

Knot Theory

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1 Introduction

In mathematics, *knot theory* is the study of knots. This field of topology focuses on issues such as

1. Given a tangled loop of string, is it really knotted or can it, with enough ingenuity and/or luck, be untangled without having to cut it?
2. More generally, given two tangled loops of string, when are they deformable into each other?
3. Is there an effective algorithm (or any algorithm to speak of) to make these determinations?¹

In this project, developed in the course of “Projecto em Matemática”, we present some basic concepts and results of knot theory. First of all, we introduce the subject, including important definitions and basic concepts. In the following section, we discuss the Reidemeister moves and their relation to equivalence between knots. Then, we develop some combinatorial techniques, including tricolorability and the linking number. In the following section, we discuss how to tabulate knots, i.e., how to describe projections and diagrams. The final section focuses on knot polynomials, one of the most important ways of telling knots apart.

¹Weisstein, Eric W. “Knot Theory.” From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/KnotTheory.html>

2 Basic Concepts

2.1 Definition of Knot

A knot is a *closed non-self-intersecting curve* embedded in \mathbb{R}^3 . In other words,

Definition 1. A knot is a simple closed polygonal curve in \mathbb{R}^3 , i.e., it is the union of the segments $[p_1, p_2], [p_2, p_3], \dots, [p_{n-1}, p_n]$ of an ordered set of distinct points (p_1, p_2, \dots, p_n) in which each segment intersects exactly two others.

2.2 Projection and diagram of a knot

In order to study knots, it is useful to consider projections of the knot on planes $P \subset \mathbb{R}^3$.

Definition 2. The image of a knot K under a projection map is called the projection of K .

In order to avoid losses of information, one should use *regular projections*, i.e.,

Definition 3. A knot projection is called a regular projection if

1. no three points on the knot project to the same point, and
2. no vertex projects to the same point as any other point in the knot.

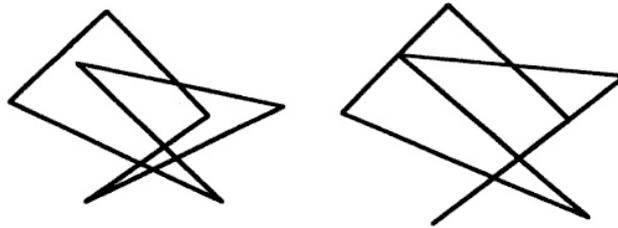


Figure 2.1: A regular projection and a non-regular projection of the same knot.

There is another loss of information when the projection map is applied: we can no longer see which portions of the knot pass over others parts. A *diagram* of a knot is the drawing of its regular projection in which are left gaps to remedy this fault. We call the arcs of this diagram *edges* and the points that correspond to two double points in the projection *crossings*. When we “travel” around the knot, if a portion of a knot passes over another part, we call it an *overcrossing* and if, on the other hand, a portion passes

under another, we call it an *undercrossing*. The diagram of a knot will be drawn as a smooth curve.²

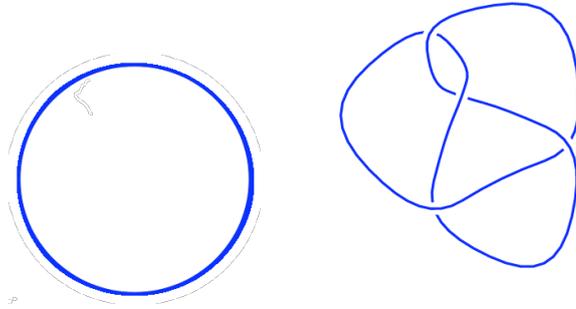


Figure 2.2: The unknot, with no crossings, and the figure-eight knot, with four crossings.

Definition 4. An oriented knot consists of a knot and an ordering of its vertices. Two orderings are called equivalent if they differ by a cyclic permutation.

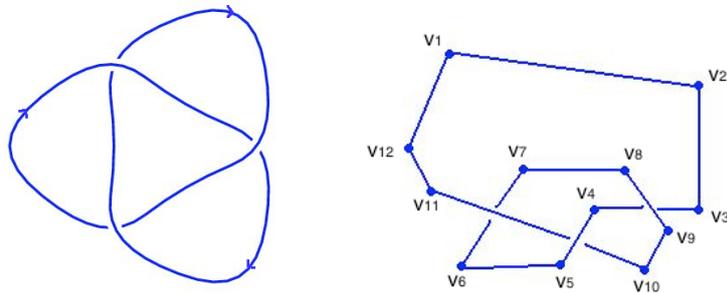


Figure 2.3: The diagram of a trefoil's projection and its drawing as a polygonal curve.

Definition 5. An alternating knot is a knot with a diagram that has crossings alternating between over and under, fixed an orientation.

Sometimes we will use in this paper the *Alexander-Briggs notation*. This consists of writing the number of crossings of a knot with a subscript to denote its archiving number within the class of knots with the given number of crossings. This numbering scheme can be found in the paper “On types

²It is our intuitive way of thinking about knots.

of knotted curve”, Annals of Mathematics, 28:562–586, 1926-1927 by James W. Alexander and G. B. Briggs. For instance, the trefoil will be denoted by 3_1 .

2.3 Links

In the study of knots, it is useful to consider objects where the number of loops knotted bigger than one. Taking this into account, it’s necessary to introduce the notion of *link*.

Definition 6. A link is a collection of disjoint knots.

A link is called *splittable* if it can be deformed so that its components lie in different sides of a plane in \mathbb{R}^3 . It is not easy to tell if a link is splittable but it is quite simple to distinguish some links by considering their number of components: if L has a different number of components than L' , they are not equivalent.

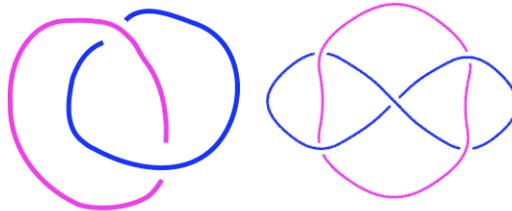


Figure 2.4: A link with two components (two unknots) and the Whitehead link.

2.4 Deformations

It is necessary to define more rigorously the notion of equivalence between two knots. First of all, it is necessary to define the notion of elementary deformations.

