Instituto Superior Técnico
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Secção de Álgebra e Análise

Differential Topology practice test

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Duration: 1h30m.
Justify your answers.

1. Let $M \subset \mathbb{R}^{n+1}$ be a compact n-dimensional submanifold with $\partial M = \emptyset$ and define for each $x \in \mathbb{R}^{n+1} \setminus M$ the map $\sigma_x \colon M \to S^n$ by

$$\sigma_x(y) = \frac{y-x}{|y-x|}.$$

Show that

(a) x and x' are in the same component of $\mathbb{R}^{n+1} \setminus M$ iff σ_x and $\sigma_{x'}$ are homotopic.

(b) x is in the unbounded component iff σ_x is nul-homotopic.

(c) x is in the bounded component iff $\deg \sigma_x = \pm 1$.

2. Let E^* be the Thom space of a rank k vector bundle E over a compact manifold M without boundary. Show that the inclusion $\mathbb{R}^k \to E$ of a fiber extends to a map $S^k \to E^*$ which is not nul-homotopic.

3. What is the minimal number of critical points of a Morse function on an orientable surface of genus g?

Solutions

1. Since $\mathbb{R}^{n+1}$ is simply connected, M is orientable and separates $\mathbb{R}^{n+1}$ into two connected components having M as boundary. Exactly one of this is unbounded.

   If x and x' are in the same component then there is a path $\gamma\colon [0,1] \to \mathbb{R}^{n+1} \setminus M$ with $\gamma(0) = x$ and $\gamma(1) = x'$. Then $(y,t) \mapsto \sigma_{\gamma(t)}(y)$ is a homotopy between σ_x and σ'_x.

   Since the degree is a homotopy invariant and a map to a sphere is nul-homotopic iff it has degree 0, it now suffices to prove the implications $\Rightarrow$ in (b) and (c).

   Since M is compact, there exists $R > 0$ so that $M \subset B_R(0)$. Taking $x \in \mathbb{R}^{n+1}$ with $|x| > R$ we see that the map $\sigma_x\colon M \to S^n$ is not surjective and hence is nul-homotopic. This proves (b).

   Finally suppose x is in the bounded component and let $z \in S^n$ be a regular value of σ_x. Then $\sigma_x^{-1}(z)$ is a finite set $\{y_1, \ldots, y_k\} \subset M$ consisting of the points where the ray L parametrized by
   $$g(t) = x + tz, \quad t > 0$$
   intersects M. The assumption that z is a regular value is equivalent to the assertion that L intersects M transversally.

   Let $y_i = x + t_i z$ with $t_1 < \ldots < t_k$. Then $g(]0,t_1[)$ is contained in the bounded component, $g(]t_1,t_2[)$ in the unbounded component and so on and finally $g(]t_k,+\infty[)$ is contained in the unbounded component. It follows that k is odd.

   Give M and S^n the orientations induced from the standard orientations on $\mathbb{R}^{n+1}$. It now suffices to show that at a point y_i where L crosses from the bounded component to the unbounded component the local degree is $\deg_{y_i} \sigma_x = 1$ and that in the other case the degree is -1. This is a direct consequence of the definition of induced orientations in terms of an outward pointing vector.

2. Suppose that the given map $f\colon S^k \to E^*$ is nul-homotopic and let $H\colon S^k \times [0,1] \to E^*$ be a nul-homotopy. We can assume without loss of generality that $H(x,1) \notin M$ (where $M \subset E$ denotes the 0-section).

   The map f is transverse to M and by making H constant in a neighborhood of $t = 0$ we can assume H is transverse to M for small t. We can pick an approximation $K\colon S^k \times [0,1] \to E^*$ to H, relative to $S^k \times [0,\epsilon]$, such that K is transverse to M and $x \mapsto K(x,1)$ does not intersect M.

   Then $L = H^{-1}(M)$ is a compact neat 1-dimensional submanifold of $S^k \times [0,1]$ but $\partial L = f^{-1}(M) \times \{0\}$ consists of a single point, which is impossible.

3. Let Σ_g denote an orientable surface of genus g. Then $H_0(\Sigma_g) = H_2(\Sigma_g) = \mathbb{Z}$ and $H_1(\Sigma_g) = \mathbb{Z}^{2g}$. It follows that a CW-complex homotopy equivalent to Σ_g has at least $2g+2$-cells (since this is the minimum rank of a chain complex having the homology of Σ_g).

   Since Σ_g is homotopy equivalent to a cell complex having one cell for each critical point of a Morse function on Σ_g we see that a Morse function must have at least $2g+2$ critical points.

On the other hand, for a standard embedding of Σ_g in $\mathbb{R}^3$ (give picture), the height function is a non-degenerate Morse function with exactly $2g+2$ critical points (one maximum, one minimum and $2g$ critical points of index 1.