Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Topology Test 2

January 15, 2009

Duration: 1h30m.
Justify your answers.

1. Show that if M is any manifold then TM is an orientable manifold.

2. Let M and N be oriented manifolds without boundary, with $\dim M + \dim N = n$, embedded disjointly in $\mathbb{R}^{n+1}$. The *linking number* $\mathrm{Lk}(M, N)$ of M and N is defined as the degree of the map $M \times N \to S^n$ defined by

$$(x, y) \mapsto \frac{x - y}{|x - y|}$$

(a) Show that if there is an oriented manifold $W \subset \mathbb{R}^{n+1} \setminus N$ with boundary $M_1 \coprod -M_2$ then $\mathrm{Lk}(M_1, N) = \mathrm{Lk}(M_2, N)$.

(b) Let M and N be disjoint oriented embedded circles in $\mathbb{R}^3$ and $W = D^2 \subset \mathbb{R}^3$ be an embedded disk with $\partial W = M$. Show that $\mathrm{Lk}(M, N) = \sharp(W, N; \mathbb{R}^3)$.

Answer one of the following questions

3. Let $f \colon S^n \to \mathbb{R}$ be a Morse function satisfying $f(x) = f(-x)$. Show that f has at least 2 critical points of index k for each $k \in \{0, 1, \ldots, n\}$. *Hint: Recall that* $H_k(\mathbb{R}P^n; \mathbb{Z}/2) = \mathbb{Z}/2$ *for* $k = 0, 1, \ldots, n$.

4. The *bordism group* $\Omega_n(X)$ of a space X is formed by the equivalence classes $[f, M]$ where M is an oriented n-manifold, $f \colon M \to X$ is a continuous map and $(f, M) \sim (f', M')$ if there is an oriented $(n + 1)$-manifold W and a map $F \colon W \to X$ such that $\partial W \simeq M \coprod -M'$ and F restricts to $f \coprod f'$ on the boundary.

Prove that $\Omega_n(X)$ has a natural structure of abelian group and that a continuous map $g \colon X \to Y$ determines a group homomorphism $g_* \colon \Omega_n(X) \to \Omega_n(Y)$ depending only on the homotopy class of g.