Definition 7. A knot J is called an elementary deformation of the knot K if one is determined by a sequence of points (p_1, p_2, \dots, p_n) and the other is determined by the sequence $(p_0, p_1, p_2, \dots, p_n)$, where

1. p_0 is a point which is not collinear with p_1 and p_n , and
2. the triangle spanned by (p_0, p_1, p_n) intersects the knot determined by (p_1, p_2, \dots, p_n) only in the segment $[p_1, p_n]$.

It must be noted that the second condition of this definition assures that the knot does not cross itself as it is deformed.

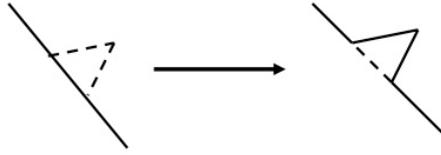


Figure 2.5: Illustration of an elementary deformation.

It is now possible to define *equivalence*:

Definition 8. Two knots K and J are called equivalent if there is a sequence of knots $K = K_0, K_1, \dots, K_n = J$, with each K_{i+1} an elementary deformation of K_i .

At this point we can define more precisely the objective of knot theory: the study of equivalence classes of knots. For instance, proving the existence of a non-trivial knot is the same as proving that there exists a knot not contained in the equivalence class of the unknot. More generally, proving that two knots are different is the same as proving that they lie in different equivalence classes.

Theorem 1. If two knots K and J have identical diagrams, then they are equivalent.

Proof. Let's assume that the knot K is represented by the ordered set (p_1, \dots, p_n) and the knot J is represented by the ordered set (q_1, \dots, q_n) . We may introduce new vertices, if necessary, in order to ensure that both knots have the same number of vertices. Therefore, since they have identical diagrams, we can perform a sequence of elementary deformations that transform (p_1, \dots, p_n) in (q_1, \dots, q_n) , as illustrated bellow.

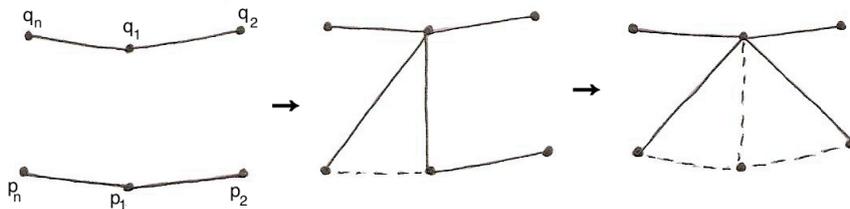


Figure 2.6

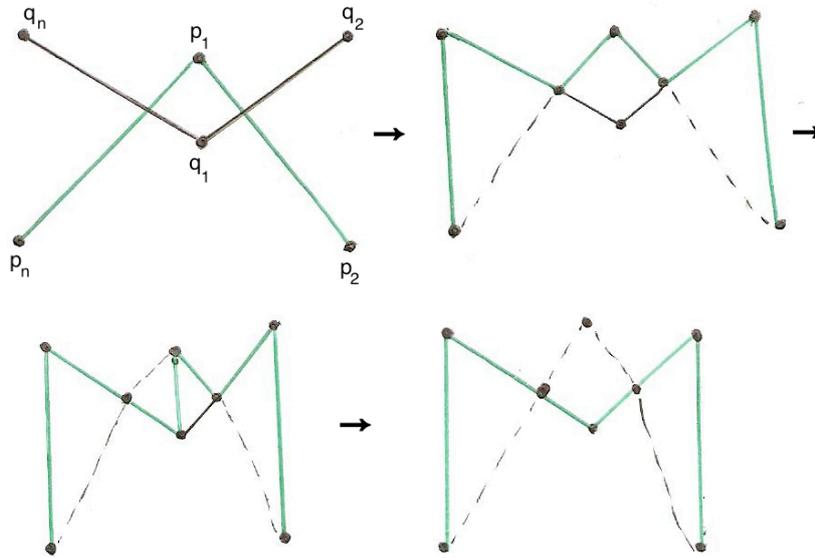


Figure 2.7: In this case it was necessary to introduce new vertices.

□

3 Reidemeister moves

One of the fundamental results of knot theory characterizes equivalence between knots in terms of their diagrams.

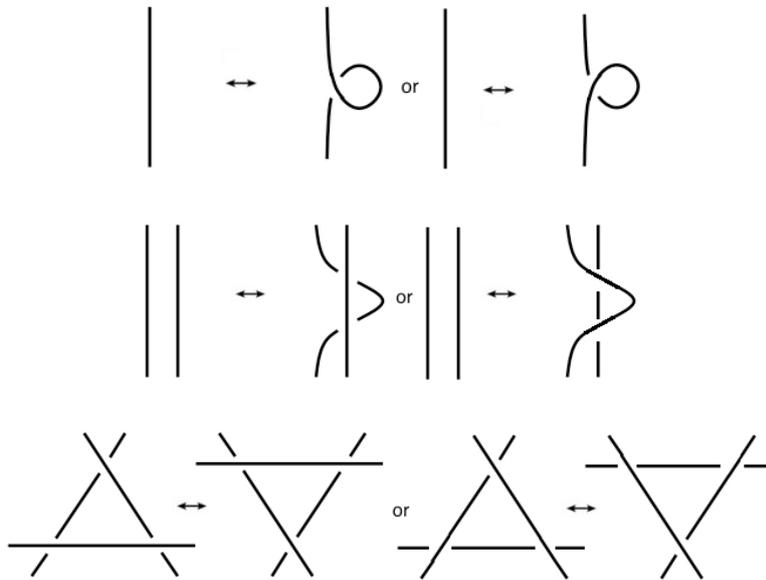


Figure 3.1: Reidemeister moves type I, II and III.

These operations can be performed on a knot's diagram without changing the knot itself.

Theorem 2. *Two knots or links are equivalent if and only if their diagrams are related by a sequence of Reidemeister moves.*

Proof. Let K and J be equivalent knots. Then, by definition 8, there is a sequence of knots $K = K_0, K_1, \dots, K_n = J$, with each K_{i+1} an elementary deformation of K_i . One can pick a projection which is regular for all the K_i . Then, without loss of generality, this proof can be reduced to the case of knots related by a single elementary deformation. Resorting, if necessary, to a small rotation we can ensure that the triangle along which the elementary deformation is performed is projected to a triangle in the plane (figure 2.5). This last triangle may contain many parts of the diagram knot, but it can be subdivided into smaller triangles so that each contains a single crossing of the diagram or a segment. In other words, we can view this elementary deformation as the composition of a series of other elementary deformations performed on smaller triangles. It is now necessary to check that only Reidemeister moves were applied between the two diagrams:

This is the case when the intersection with the deformation triangle is a segment and this segment is adjacent to the segments being deformed.

