INSTITUTO SUPERIOR TÉCNICO
Mestrado em Engenharia Física Tecnológica
Ano Lectivo: 2010/2011 Semestre: 1º

MATEMÁTICA COMPUTACIONAL

Resolução do Teste de 30 de Outubro de 2010

[1][40]

$$z_1 = x + 1, \qquad z_2 = \sqrt{z_1}, \qquad z_3 = \sqrt{x}, \qquad z = f(x) = z_2 - z_3$$

$$\delta^L_{\tilde{z}_1} = \frac{x}{z_1}\,\delta_{\tilde{x}} + \delta_1$$

$$\delta^L_{\tilde{z}_2} = \frac{1}{2}\,\delta^L_{\tilde{z}_1} + \delta_2$$

$$\delta^L_{\tilde{z}_3} = \frac{1}{2}\,\delta_{\tilde{x}} + \delta_3$$

$$\begin{aligned}\delta^L_{\tilde{z}} &= \frac{z_2}{z}\,\delta^L_{\tilde{z}_2} - \frac{z_3}{z}\,\delta^L_{\tilde{z}_3} + \delta_4 \\ &= \frac{z_2}{z}\left(\frac{x}{2z_1}\,\delta_{\tilde{x}} + \frac{1}{2}\,\delta_1 + \delta_2\right) - \frac{z_3}{z}\left(\frac{1}{2}\,\delta_{\tilde{x}} + \delta_3\right) + \delta_4 \\ &= \frac{1}{2z}\left(\frac{xz_2}{z_1} - z_3\right)\delta_{\tilde{x}} + \frac{z_2}{z}\left(\frac{1}{2}\,\delta_1 + \delta_2\right) - \frac{z_3}{z}\,\delta_3 + \delta_4 \\ &= -\frac{1}{2}\sqrt{\frac{x}{x+1}}\,\delta_{\tilde{x}} + \frac{\sqrt{x+1}}{2f(x)}\,\delta_1 + \frac{\sqrt{x+1}}{f(x)}\,\delta_2 - \frac{\sqrt{x}}{f(x)}\,\delta_3 + \delta_4 \\ &=: p_f(x)\,\delta_{\tilde{x}} + q_1(x)\,\delta_1 + q_2(x)\,\delta_2 + q_3(x)\,\delta_3 + \delta_4\end{aligned}$$

O problema é estável ou bem posto para qualquer $x \in \mathbb{R}$ pois $|p_f(x)| \in \left[0, \frac{1}{2}\right[$ em todo o domínio de f.

O algoritmo para o cálculo de $f(x)$ é numericamente instável para $x \gg 1$ pois $f(x) \approx \dfrac{1}{2\sqrt{x}}$ e portanto $q_1(x) \approx x,\;\; q_2(x) \approx 2x,\;\; q_3(x) \approx -2x$ para $x \gg 1$.

[2]
(a)[35]

$$f(x) = 0 \;\;\Leftrightarrow\;\; x = g(x), \qquad g(x) = \frac{e^{x/2}}{\sqrt{3}}$$

$$g'(x) = \frac{1}{2}\,g(x), \qquad g''(x) = \frac{1}{4}\,g(x)$$

Condições suficientes de convergência do método do ponto fixo com função iteradora g para z_2, $\;\forall x_0 \in I_2 = [0.8, 1.0]$:

(i) $g \in C^1(I_2)$,

pois $x \mapsto e^x$ é $C^\infty(\mathbb{R})$.

(ii) $\max_{x \in I_2} |g'(x)| = |g'(1.0)| = 0.475945 < 1$,

pois g' é positiva e crescente em I_2.

(iii) $g(I_2) \subset I_2$,

pois $g(0.8) = 0.861305 \in I_2, \quad g(1.0) = 0.951890 \in I_2$,

e g é crescente em I_2.

Método do ponto fixo:

$x_m = g(x_{m-1}), \quad m \in \mathbb{N}_0, \qquad x_0 \in I_2$

$|z_2 - x_m| \leq \dfrac{L}{1-L}\,|x_m - x_{m-1}| =: B_m, \qquad m \geq 1$

$L = \max_{x \in I_2} |g'(x)| = 0.475945$

m	x_m	B_m
0	0.9	
1	0.905465	0.496×10^{-2}
2	0.907943	0.225×10^{-2}
3	0.909069	0.102×10^{-2}
4	0.909581	0.465×10^{-3}

$z_2 = 0.909581 + \Delta, \qquad |\Delta| < B_4$

(b)[35]

Condições suficientes de convergência do método de Newton para z_3, $\forall x_0 \in I_3 = [3.6, 3.8]$:

(0) $f \in C^2(I_3)$

(i) $f(3.6) = -2.28177, \;\; f(3.8) = 1.38118, \;\; f(3.6)f(3.8) < 0$

(ii) $f'(x) = e^x - 6x > 0, \;\; \forall x \in I_3$

(iii) $f''(x) = e^x - 6 > 0, \;\; \forall x \in I_3$

(iv) $\left|\dfrac{f(3.6)}{f'(3.6)}\right| = 0.152136 < 0.2, \qquad \left|\dfrac{f(3.8)}{f'(3.8)}\right| = 0.0630644 < 0.2$

