}$$

$$y_1 = 2 - 5h + 11h^2 - \frac{5h^3}{3} - \frac{50h^4}{9}$$

(b)[15]

$$W'(x) = F(x, W(x))$$

$$W(x) = \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}, \qquad F(x,W(x)) = \begin{bmatrix} z(x) \\ g(x,y(x),y'(x)) \end{bmatrix}$$

$$g(x,y,z) = x(1-y^2)z - y$$

Método de Taylor de ordem 2 (passo h):

$$W_{n+1} = W_n + hF(x_n,W_n) + \frac{h^2}{2}(d_F F)(x_n,W_n), \qquad n \geq 0$$

$$\begin{aligned}(D_F F)(x,W) &= \left(\frac{\partial}{\partial x} + F(x,W)\cdot\nabla_W\right)F(x,W) \\ &= \left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + g(x,y,z)\frac{\partial}{\partial z}\right)\begin{bmatrix} z \\ g(x,y,z)\end{bmatrix} \\ &= \begin{bmatrix} g(x,y,z) \\ g_x(x,y,z) + zg_y(x,y,z) + g(x,y,z)g_z(x,y,z)\end{bmatrix} \\ &= \begin{bmatrix} g(x,y,z) \\ s(x,y,z)\end{bmatrix}\end{aligned}$$

$$s(x,y,z) = (1-y^2)z + z(-1-2xyz) + g(x,y,z)x(1-y^2)$$

$$\begin{bmatrix} y_{n+1} \\ z_{n+1}\end{bmatrix} = \begin{bmatrix} y_n \\ z_n\end{bmatrix} + h\begin{bmatrix} z_n \\ g(x_n,y_n,z_n)\end{bmatrix} + \frac{h^2}{2}\begin{bmatrix} g(x_n,y_n,z_n) \\ s(x_n,y_n,z_n)\end{bmatrix}$$

$$y_1 = y_0 + hz_0 + \frac{h^2}{2}g(x_0,y_0,z_0) = 2 + 2h - 4h^2$$

$$z_1 = z_0 + hg(x_0,y_0,z_0) + \frac{h^2}{2}s(x_0,y_0,z_0) = 2 - 8h$$