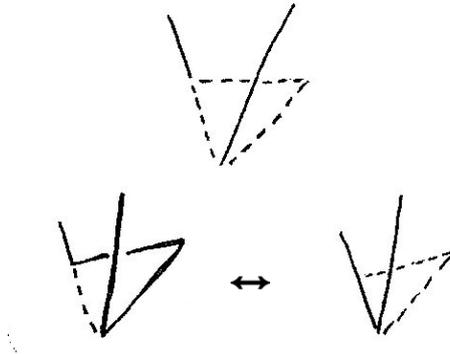


Figure 3.2: Reidemeister move type I. If we change the crossing, the process is similar.

This is the case when the intersection with the deformation triangle is a segment not adjacent to the segments being deformed.

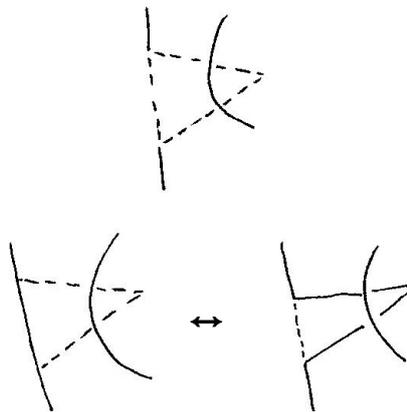


Figure 3.3: Reidemeister move type II. If we change the crossings, the process is similar.

This is the case when the intersection with the triangle is a crossing.

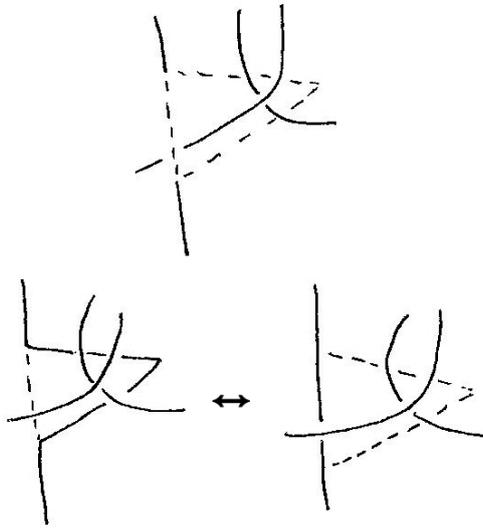


Figure 3.4: Reidemeister move type III. If we change the crossings, the process is similar.

□

When proving the equivalence of two knot diagrams one often uses *planar isotopies*. Informally, these are the deformations of a knot's diagram to another by stretching and bending the segments of the projection (never crossing other segments). Formally, they correspond to sequences of elementary deformations where the projection of the deformation triangle does not intersect the rest of the knot diagram.

Example 1 (Two diagrams representing the trefoil knot, and a sequence of Reidemeister moves relating them).³

³Exercise 1.10 of *The Knot Book*.

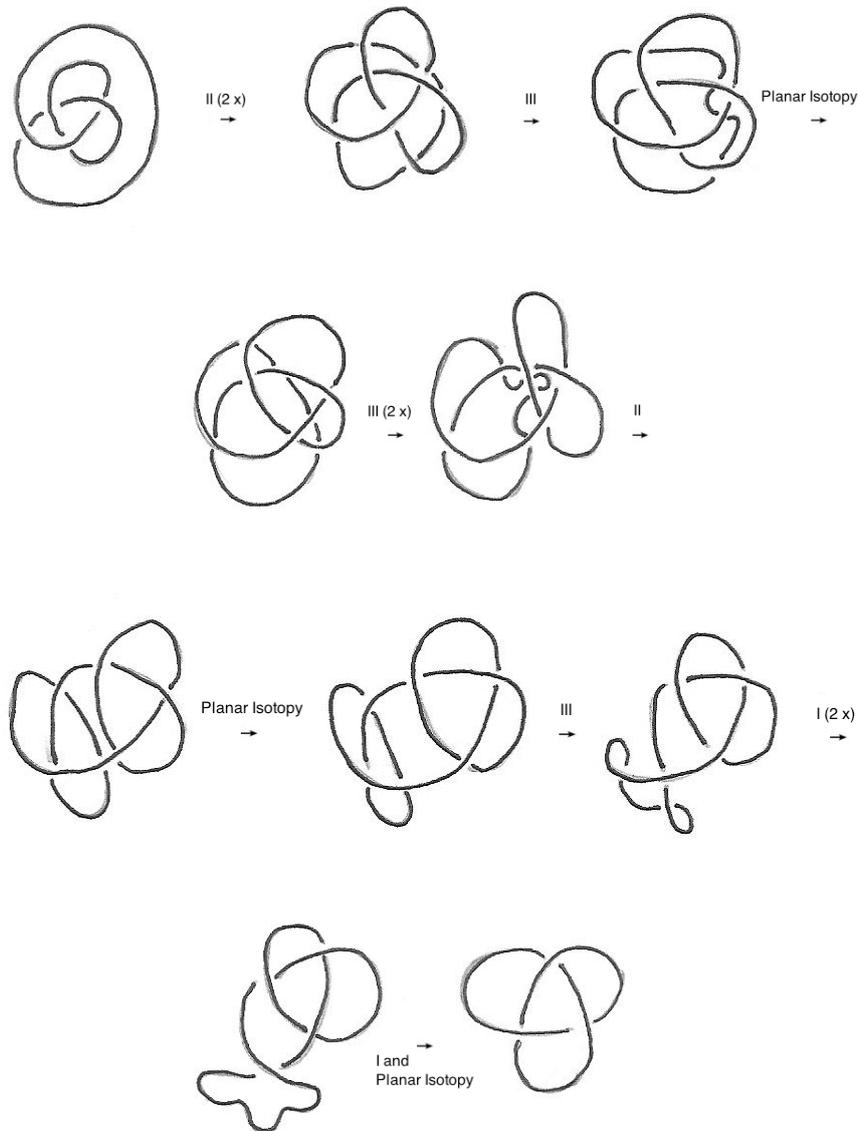


Figure 3.5: The first knot in this image is equivalent to the trefoil knot.

