Differential Geometry - 1st Semester 2011/12

Ficha 3 (due Wednesday October 19)

1) Let $N \subset M$ be an embedded submanifold, and $p \in N$. Show that as a subspace of T_pM, the tangent space T_pN is given by

$$T_pN = \{X \in T_pM : Xf = 0 \text{ whenever } f \in C^\infty(M) \text{ and } f_{|N} = 0\ .\}$$

2) Let $N \subset M$ be an immersed submanifold of M. Let X, Y be smooth vector fields on M. Show that if X and Y are tangent to N, so is $[X, Y]$. You may want to use the result from **1)**.

3) Show that the 2-dimensional distribution on $M = \mathbb{R}^3$, defined by the vector fields

$$X_1 = \frac{\partial}{\partial x} \quad , \quad X_2 = \mathrm{e}^{-x}\,\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\ ,$$

does not have integral manifolds.

4) Consider the following 2-dimensional distribution on $M = \mathbb{R}^3 \setminus \{0\}$,

$$D_p = \operatorname{span}(X_p, Y_p, Z_p)\ ,$$

where

$$X = y\,\frac{\partial}{\partial z} - z\,\frac{\partial}{\partial y} \quad , \quad Y = z\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial z} \quad , \quad Z = x\,\frac{\partial}{\partial y} - y\,\frac{\partial}{\partial x}\ .$$

Show that D cannot be generated globally by two vector fields.

5) Consider a distribution on $M = \mathbb{R}^3$ generated by the vector fields

$$X_1 = \frac{\partial}{\partial x} + \cos x\,\cos y\,\frac{\partial}{\partial z} \quad , \quad X_2 = \frac{\partial}{\partial y} - \sin x\,\sin y\,\frac{\partial}{\partial z}\ .$$

Verify that the distribution is involutive. Determine the foliation that integrates it. *Hint:* consider leaves of the form

$$L_\alpha = \{(x, y, f(x, y) + \alpha) : x, y \in \mathbb{R}\}$$

based on surfaces $F_\alpha(x, y, z) = z - f(x, y) = \alpha \in \mathbb{R}$.