Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Topology Make-up Exam

January 30, 2009

Duration: 1h30m + 1h30m.

Justify your answers.

Part 1

1. Let M be a manifold and $A \subset M$ be a submanifold with codimension greater or equal to 2. Prove that if $M \setminus A$ is simply connected then M is simply connected.

2. Let M, N be manifolds, $V \subset M$ be open and consider the restriction map

$$\rho_V \colon C^r(M,N) \to C^r(V,N)$$

sending f to $f_{|V}$ (where $0 \leq r \leq \infty$).

(a) Show ρ_V is continuous for the weak topologies.

(b) Give an example showing this is not necessarily the case for the strong topologies.

3. The *limit set* $L(f)$ of a map $f\colon M \to N$ between smooth manifolds is

$$L(f) = \{y \in N \mid \exists\ (x_n) \in M \text{ without convergent subsequences so that} f(x_n) \to y\}.$$

Let $1 \leq r \leq \infty$ and $\mathcal{L} = \{f \in C^r_S(M,N) \colon f(M) \cap L(f) = \emptyset\}$.

(a) Show that given $f \in \mathcal{L}$ there exists $V \subset N$ open so that $f\colon M \to V$ is proper.

(b) Show that if $\dim N > 2 \dim M$ and $\partial M = \partial N = \emptyset$ then embeddings are dense in $\mathcal{L}$ (in the strong topology).

Part 2

4. Let N be a simply connected manifold and $M \subset N$ be a compact codimension 1 submanifold, $\partial M = \partial N = \emptyset$.

(a) Show that M is the level set of a regular value of a smooth function $f\colon N \to \mathbb{R}$ if and only if M is orientable.

(b) What happens if we remove the condition that the value be regular?

5. Let M and N be compact oriented n-manifolds without boundary and assume N is connected. Prove that the degree of a map $f\colon M \to N$ equals the intersection number of the graph of f with $M \times y$ in $M \times N$ for any $y \in N$

6. Let M and N be smooth compact manifolds without boundary and $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ be Morse functions with k and l critical points respectively. Show that if E is a smooth fiber bundle over N with fibre M then E admits a Morse function with kl critical points.