4 Combinatorial Techniques

4.1 Tricolorability

A fundamental question in knot theory is how to distinguish knots. In this section, we present very simple combinatorial techniques that sometimes allow us to achieve this goal.

The first combinatorial technique that we present is the method of *tricolorability*.

Definition 9. *A diagram of a knot or a link is tricolorable if each of the arcs can be colored in one of three different colors, so that at each crossing, either three different colors come together or all the segments have the same color.*

Once we prove that this property depends only on the equivalence class of the knot, it is possible to say, for instance, that the trefoil is different from the unknot. The unknot, according to the previous definition, is not tricolorable: it is only possible to color the trivial knot with one color. On the other hand, as seen in the figure 4.2, the trefoil is tricolorable.

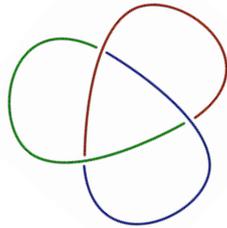


Figure 4.1: The trefoil knot is tricolorable.

Example 2. ⁴

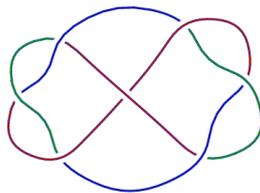


Figure 4.2: The knot 7_4 is tricolorable.

Theorem 3. *If a diagram of a knot K is tricolorable, then every diagram of K is tricolorable.*

⁴Exercise 1.22 of *The Knot Book*.

Proof. To prove this last result it is only necessary to show that performing a Reidemeister move - RM - on a tricolorable diagram doesn't change its tricolorability.

RM type I.

(\rightarrow) In this case, the original arc has one color. When the type I move is applied in this direction, a crossing is added and, as a result, there are now two arcs. Leaving these two arcs with the same color preserves tricolorability.

(\leftarrow) In this case, the two arcs have the same color. When the move is applied, the result is just one arc so there is nothing to check. The diagram is still tricolorable.

RM type II.

(\rightarrow) In this case, there are two options: either the two original arcs have the same color and when the type II move is applied we can just leave them with that color, or the two original arcs have two different colors, when the type II move is applied we color the new arc with the third color, as in figure 4.3.

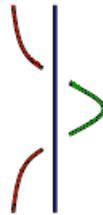


Figure 4.3: The colors chosen for this example are illustrative only.

(\leftarrow) In this case, when type II move is performed, we obtain two arcs that do not intersect. If there were only one color, then the two new arcs will preserve that color. If there were three different colors then the color of the segments on the left are the same. Therefore, the new arc on the left will preserve that color and the arc on the right will preserve the color of the segment in the middle. The diagram is still tricolorable.

RM type III. In this case, it is necessary to check five different cases:

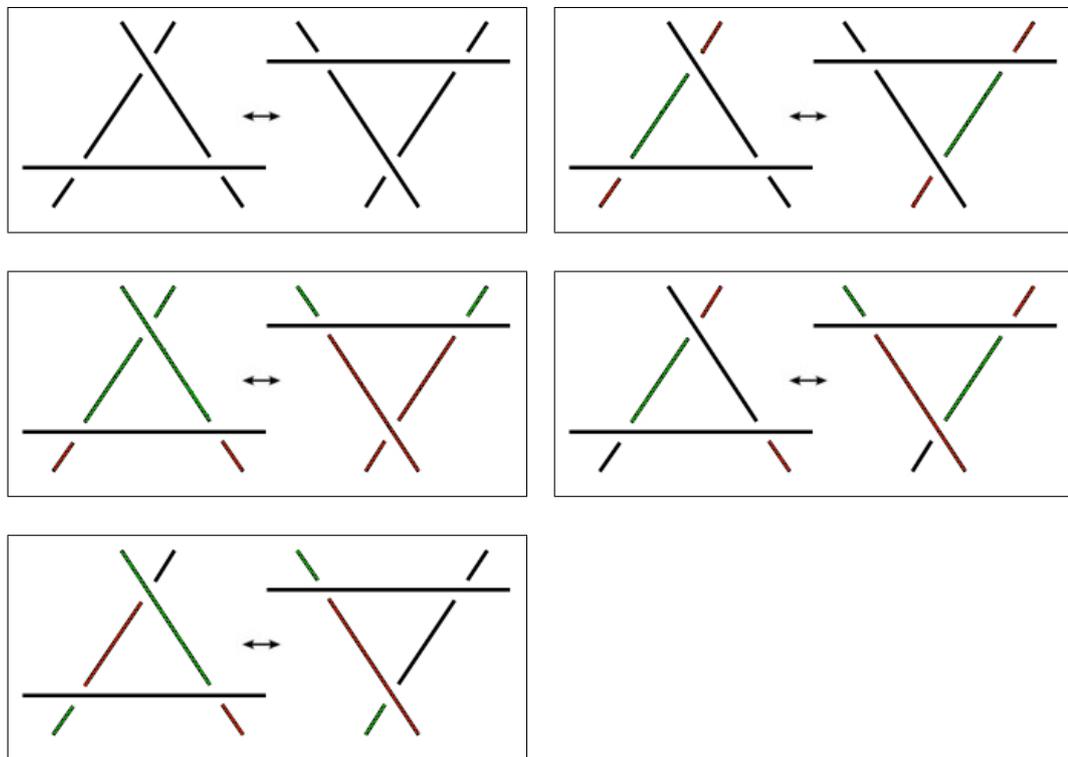


Figure 4.4: All this five scenarios prove that the resulting diagram after performing type III move is still tricolorable.

□

4.2 Linking number

Given an oriented link with two components, say W , we can calculate its *linking number* in order to “measure” how linked up this components are. In order to do this, we assign signs as follow: +1 to right-handed crossings and -1 to left-handed crossings.

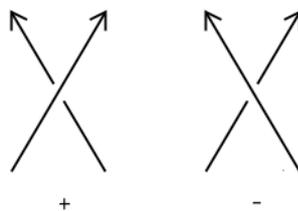


Figure 4.5: Right-handed crossing and left-handed crossing.

Definition 10. The linking number of W is defined to be the sum of the signs of the crossings where the two components meet, divided by 2.

