Portugal - Talk 3

Note Title 6/25/2011

Last time: Defined TAF

Reminder

$F \,|^{2}\, \mathbb{Q}$ quad imag extension $u\bar{u} \,|\, p$

$V =$ F-v.s. dim n

$\langle -,- \rangle : V \otimes V \to \mathbb{Q}$ alternating hermitian form signature $(1, n-1)$

$L \subset V$ lattice $\qquad U =$ grp of isometries of $(V, \langle -,- \rangle)$

$$Sh_U = \{(A, i, \lambda)\}$$

$A =$ ab var dim n

$i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$

$\lambda : A \to A^\vee$ polarization

$(*)\quad A[p^\infty] = \underbrace{A[u^\infty]}_{\text{dim } 1} \oplus \underbrace{A[\bar{u}^\infty]}_{\text{dim } n-1}$

$(**)\quad \forall\, \ell \neq p \quad (T_\ell A, \langle -,- \rangle_\ell) \sim (L_\ell, \langle -,- \rangle_\ell)$

$A_{univ}[u^\infty] \downarrow Sh_U \rightsquigarrow \mathcal{E}_U$ sheaf of spectra$/Sh_U \rightsquigarrow TAF_U = \Gamma \mathcal{E}_U$

Examples

$n = 1$

(A, i, λ)

$A = C$ elliptic curve, λ essentially unique

Fact: C supersingular $\Rightarrow$ $\mathrm{End}(C)$ = order in $\mathbb{Q}$ quaternion alg

$C/$ field char p $\quad$ C ordinary $\Rightarrow$ $\mathrm{End}(C)$ = order in F quadratic imag exten

$$\{C \text{ w/ cx mult}\} \cong Cl(F)$$

$$\underset{\text{aut } \mathcal{O}_F^\times}{\mathbb{C}/I} \longleftrightarrow I$$

So $\quad Sh(\mathbb{C}) = Cl(F)/\!/\mathcal{O}_F^\times$ (finite set of pts)

$$\mathcal{O}_F^\times = \mu_2, \mu_4, \mu_6$$

$$TAF \simeq \prod_{Cl(F)} (K_p^\wedge)^{h\mathcal{O}_F^\times}$$

$\mathcal{O}_F^\times = \mu_2$ $\Rightarrow$ product of $(KO_p^\wedge)$'s finite

$n=2$

Prop:
$$\coprod_{Cl(F)} \mathcal{M}_{ell} \xrightarrow[\simeq]{} Sh_U$$
$$(I, C) \longmapsto C \otimes I$$

2-dim'l ab var w/ cx mult.

$$\Longrightarrow \quad TAF = \prod_{Cl(F)} TMF$$

Variants: Hill-Lawson

B = Quaternion alg $/\mathbb{Q}$

classified by $\mathrm{Inv}_v B \in \mathbb{Z}/2$ $\quad v \in$ places of $\mathbb{Q}$

$\sum_i \mathrm{Inv}_v = 0$

$\mathrm{disc} := \prod_{\substack{p \text{ finite} \\ B \text{ ramifies at } p}} p$

$p \nmid \mathrm{disc}$

$\Rightarrow B^\wedge_p \simeq M_2(\mathbb{Z}_p)$

$\Rightarrow A[p^\infty] \simeq \varepsilon A[p^\infty]^2$

$\dim \varepsilon A[p^\infty] = 1$

$Sh_B = (A, i)$

A = 2-dim'l ab var

$i: \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$

($\exists$ essentially unique polarization)

TAF_B "twists of TMF"

Cx points

$\mathcal{O}_B \subset B$ maximal order

"Fuchsian gp"

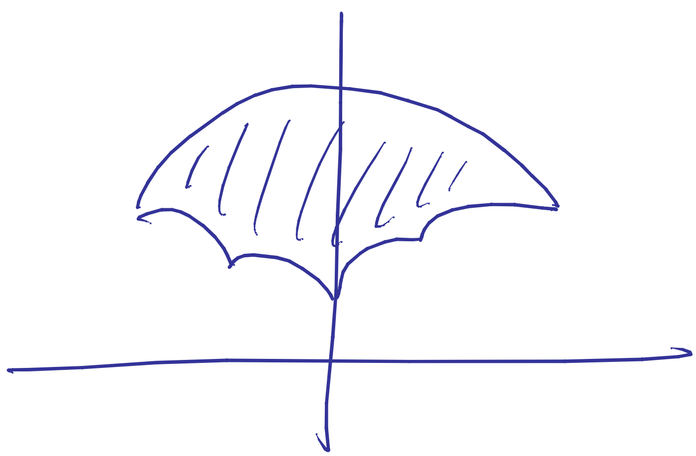
$\mathcal{O}_B^{N=1} \hookrightarrow SL_2(\mathbb{R}) \curvearrowright \mathcal{H}$ frac linear trans

$B \otimes \mathbb{R} = M_2(\mathbb{R})$

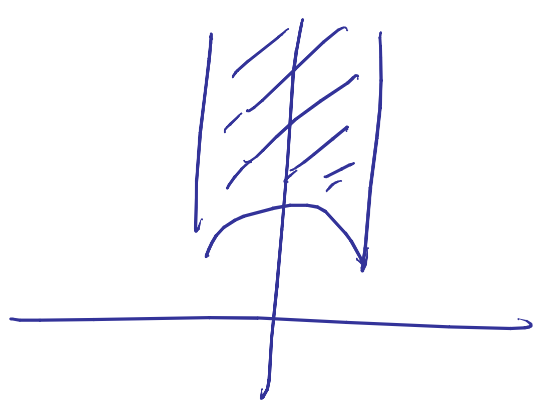
e.g. disc $= 6$

$Sh_B(\mathbb{C}) = \mathcal{O}_B^{N=1} \backslash \mathcal{H}$ cpt!

