Análise Matemática III - Turma especial

Ficha 10

A entregar até quarta-feira, 28 de Novembro às 15h

1 *Considere as seguintes formas diferenciais:*

$\alpha = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \ldots + dx^{2n-1} \wedge dx^{2n} \in \Omega^2(E^{2n})$,
$\beta = xdx + ydy \in \Omega^1(E^2)$
$\gamma = \frac{-y}{\sqrt{x^2+y^2}}dx + \frac{x}{\sqrt{x^2+y^2}}dy \in \Omega^1(E^2 \setminus 0)$
$\omega = e^{xz}dx + x\cos z\ dy + y^2dz \in \Omega^1(E^3)$
$\eta = xdx \wedge dy - zdx \wedge dz + xyzdy \wedge dz \in \Omega^2(E^3)$
$\zeta = \frac{xz}{\sqrt{x^2+y^2}}dx + \frac{yz}{\sqrt{x^2+y^2}}dy - \sqrt{x^2+y^2}dz \in \Omega^1(U)$ *com* $U = E^3 \setminus \{(0,0,z)|z \in \mathbb{R}\}$

e as seguintes transformações:

$\mathbf{h} : \mathbb{R} \to E^2$ *definida por* $\mathbf{h}(t) = (t, t^2)$,
$\mathbf{g} :]0, +\infty[\times]0, 2\pi[\to E^2$ *definida por* $\mathbf{g}(r, \theta) = (r\cos\theta, r\sin\theta)$,
$\mathbf{f} :]0, +\infty[\times]0, 2\pi[\times]0, \pi[\to E^3$ *definida por* $\mathbf{f}(r, \theta, \phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$.

Calcule

(a) $\beta \wedge \gamma$, $\eta \wedge \eta$,

(b) $\alpha \wedge \ldots \wedge \alpha$ *(produto exterior com n factores),*

(c) $\mathbf{g}^*\beta$, $\mathbf{g}^*\gamma$, $\mathbf{f}^*\zeta$, $\mathbf{h}^*\beta$,

(d) $d\omega, d\eta, \mathbf{f}^*d\zeta$.

2 *Seja* $\mathbf{f} : U \subset E^n \to V \subset E^n$ *uma transformação diferenciável e*

$$\omega(\mathbf{y}) = h(\mathbf{y})dy^1 \wedge \ldots dy^n \in \Omega^n(V)$$

uma forma diferencial. Mostre que

$$\mathbf{f}^*\omega(\mathbf{x}) = h(\mathbf{f}(\mathbf{x})) \det D\mathbf{f}(\mathbf{x})dx^1 \wedge \ldots dx^n.$$

3 *Dado um campo vectorial* $\mathbf{F} : U \subset E^3$ *de classe* C^∞, *definido por*

$$\mathbf{F}(x, y, z) = \big(F^1(x, y, z), F^2(x, y, z), F^3(x, y, z)\big)$$

definem-se as formas

$$w_F = F^1dx + F^2dy + F^3dz \in \Omega^1(U),$$

$$\Omega_F = F^1dy \wedge dz + F^2dz \wedge dx + F^3dx \wedge dy \in \Omega^2(U),$$

o campo vectorial

$$\operatorname{rot}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F^1 & F^2 & F^3 \end{vmatrix}$$

e o campo escalar

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}.$$

Seja $f : U \subset E^n \to \mathbb{R}$ *um campo escalar de classe* C^∞. *Mostre que*

(a) $df = \omega_{\nabla f}$ *onde* ∇f *designa o gradiente de* f,

(b) $d\omega_{\mathbf{F}} = \Omega_{\nabla \times \mathbf{F}}$,

(c) $d\Omega_{\mathbf{F}} = \nabla \cdot \mathbf{F} dx \wedge dy \wedge dz$,

(d) $\nabla \times (\nabla f) = 0$ *e* $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

4 *Decida se as seguintes formas* $\omega \in \Omega^*(E^3)$ *são exactas e em caso afirmativo calcule um potencial, isto é, uma forma* α *tal que* $d\alpha = \omega$.

(a) $yzdx + xzdy + xydz$,

(b) $zdx \wedge dy - ydx \wedge dz + xdy \wedge dz$,

(c) $yzdx \wedge dz + xzdy \wedge dz$,

(d) $x^2ye^z dx \wedge dy \wedge dz$.