Example 3.⁵

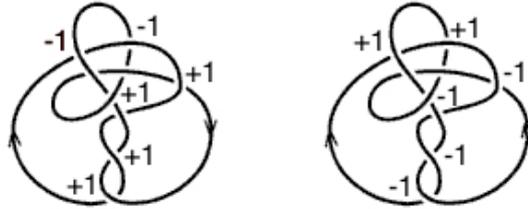


Figure 4.6: With the orientations on the left, this link has linking number $+1$. Reversing the orientation of one of the components one obtains the linking number -1 .

The linking number is an *invariant* of the oriented link. This is, fixed an orientation, the linking number is unchanged by ambient isotopy - the movement of the knot through the three-dimensional space without letting it pass through itself. Therefore, if two links have distinct linking number they must be different. However, we cannot tell apart two links with the same linking number. For example, the linking number of the Whitehead link is 0, so it is the same as the linking number of the trivial link with two components. However, it can be shown that the Whitehead link is not trivial. In order to prove that if two links have distinct linking number they must be different it is only necessary to show that performing a Reidemeister move won't change this number.

RM type I. This Reidemeister move doesn't affect in any way the linking number because the crossing that is added or removed is with the arc-itself.

RM type II.

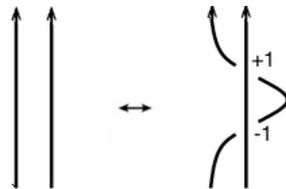


Figure 4.7

⁵Exercise 1.15 of *The Knot Book*.

(\rightarrow) When this move is applied, a +1 and a -1 are added. So, the linking number is not affected.

(\leftarrow) When this move is applied, a +1 and a -1 are subtracted. So, as before, the linking number is not affected.

RM type III. Performing this Reidemeister move in any direction doesn't change the number of +1s and -1s. So, the linking number isn't affected.

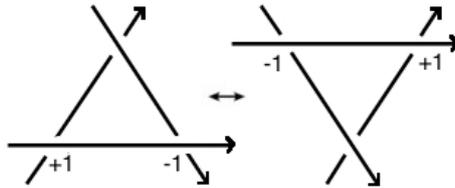


Figure 4.8

In figure 4.8, we assume that the horizontal segment belongs to a different component than the others. The other cases are similar.

It must be noted that if the orientation of one component is changed, the linking number of the knot is multiplied by -1 (as seen in example 3).

5 Tabulating Knots

5.1 The Dowker Notation for Knots

The Dowker notation is one way of describing a knot's projection. This method, in the case of alternating knots, can be described by the following algorithm:

1. Choose an orientation;
2. Choose an arbitrary crossing and label it 1;
3. Follow the undergoing strand to the next crossing and label it 2;
4. Continue to follow the same strand and label consecutively the crossings until each one of them has two numbers;
5. Each crossing will have an even and an odd number (we will explain the reason why later) in a total of $2n$ numbers; write in a row the odd numbers $1\ 3\ \dots\ 2n - 1$ and underneath it the corresponding even numbers.

The row with the even numbers is the knot's Dowker representation.

Example 4.⁶

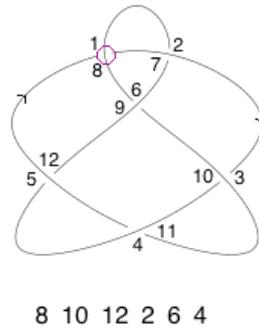


Figure 5.1: A sequence of integers that represents 6_2 .

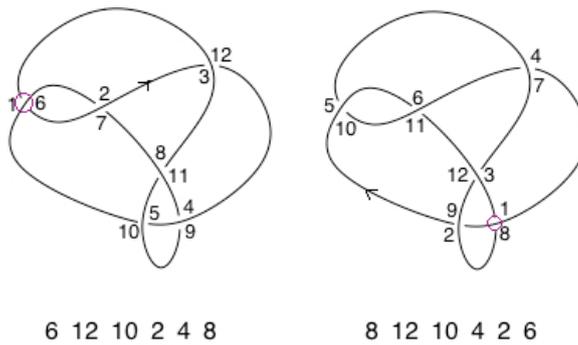


Figure 5.2: Two sequences of integers that represent 6_3 .

A set consisting of pairs of numbers is *drawable* if it fulfills the following condition:

Any two loops defined by a given interval must either share one or more segments or intersect in an even number of points, not including the initial and the end point.

It was noted above that every crossing has one even and one odd number.⁷ If it were not the case, i.e., if i and j were both even or both odd, then

⁶Exercise 2.3 of *The Knot Book*.

⁷Exercise 2.2 of *The Knot Book*.

we would have two loops, namely $i, i + 1, \dots, j - 1, j$ and $j, j + 1, \dots, 2n - 1, 2n, 1, \dots, i$ that do not share any segment but intersect in an even number of points including the initial and end points, in contradiction with the condition above.

Example 5.⁸

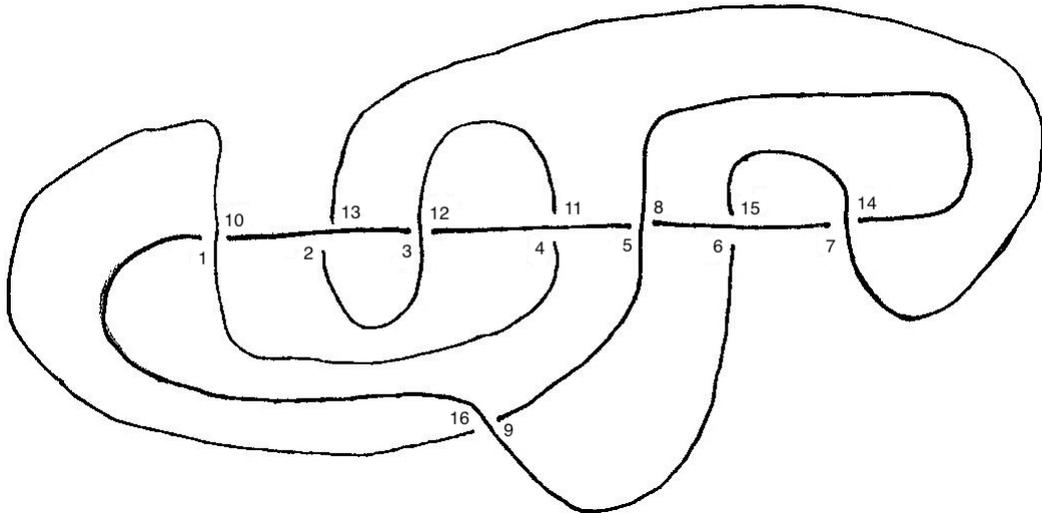


Figure 5.3: The alternating knot corresponding to the sequence 10 12 8 14 16 4 2 6.