Método de Newton:

$$x_m = G(x_{m-1}), \quad G(x) = x - \frac{f(x)}{f'(x)}, \quad m \geq 0, \qquad x_0 \in I_3$$

$$|z_3 - x_m| \leq B_m, \qquad B_m = K B_{m-1}^2, \qquad K = \frac{\max_{x\in I_3} |f''(x)|}{2\min_{x\in I_3} |f'(x)|}$$

$$\left.\begin{array}{l} \min\limits_{x\in I_3} |f'(x)| = f'(3.6) \\ \max\limits_{x\in I} |f''(x)| = f''(3.8)| \end{array}\right\} \quad \Longrightarrow \quad K = 1.29019$$

$$x_0 = 3.7, \qquad |z_3 - x_0| \leq 0.1 = B_0$$

$$K|z_3 - x_0| \leq 0.129019 < 1$$

m	x_m	B_m
0	3.7	0.1
1	3.73413	0.129×10^{-1}
2	3.73308	0.215×10^{-3}

$$z_3 = 3.73308 + \Delta, \qquad |\Delta| \leq B_2.$$

(c)[20]

$$\lim_{m\to\infty} B_m = 0$$

$$K_\infty^{[r]} = \frac{B_{m+1}}{B_m^r} = \delta^{q_{m+1} - r q_m}$$

$$q_{m+1} - r q_m = \frac{1}{\sqrt{5}} \left[r_0^{m+1}(r_0 - r) - r_1^{m+1}(r_1 - r) \right]$$

$$\lim_{m\to\infty} K_\infty^{[r]} = \begin{cases} 0, & r < r_0 \\ 1, & r = r_0 \\ \infty, & r > r_0 \end{cases}$$

Conclui-se pois que a sucessão $\{B_m\}$ tem ordem de convergência r_0.

[3]
(a)[35]

$$\|\delta_{x_\varepsilon}\|_1 \leq \frac{\text{cond}_1(A_0)}{1 - \text{cond}_1(A_0)\|\delta_{A_\varepsilon}\|_1} \left(\|\delta_{A_\varepsilon}\|_1 + \|\delta_{b_\varepsilon}\|_1 \right)$$

$$\delta_{x_\varepsilon} = \frac{x_0 - x_\varepsilon}{\|x_0\|_1}, \qquad \delta_{b_\varepsilon} = \frac{b_0 - b_\varepsilon}{\|b_0\|_1}, \qquad \delta_{A_\varepsilon} = \frac{A_0 - A_\varepsilon}{\|A_0\|_1}$$

$$\text{cond}_1(A_0) = \|A_0\|_1 \|A_0^{-1}\|_1, \qquad \text{cond}_1(A_0)\|\delta_{A_\varepsilon}\|_1 < 1$$

$$A_0 = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}, \qquad b_0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \qquad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A_0^{-1} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

$$\|A_0\|_1 = \max\{6+2, 2+3\} = 8$$

$$\|A_0^{-1}\|_1 = \frac{1}{14}\max\{3+2, 2+6\} = \frac{4}{7}$$

$$\mathrm{cond}_1(A_0) = \frac{32}{7}$$

$$\|b_0\|_1 = 5, \qquad \|x_0\|_1 = 2$$

$$A_\varepsilon - A_0 = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}, \qquad b_\varepsilon - b_0 = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}$$

$$\|A_\varepsilon - A_0\|_1 = |\varepsilon|, \qquad \|b_\varepsilon - b_0\|_1 = |\varepsilon|$$

$$\|\delta_{A_\varepsilon}\|_1 = \frac{|\varepsilon|}{8}, \qquad \|\delta_{b_\varepsilon}\|_1 = \frac{|\varepsilon|}{5}$$

$$\|x_\varepsilon - x_0\|_1 \le 2\,\frac{\frac{32}{7}}{1 - \frac{32}{7}\frac{|\varepsilon|}{8}}\left(\frac{|\varepsilon|}{8} + \frac{|\varepsilon|}{5}\right) \le \frac{\frac{104}{35}|\varepsilon|}{1 - \frac{4}{7}|\varepsilon|}$$

(b)[35]

$$A_1 = \begin{bmatrix} 7 & -2 \\ -2 & 3 \end{bmatrix} = L + D + U = M_{GS} + N_{GS}, \qquad b_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$M_{GS} = L + D = \begin{bmatrix} 7 & 0 \\ -2 & 3 \end{bmatrix}, \qquad N_{GS} = U = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$x^{(m+1)} = D^{-1}\left(-Lx^{(m+1)} - Ux^{(m)} + b_1\right), \qquad m \ge 0$$

$$x^{(m+1)} = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\left(-\begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}x^{(m+1)} - \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}x^{(m)} + \begin{bmatrix} 5 \\ 1 \end{bmatrix}\right)$$

$$x^{(m+1)} = \begin{bmatrix} x_1^{(m+1)} \\ x_2^{(m+1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{7}\left(5 + 2x_2^{(m)}\right) \\ \frac{1}{3}\left(1 + 2x_1^{(m+1)}\right) \end{bmatrix}, \qquad m \ge 0$$

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad x^{(1)} = \begin{bmatrix} \frac{5}{7} \\ \frac{17}{21} \end{bmatrix}, \qquad x^{(2)} = \begin{bmatrix} \frac{139}{147} \\ \frac{425}{441} \end{bmatrix}$$

$$\|x_1 - x^{(2)}\|_\infty \leq \frac{c}{1-c}\|x^{(2)} - x^{(1)}\|_\infty, \qquad c = \|C_{GS}\|_\infty$$

$$C_{GS} = -M_{GS}^{-1}N_{GS} = -\frac{1}{21}\begin{bmatrix} 3 & 0 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 0 & 6 \\ 0 & 4 \end{bmatrix}$$

$$\|C_{GS}\|_\infty = \frac{2}{7}$$

$$x^{(2)} - x^{(1)} = \begin{bmatrix} \frac{34}{147} \\ \frac{68}{441} \end{bmatrix} \quad \Rightarrow \quad \|x^{(2)} - x^{(1)}\|_\infty = \frac{34}{147}$$

$$\|x_1 - x^{(2)}\|_\infty \leq \frac{68}{735}$$