Differential Geometry - 1st Semester 2011/12

Ficha 6 (due Wednesday December 14)

1) Let $\xi = (\pi, E, M)$ be a real vector bundle of rank r with a connection ∇. Let $\{s_a\}$ and $\{s'_a\}$ denote two local frames that are related by a matrix $A \in C^\infty(U_\alpha, GL(r))$ in a trivializing chart (U_α, ϕ_α), i.e. $s'_a = A^b{}_a\, s_b$.

a) Show that the associated connection one-forms ω and ω' are related by

$$\omega' = A^{-1}\,\omega\, A + A^{-1} dA\ .$$

b) Show that the associated 2-form curvatures Ω and Ω' are related by

$$\Omega' = A^{-1}\,\Omega\, A\ .$$

2) Given two real vector bundles $\xi = (\pi, E, M)$ and $\eta = (\tau, F, M)$, show that there exists a vector bundle $Hom(E, F)$ whose fibers are spaces of homomorphisms $Hom(E_p, F_p)$ $(p \in M)$. Determine the transition functions of $Hom(E, F)$ in terms of the transition functions of ξ and η. Show that there is a bundle isomorphism $Hom(E, F) \simeq E^* \otimes F$.

3) Consider the complex line bundle $\xi = (\pi, \mathbb{CP}^2 \backslash \{[0, 0, 1]\}, \mathbb{CP}^1)$, where $\pi([z, w, t]) = [z, w]$. ($\mathbb{CP}$ = complex projective space). Take

$$\begin{aligned} U_1 &= \{[z, 1] \in \mathbb{CP}^1 : z \in \mathbb{C}\} \quad , \quad \phi_1([z, 1, t]) = ([z, 1], t)\ , \\ U_2 &= \{[1, w] \in \mathbb{CP}^1 : w \in \mathbb{C}\} \quad , \quad \phi_2([1, w, t]) = ([1, w], t)\ , \end{aligned}$$

as trivializing charts.

Show that:

a) there exists a cocycle $g_{21} : U_1 \cap U_2 \to \mathbb{C}\backslash\{0\}$ given by

$$g_{21}([z, 1]) = \frac{\bar{z}}{|z|}\ ;$$

b) for this cocyle, the 1-forms

$$\omega^1 = \tfrac{1}{2}\,\frac{z\, d\bar{z} - \bar{z}\, dz}{1 + |z|^2} \quad , \quad \omega^2 = \tfrac{1}{2}\,\frac{w\, d\bar{w} - \bar{w}\, dw}{1 + |w|^2}$$

define a connection ∇ on ξ;

c) in U_1, the curvature 2-form reads

$$\Omega = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}\ .$$

4) Consider two complex vector bundles E_i over M $(i = 1, 2)$. Show that the total Chern class satisfies

$$c(E_1 \oplus E_2) = c(E_1) \cup c(E_2)\ .$$