We can give an upper bound on the number of possible alternating knot projections with, for example, seven crossings.⁹ This is given by counting how many sequences of integers 2 4 6 8 10 12 14 there are: 7!. The Dowker notation can be generalized to non-alternating knots. The procedure is the same as for alternating knots with the difference that, if a crossing is assigned an even integer on the understrand that number is given a negative sign. Otherwise, the even integer is left with a positive sign.

⁸Exercise 2.5 of *The Knot Book*.

⁹Exercise 2.7 of *The Knot Book*.

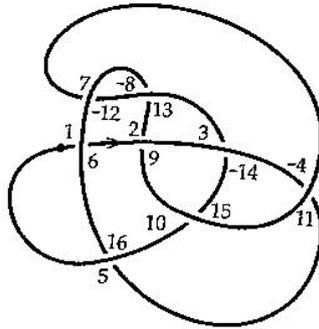


Figure 5.4: This non-alternating knot has the sequence 6 -14 16 -12 2 -4 -8 10.

We can now calculate an upper bound on the number of alternating and non-alternating knots with seven crossings. As it was seen before, there are $7!$ sequences corresponding to the alternating knots. As each even number may have a positive or a negative sign, there are $7!2^7$ possibilities. It must be noted that, in fact, there aren't these many knots: there are sequences that represent the same knot and sequences which are not possible, as discussed above.

5.2 Conway's Notation

Definition 11. A tangle in a knot (or link) projection is a region in the projection plane surrounded by a circle such that the knot (or link) crosses the circle exactly four times.

The equivalence between tangles can be defined by

Definition 12. Two tangles are said to be equivalent if they are related by a sequence of Reidemeister moves taking place in the interior of the circle.

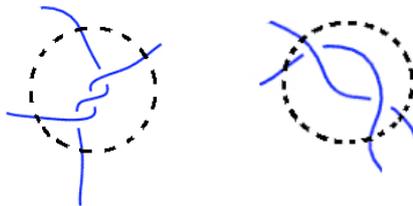


Figure 5.5: Tangles.

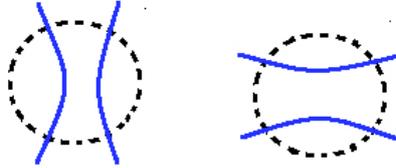


Figure 5.6: The ∞ tangle and the 0 tangle.

An n -tangle is a tangle with n twists in which the overstrand has always positive slope (if we think of it as a small line segment in the plane). In the same way, in an $-n$ -tangle, the overstrand always has negative slope.

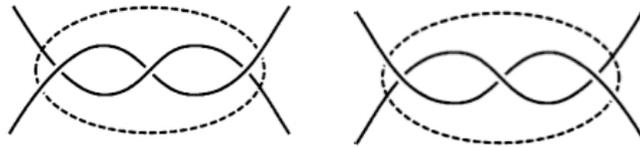


Figure 5.7: 3-tangle and -3 -tangle

The *multiplication* of two tangles, say T_1 and T_2 , is defined as illustrated in figure 5.8¹⁰: first T_1 is reflected across its NW to SE diagonal line and then it is “glued” to T_2 . This operation is not associative.

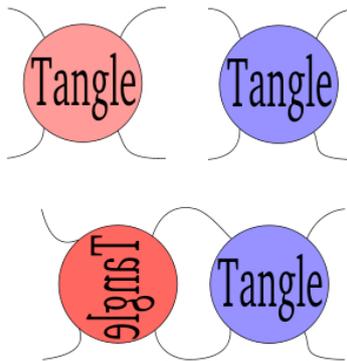


Figure 5.8

The operation of *addition* of tangles is defined as illustrated in 5.9¹¹

¹⁰http://en.wikipedia.org/wiki/File:Tangle_Operations.svg

¹¹http://en.wikipedia.org/wiki/File:Tangle_Operations.svg

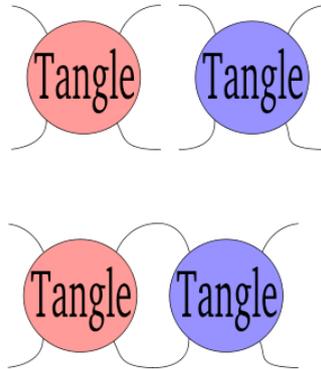


Figure 5.9

A *rational tangle* is a multiplication of $\pm n$ -tangles in the order $((((t_1 t_2) t_3) t_4) \dots t_n)$.

The 0 tangle is an additive identity for tangles¹². There is, also, a left multiplicative identity which is the ∞ tangle. However, there isn't a right multiplicative identity.¹³ We can show this by contradiction: Suppose that there exists a right multiplicative identity, say I_r . So, the product of the 0 tangle with I_r would have to be 0. By definition of multiplication, the first thing to do is to reflect the 0 tangle across its NW to SE diagonal. The result is the ∞ tangle. Now we have to "glue" this two tangles. It is impossible to obtain as a result the 0 tangle as we can see in figure 5.10.

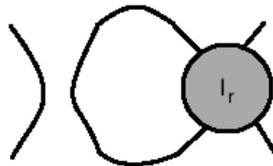


Figure 5.10

A rational tangle $t_1 t_2 \dots t_n$ can be assigned the following continued fraction

$$t_n + \frac{1}{t_{n-1} + \frac{1}{\dots + \frac{1}{t_2 + \frac{1}{t_1}}}}$$

¹²Exercise 2.21 of *The Knot Book*.

¹³Exercise 2.22 of *The Knot Book*.