Fund Domain



Contrast w/ $SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$

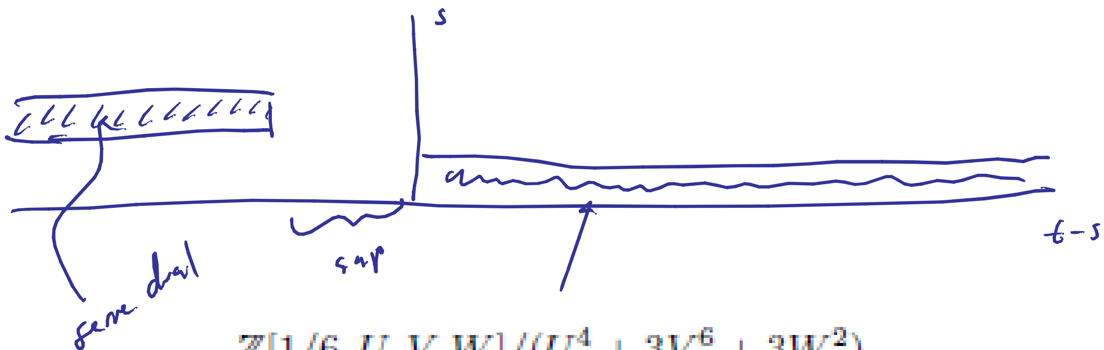


$SL_2(\mathbb{Z}) \backslash \mathcal{H} = M_{ell}(\mathbb{C})$

(may be compactified)

Hill-Lawson: Computations of TAF_B!

disc 6: $H^*(Sh_B, \omega^{\otimes *}) \Rightarrow \pi_* TAF_B$



$\mathbb{Z}[1/6, U, V, W]/(U^4 + 3V^6 + 3W^2)$.

$$taf_B = \tau_{\geq 0} TAF_B$$

$(\overline{\mathcal{M}}_{ell}$ gives $tmf)$ have to work!

Thm (Hill-Lawson)

disc 14!, $p = 3$

$$taf_B \simeq BP\langle 2\rangle \Rightarrow BP\langle 2\rangle \ E_\infty \text{ at } p=3 \ !$$

Also: $Sh_B \circlearrowleft w_\ell$ $\forall \ \ell$, B_ℓ ramified

involutions

$$Sh_B /\!/ w_\ell \rightsquigarrow taf_B^{w_\ell}$$

disc 10:

$p=3$ $\qquad taf_B^{w_2, w_5} \simeq tmf$ (weird...)

Similarly: "twists of TAF_u"

Let B = central simple F-alg: split at $u, \bar{u}$

$\dim_F B = n^2$

Endow B w/ involution $*$

$$*\big|_F = \bar{(\ )}$$

V = free B-mod rank 1, $\langle -,- \rangle$

Consider $Sh_B = \{(A, i, \lambda)\}$

A = ab var dim n^2

$i : \mathcal{O}_B \hookrightarrow \mathrm{End}(A)$

$\lambda : A \to A^\vee$ polarization

$$B_u \simeq M_n(\mathbb{Z}_p)$$

$$\Rightarrow \quad A[u^\infty] \simeq \varepsilon A[u^\infty]^n$$

$$\dim(\varepsilon A[u^\infty]) = 1$$

B = division alg $\Rightarrow$ Sh_B cpt.

Use to get "taf_B"'s ?

Need Comp computations

K(n)-local thy: Sh_n

$$1 \le ht\, A[u^\infty] \le n$$

$$Sh_n^{(k)} = \{(A, i, \lambda) \mid ht\ A[u^\infty] = k\}$$

dim$/\mathbb{Z}_p$ is $n-k$

$k = n$, $\quad A[u^\infty]_0 \xrightarrow{\simeq} A[u^\infty]$

$$Sh_n^{(n)} \simeq \coprod_{(A,i,\lambda) \in Sh_n^{(n)}(\bar{\mathbb{F}}_p)} Spf(\mathcal{O}_{Def\, A[u^\infty]}) // Aut(A,i,\lambda) // Gal(\bar{\mathbb{F}}_p/\mathbb{F}_p)$$

$$\Rightarrow \quad TAF_{K(n)} \simeq \left(\prod_{(A,i,\lambda) \in Sh_n^{(n)}(\bar{\mathbb{F}}_p)} E_n^{h Aut(A,i,\lambda)} \right)^{h Gal}$$

$$TAF_{K(n)} \simeq \prod_i E_n^{hG_i}$$

EO_n & TAF

Q: what is relationship between EO_n & TAF

When $\exists i$ s.t. G_i is maximal finite in $\mathbb{S}_n$?

<u>A</u>:

<u>Thm</u> (B-Hopkins)

$p \leq 7$, $n = (p-1)p^{r-1}$ $\Rightarrow$ Yes!

p odd, not in above situation $\Rightarrow$ No!

$p = 2$, not in above situation $\Rightarrow$?

<u>Consolation prize</u>:

$n = (p-1)p^{r-1} \Rightarrow C_{p^r} \hookrightarrow \mathbb{S}_n$

Can always find $G_i \supset C_{p^r}$.

So TAF is just as powerful as EO_n

$p = 2$

$n = 4$ $\qquad G = C_8$

<u>Hill-Hopkins-Ravenel</u> solution to Kervaire

$$S \longrightarrow E_4^{hC_8} = EO_4$$

- $E_4^{hC_8}$ detects Kervaire classes if they exist
- $\pi_* E_4^{hC_8}$ 256 periodic
- $\pi_{-2} = 0$ <u>Gap</u>

(EO_4 is a summand of TAF)