Differential Geometry - 1st Semester 2011/12

Ficha 4 (due Wednesday November 2)

1) **a)** Show that $SL(n, \mathbb{R})$ is a Lie group.
Hint: Consider the level set $f^{-1}(1)$, where the map $f : GL(n, \mathbb{R}) \to \mathbb{R}$ is given by $f(A) = \det A$.

b) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider conjugation by a group element $g \in G$, i.e. consider the map $\psi_g : G \to G$ with $h \mapsto g\,h\,g^{-1}$. This induces the Lie algebra automorphism $ad(g) : \mathfrak{g} \to \mathfrak{g}$ with $ad(g)(X) = (\psi_g)_* X$. Determine $ad(g)(X)$.

2) Let X, Y denote smooth vector fields on a smooth manifold M, and $w \in \Omega(M)$ a differential form.

a) Show that

$$L_{[X,Y]} w = L_X \left(L_Y w \right) - L_Y \left(L_X w \right) \ .$$

b) Let $w \in \Omega^1(M)$. Using local coordinates $(U, (x^1, \dots, x^m))$, show that

$$dw(X, Y) = X \left(w(Y) \right) - Y \left(w(X) \right) - w \left([X, Y] \right) \ .$$

3) Let M be a smooth manifold of dimension m, and D a k-dimensional smooth distribution on M. Then, in the neighborhood of any $p \in M$ there exist $(m - k)$ linearly independent 1-forms $w^1, \dots w^{m-k}$ such that $w^i(X) = 0$ when $X \in D$.

a) Show that D on M is involutive $\Longleftrightarrow$
$dw^i = \sum_{j=1}^{m-k} \alpha^i{}_j \wedge w^j$, where $\alpha^i{}_j \in \Omega(M)$ denote 1-forms $(i, j = 1, \dots, (m - k))$.

b) Show that $I(D) = \{w \in \Omega(M) : w \text{ annihilates } D\}$ is an ideal of the external algebra $\Omega(M)$.

4) Let M be a symplectic manifold with symplectic form ω. Show that:

a) M has even dimension $d = 2n$.

b) ω^n is a volume form on M (hence, any symplectic manifold is orientable).

c) Any smooth function $f \in C^\infty(M)$ is constant along the flow generated by the Hamiltonian vector field X_f.

d) The flux of the Hamiltonian vector field X_f preserves the symplectic form, i.e. $L_{X_f} \omega = 0$.

e) The Poisson bracket $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defines a Lie algebra structure on $C^\infty(M)$.

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f) The map $f \in C^\infty(M) \mapsto X_f$ (with X_f Hamiltonian vector field on M) is a homomorphism of Lie algebras.

g) Consider local coordinates (x^i, y^i) on an open set $U \subset M$. Then, these coordinates are Darboux $\Longleftrightarrow \{x^i, x^j\} = \{y^i, y^j\} = 0\,,\ \{x^i, y^j\} = -\delta^{ij}$.

5) Let N be a smooth manifold. Show that the cotangent bundle T^*N is a symplectic manifold (its canonical symplectic form is $\omega = d\theta$, where θ denotes the 1-form on T^*N given as follows: let $p \in N$, $\eta \in T_p^*N$ and $(p, \eta) \in T^*N$. Then $\theta_{(p,\eta)} = \pi^*\eta$, where $\pi : T^*N \to N$ is the canonical projection.).