Theorem 4. *Two tangles in Conway's notation are equivalent if and only if the rational number obtained from the corresponding continued fraction is the same.*

Example 6. ¹⁴

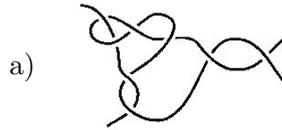


Figure 5.11: -2 -2 -2 -2 -2

The respective continuous fraction:
$$-2 + \frac{1}{-2 + \frac{1}{-2 + \frac{1}{-2 + \frac{1}{-2}}}} = -\frac{29}{12}$$

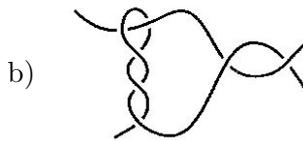


Figure 5.12: 2 -3 2

The respective continuous fraction:
$$2 + \frac{1}{-3 + \frac{1}{2}} = \frac{8}{5}$$

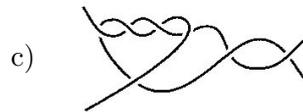


Figure 5.13: -4 1 2

The respective continuous fraction:
$$2 + \frac{1}{1 + \frac{1}{-4}} = \frac{10}{3}$$

¹⁴Exercise 2.13 of *The Knot Book*.



Figure 5.14: 1 1 1 1 1

The respective continuous fraction: $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}} = \frac{8}{5}$

We can conclude, by theorem 4, that tangles b) and d) are the same.

An *algebraic tangle* is a tangle obtained by the operations of addition and multiplication on rational tangles. For instance, the knot 8_5 can be written as $(3 \times 0) + (3 \times 0) + (2 \times 0)$.

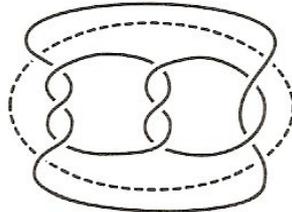


Figure 5.15: 8_5 has Conway notation 3, 3, 2.

For knots with few crossings, we can describe them in this notation. However, Conway's notation is not necessarily valid to every knot.

6 Polynomials

The polynomials that we will present in this section are Laurent polynomials, which can have both positive and negative powers.

We will describe this polynomials in terms of skein relations, i.e.,

Definition 13. A skein relation is an equation that relates the polynomial of a knot (or link) to the polynomial of knots (or links) obtained by changing the crossings in a projection of the original link.

One can prove that the value of the polynomial obtained by applying the skein relation is independent of the choice of the order of the crossings.

A *resolving tree* is the process of repeatedly choosing a crossing, applying the skein relations and obtain two simpler links.

Example 7 (The resolving tree of the trefoil).

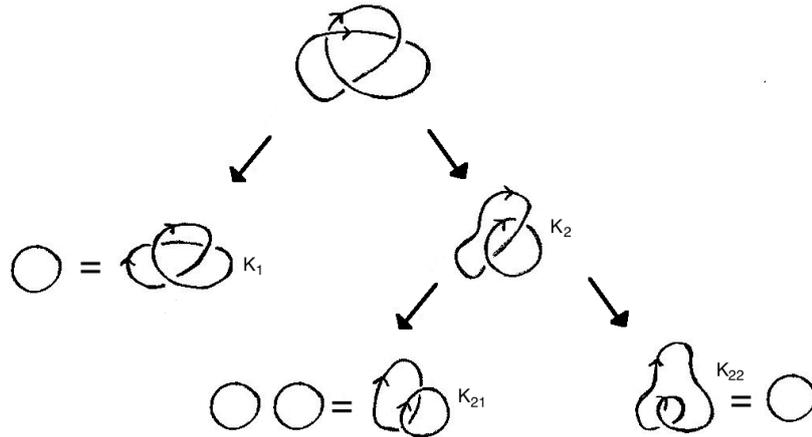


Figure 6.1

6.1 The Bracket Polynomial and the Jones Polynomial

The rules to calculate the bracket polynomial of L , $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$:

- Rule 1: $\langle \bigcirc \rangle = 1$
- Rule 2:

$$\langle \begin{array}{c} \diagup \\ \diagdown \end{array} \rangle = A \langle \begin{array}{c} \text{---} \\ \text{---} \end{array} \rangle + A^{-1} \langle \bigcirc \bigcirc \rangle$$

$$\langle \begin{array}{c} \diagdown \\ \diagup \end{array} \rangle = A \langle \bigcirc \bigcirc \rangle + A^{-1} \langle \begin{array}{c} \text{---} \\ \text{---} \end{array} \rangle$$

- Rule 3: $\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2}) \langle L \rangle$

Example 8 (The bracket polynomial of the trivial link of n components.).
¹⁵

By rule 3, we have
 $\langle \bigcirc \cup \bigcirc \rangle = (-A^2 - A^{-2}) \langle \bigcirc \rangle = (-1)(A^2 + A^{-2})1.$

¹⁵Exercise 6.1 of *The Knot Book*.

The bracket polynomial of the usual projection of the trivial link of n components will be

$$\langle \bigcirc \cup \bigcirc \cup \dots \cup \bigcirc \rangle = (-1)^{n-1} (A^2 + A^{-2})^{n-1}.$$

Example 9 (The bracket polynomial of the trefoil).¹⁶

¹⁶Exercise 6.2 of *The Knot Book*.

$$\begin{aligned}
\langle \text{G} \rangle &= A \langle \text{G} \rangle + A^{-1} \langle \text{G} \rangle = \\
&= A (A \langle \text{G} \rangle + A^{-1} \langle \text{G} \rangle) \\
&\quad + A^{-1} (A \langle \text{G} \rangle + A^{-1} \langle \text{G} \rangle) \\
&= A (A (A \langle \text{G} \rangle + A^{-1} \langle \text{G} \rangle)) \\
&\quad + A (A^{-1} (A \langle \text{G} \rangle + A^{-1} \langle \text{G} \rangle)) \\
&\quad + A^{-1} (A (A \langle \text{G} \rangle + A^{-1} \langle \text{G} \rangle)) \\
&\quad + A^{-1} (A^{-1} (A \langle \text{G} \rangle + A^{-1} \langle \text{G} \rangle)) = \\
&= A (A (A (A^2 + A^{-2})^2 + A^{-1} (- A^2 - A^{-2}))) \\
&\quad + A (A^{-1} (A (- A^2 - A^{-2}) + A^{-1})) \\
&\quad + A^{-1} (A (A (- A^2 - A^{-2}) + A^{-1})) \\
&\quad + A^{-1} (A^{-1} (A + A^{-1} (- A^2 - A^{-2}))) = \\
&= A^7 - A^3 - A^{-5}
\end{aligned}$$

