Differential Geometry - 1st Semester 2011/12

Ficha 2 (due Monday October 3)

1) Let $O(n,\mathbb{R}) = \{A \in M_n(\mathbb{R}) : AA^T = I\}$ be the set of $n \times n$ orthogonal matrices. Show that $O(n,\mathbb{R})$ is an embedded submanifold. Show that the tangent space $T_I O(n)$ can be identified with the space of $n \times n$ antisymmetric matrices.

2) Consider the map $\Phi : \mathbb{P}^2 \to \mathbb{R}^4$ defined by

$$\Phi([x,y,z]) = \frac{1}{x^2+y^2+z^2}\,(x^2-z^2, yz, xz, xy)\ .$$

Show that $(\mathbb{P}^2, \Phi)$ is an embedded submanifold of $\mathbb{R}^4$.

3) Let M and N be smooth manifolds, and $\Phi : M \to N$ a smooth map. Let $q \in N$ be a regular value. Show that

$$T_p \Phi^{-1}(q) = \{v \in T_p M : d_p\Phi \cdot v = 0\}\ .$$

4) Let M be a smooth manifold, $A \subset M$ and $i : A \hookrightarrow M$ the canonical inclusion. Show that (A, i) is an embedded submanifold of M of dimensions $k \iff$ for each $p \in M$, there exists a coordinate system $(U, (x^1, \ldots, x^m))$ centered at p such that

$$A \cap U = \{p \in U : x^{k+1}(p) = \cdots = x^m(p) = 0\}\ .$$