INSTITUTO SUPERIOR TÉCNICO
Mestrado em Engenharia Física Tecnológica
Ano Lectivo: 2010/2011

MATEMÁTICA COMPUTACIONAL

Resolução do Teste de 15 de Dezembro de 2010

[1]

(a)[20]

O teorema do ponto fixo de Banach diz-nos que g terá um e um só ponto fixo em D se forem satisfeitas as seguintes condições:

- $g(D) \subset D$ • $g \in C^1(D)$ • $\max_{x \in D} \|J_g(x)\|_\infty < 1$

Verifiquemos pois estas condições:

- $$\left\{ \begin{array}{ll} g_1(x) \in \left[-\frac{7}{12}, \frac{7}{12}\right] \subset \left[-\frac{2}{3}, \frac{2}{3}\right], & \forall x \in D \\ g_2(x) \in \left[-\frac{9}{20}, \frac{9}{20}\right] \subset \left[-\frac{2}{3}, \frac{2}{3}\right], & \forall x \in D \end{array} \right. \quad \Longrightarrow \quad g(D) \subset D$$

- g_1 e g_2 são polinómios trigonométricos nas variáveis x_1 e x_2, logo são infinitamente diferenciáveis.

- $$J_g(x) = \begin{bmatrix} -\frac{1}{3}\sin x_1 & \frac{1}{4}\cos x_2 \\ \frac{1}{4}\cos x_1 & -\frac{1}{5}\sin x_2 \end{bmatrix}$$

$$\|J_g(x)\|_\infty = \max\left\{\frac{1}{3}|\sin x_1| + \frac{1}{4}|\cos x_2|, \frac{1}{4}|\cos x_1| + \frac{1}{5}|\sin x_2|\right\}$$

$$\|J_g(x)\|_\infty \leq \max\left\{\frac{7}{12}, \frac{9}{20}\right\} = \frac{7}{12} = 0.583333 < 1$$

Concluímos que g tem um único ponto fixo em D e, portanto, que o sistema de equações $x = g(x)$ tem uma única solução em D.

(b)[30]

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad x^{(1)} = g\left(x^{(0)}\right) = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$x^{(2)} = g\left(x^{(1)}\right) = \begin{bmatrix} \frac{1}{3}\cos\frac{1}{3} + \frac{1}{4}\sin\frac{1}{5} \\ \frac{1}{4}\sin\frac{1}{3} + \frac{1}{5}\cos\frac{1}{5} \end{bmatrix} = \begin{bmatrix} 0.364653 \\ 0.277812 \end{bmatrix}$$

$$\|z - x^{(2)}\|_\infty \leq \frac{L}{1-L}\|x^{(2)} - x^{(1)}\|_\infty$$

$$x^{(2)} - x^{(1)} = \begin{bmatrix} 0.0313196 \\ 0.077812 \end{bmatrix}$$

$$\max_{x \in D} \|J_g(x)\|_\infty \leq \frac{7}{12} = L$$

$$\|z - x^{(2)}\|_\infty \leq \frac{7}{5} \times 0.077812 = 0.108937$$

[2]

(a)[25]

Fórmula de Newton com diferenças divididas:

i	x_i	$f[x_i]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$	$f[\cdot,\cdot,\cdot,\cdot]$
0	0.0	-1.0			
			1.0384		
1	1.25	0.298		-0.2368	
			0.4464		-0.00512
2	2.5	0.856		-0.2560	
			-0.1936		
3	3.75	0.614			

$$\begin{aligned} p_3(x) &= f[x_0] + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) \\ &\quad + f[x_0,x_1,x_2,x_3](x-x_0)(x-x_1)(x-x_2) \end{aligned}$$

$$p_3(x) = -1.0 + 1.0384x - 0.2368x(x-1.25) - 0.00512x(x-1.25)(x-2.5)$$

(b)[30]

Melhor aproximação mínimos quadrados:

$$q_1^*(x) = a^*\phi_0(x) + b^*\phi_1(x), \qquad \phi_0(x) = 1, \qquad \phi_1(x) = x^2$$

$$\begin{bmatrix} \langle\bar\phi_0,\bar\phi_0\rangle & \langle\bar\phi_0,\bar\phi_1\rangle \\ \langle\bar\phi_1,\bar\phi_0\rangle & \langle\bar\phi_1,\bar\phi_1\rangle \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} \langle\bar f,\bar\phi_0\rangle \\ \langle\bar f,\bar\phi_1\rangle \end{bmatrix}, \qquad \langle\bar\phi,\bar\psi\rangle = \sum_{i=1}^{4}\bar\phi_i\bar\psi_i$$

$$\bar{\phi}_0 = [\phi_0(x_i)] = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \bar{\phi}_1 = [\phi_1(x_i)] = \begin{bmatrix} 0.0 \\ 1.5625 \\ 6.25 \\ 14.0625 \end{bmatrix}$$

$$\bar{f} = [f(x_i)] = \begin{bmatrix} -1.0 \\ 0.298 \\ 0.856 \\ 0.614 \end{bmatrix}$$

$$\langle \bar{\phi}_0, \bar{\phi}_0 \rangle = 4$$

$$\langle \bar{\phi}_0, \bar{\phi}_1 \rangle = 21.875 = \langle \bar{\phi}_1, \bar{\phi}_0 \rangle$$

$$\langle \bar{\phi}_1, \bar{\phi}_1 \rangle = 239.258$$

$$\langle \bar{f}, \bar{\phi}_0 \rangle = 0.768$$

$$\langle \bar{f}, \bar{\phi}_1 \rangle = 14.45$$

$$\begin{bmatrix} 4 & 21.875 \\ 21.875 & 239.258 \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} 0.768 \\ 14.45 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} -0.276571 \\ 0.0856815 \end{bmatrix}$$

(c)[25]

Fórmula de Simpson composta ($F \in C([a, b])$):

$$I(F) = \int_a^b F(x)\, dx \quad \approx$$

$$I_2^{(M)}(F) = \frac{h_M}{3} \left[F(X_0) + F(X_M) + 4 \sum_{j=1}^{M/2} F(X_{2j-1}) + 2 \sum_{j=1}^{M/2-1} F(X_{2j}) \right]$$

$$h_M = \frac{b-a}{M}, \qquad X_j = a + j h_M, \quad j = 0, 1, \ldots, M, \qquad M \text{ par}$$

$$\begin{cases} a = 0, \quad b = 5, \qquad M = 4, \qquad h_M = \dfrac{5}{4} \\ X_j = 1.25j, \quad j = 0, 1, 2, 3, 4, \qquad F(X_j) = f(x_j) \end{cases}$$

$$I_2^{(4)}(f) = \frac{5}{12} \left[f(0.0) + 4f(1.25) + 2f(2.5) + 4f(3.75) + f(5.0) \right] = 1.95125$$

(d)[20]

$$E_1^{(M)}(f) = \frac{b-a}{2} \left(\frac{h_M}{2} \right)^4 E_1(f), \qquad E_1(f) = \frac{1}{135} f^{(4)}(\xi)$$

$$h_M = \frac{b-a}{M}, \qquad \xi \in]a, b[$$

$$\left|E_1^{(M)}(f)\right| = \frac{5}{2}\left(\frac{5}{2M}\right)^4 \frac{1}{135}\left|f^{(4)}(\xi)\right|$$

$$\left|f^{(4)}(\xi)\right| \leq \max_{x\in[0,5]}\left|f^{(4)}(x)\right| = 17$$

$$\left|E_1^{(M)}(f)\right| \leq \frac{17}{54}\left(\frac{5}{2M}\right)^4 < \varepsilon \quad \Leftrightarrow \quad M > \frac{5}{2}\left(\frac{17}{54\varepsilon}\right)^{1/4}$$

$$\varepsilon = 10^{-6}: \qquad M > 59.218 \quad \Rightarrow \quad M = 60$$

[3] $f(x,y) = 4e^{-y} =: F(y)$

(a)[30] Método de Taylor de $2^{\underline{a}}$ ordem (passo h):

$$y_1 = y_0 + hf(x_0,y_0) + \frac{h^2}{2}(d_f f)(x_0,y_0)$$

$$(d_f f)(x,y) = \left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)(x,y) = F(y)F'(y) = -[F(y)]^2$$

$$y_1 = hF(0) - \frac{h^2}{2}[F(0)]^2 = 4h - 8h^2$$

(b)[20]

$$|Y(h) - y_1| \leq \frac{\tau(h)}{K}\left(e^{Kh} - 1\right)$$

K: constante de Lipschitz da função incremento do método de Taylor, φ, definida por:

$$\varphi(x,y) = f(x,y) + \frac{h}{2}(d_f f)(x,y)$$

$$\tau(h) = \max_{x\in[0,h]}|\tau(x,Y(x),h)| = \frac{h^2}{6}\max_{x\in[0,h]}|Y'''(x)|$$

$$\varphi(x,y) = F(y) - \frac{h}{2}[F(y)]^2$$

$$\frac{\partial\varphi(x,y)}{\partial y} = [-1 + hF(y)]F(y)$$

$$\max_{x\in[0,h]}|F(Y(x))| = 4, \quad \text{pois} \quad Y(x) \geq 0, \ \ \forall x \geq 0$$

$$\left|\frac{\partial\varphi(x,y)}{\partial y}\right| \leq 4(1+4h) =: K$$

$$Y'(x) = F(Y(x)), \qquad Y''(x) = -[F(Y(x))]^2, \qquad Y'''(x) = 2[F(Y(x))]^3$$

$$\tau(h) \leq \frac{h^2}{6}\,128 = \frac{64h^2}{3}$$

$$|Y(h) - y_1| \leq \frac{16h^2}{3(1+4h)}\,\left[e^{4h(1+4h)} - 1\right]$$

ALTERNATIVA:

$$Y(h) = Y(0) + hY'(0) + \frac{h^2}{2}\,Y''(0) + \frac{h^3}{6}\,Y'''(\theta h), \qquad \theta \in]0,1[$$

$$Y'(x) = F(Y(x)), \qquad Y''(x) = -[F(Y(x))]^2, \qquad Y'''(x) = 2[F(Y(x))]^3$$

$$Y(0) = 0, \qquad Y'(0) = 4, \qquad Y''(0) = -16, \qquad Y'''(\theta h) = 128\,e^{-3Y(\theta h)}$$

$$Y(h) = 4h - 8h^2 + \frac{64h^3}{3}\,e^{-3Y(\theta h)}$$

$$Y(h) - y_1 = \frac{64h^3}{3}\,e^{-3Y(\theta h)}$$

$$Y(x) \geq 0, \ \ \forall x \geq 0$$

$$|Y(h) - y_1| \leq \frac{64h^3}{3}$$