This polynomial is invariant under the type II Reidemeister move

$$\begin{aligned}
 \langle \text{II} \rangle &= A \langle \text{II} \rangle + A \langle \text{II} \rangle = \\
 A(A \langle \text{II} \rangle + A^{-1} \langle \text{II} \rangle) + A(A^{-1} \langle \text{II} \rangle + A \langle \text{II} \rangle) &= \\
 A(A \langle \text{II} \rangle + A^{-1}(-A^2 - A^{-2}) \langle \text{II} \rangle) + & \\
 + A^{-1}(A \langle \text{II} \rangle + A \langle \text{II} \rangle) &= \langle \text{II} \rangle
 \end{aligned}$$

and the type III Reidemeister move.

$$\begin{aligned}
 \langle \text{III} \rangle &= A \langle \text{III} \rangle + A^{-1} \langle \text{III} \rangle \\
 &= A \langle \text{III} \rangle + A^{-1} \langle \text{III} \rangle = \langle \text{III} \rangle
 \end{aligned}$$

However, it is not invariant under the type I Reidemeister move type I because

$$\begin{aligned}
 \langle \text{I} \rangle &= A \langle \text{I} \rangle + A^{-1} \langle \text{I} \rangle \\
 &= A(-A^2 - A^{-2}) \langle \text{I} \rangle + A^{-1} \langle \text{I} \rangle \\
 &= -A^3 \langle \text{I} \rangle
 \end{aligned}$$

Therefore, the bracket polynomial is not a knot invariant.

From the bracket polynomial we can compute the *Jones polynomial* which can be proved to be an actual invariant of knots. So, in order to this, we must define

Definition 14. The writhe of an oriented link, denoted by $w(L)$, is the sum of all the signs of its crossings.

The *writhe* of a link is invariant under the Reidemeister II and III; the proof the invariance of the writhe is similar to the proof of the invariance of the linking number. However, the Reidemeister move changes $w(L)$ by +1 or -1.

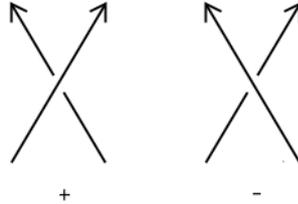


Figure 6.2: Signs +1 and -1 respectively.

Definition 15. The X polynomial is a polynomial of oriented links and it is defined by $X(L) = (-A^3)^{-w(L)} \langle L \rangle$.

This polynomial is invariant under Reidemeister moves II and III since $\langle L \rangle$ and $w(L)$ are unaffected by them. It is, also, invariant under Reidemeister move type I because

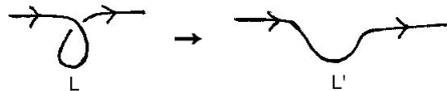


Figure 6.3: L' and L , respectively.

$$\begin{aligned}
 X(L') &= \\
 &= (-A^3)^{-w(L')} \langle L' \rangle \\
 &= (-A^3)^{-(w(L)+1)} \langle L' \rangle \\
 &= (-A^3)^{-(w(L)+1)} ((-A)^3 \langle L \rangle) \\
 &= (-A^3)^{-w(L)} \langle L \rangle \\
 &= X(L)
 \end{aligned}$$

Example 10 (The X polynomial of the trefoil).¹⁷ Let L denote the trefoil knot.

$$X(L) = (-A)^3 \langle L \rangle = -A^9(A^7 - A^3 - A^{-5}) = -A^{16} + A^{12} + A^4$$

The *Jones Polynomial* is the same as the X polynomial. But, as a matter of tradition, we make the following substitution

Definition 16. The *Jones polynomial*, denoted by $V_L(t)$, is obtained from the X polynomial replacing A by $t^{-1/4}$.

¹⁷Exercise 6.5 of *The Knot Book*.

Example 11 (The Jones polynomial of the trefoil).¹⁸ $V_{trefoil}(t) = -t^{-4} + t^{-3} + t^{-1}$.

6.2 The Alexander Polynomial

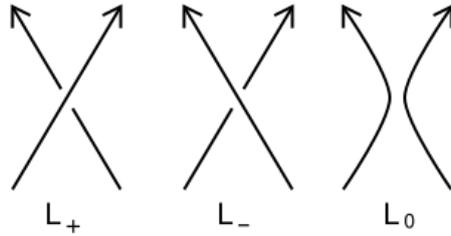


Figure 6.4: Diagrams appearing in the skein relations for the Alexander polynomial.

The rules to calculate the Alexander polynomial for a knot K , $\Delta(K)$:

- Rule 1: $\Delta(\bigcirc) = 1$
- Rule 2: $\Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0) = 0$

Example 12 (The Alexander polynomial of the figure-eight).¹⁹

$$\Delta(\text{figure-eight with crossing}) - \Delta(\text{figure-eight with crossing}) + (t^{1/2} - t^{-1/2})\Delta(\text{figure-eight without crossing}) = 0$$

In order to calculate the Alexander polynomial of the link with coefficient $(t^{1/2} - t^{-1/2})$, we have

¹⁸Exercise 6.7 of *The Knot Book*.

¹⁹Exercise 6.14 of *The Knot Book*.

$$-\Delta(\text{diagram 1}) + \Delta(\text{diagram 2}) + (t^{1/2} - t^{-1/2}) \Delta(\text{diagram 3}) = 0$$

$$\Delta(\text{diagram 1}) = 0 + (t^{1/2} - t^{-1/2})$$

So, the Alexander polynomial of the figure-eight knot is

$$\Delta(\text{figure-eight knot}) - 1 + (t^{1/2} + t^{-1/2})^2 = 0$$

$$\Delta(\text{figure-eight knot}) = 3 - t - t^{-1}$$

There is another way of finding the Alexander polynomial.

1. Number the n arcs of an oriented knot;
2. Number, separately, the n crossings;
3. Define an $n \times n$ matrix in the following way:
 - a. If the crossing numbered l is as figure 6.5 (1), then enter $1 - t$ in column i of row l ; enter a -1 in column j of row l ; and enter a t in column k of the same row.
 - b. If the crossing numbered l is as figure 6.5 (2), then enter $1 - t$ in column i of row l ; enter a t in column j of row l ; and enter a -1 in column k of the same row.
 - c. If any two of i, j or k are equal, the sum of the entries described above is put in the appropriate column: for example, if $j = k$ in a left-handed crossing, we must enter in column j the input $j + k = t - 1$.
 - d. All the remaining entries of row l are 0.

If we remove the last row and the last column of the matrix described above, we obtain the *Alexander matrix*.

Definition 17. *The determinant of the Alexander matrix is called the Alexander polynomial.*

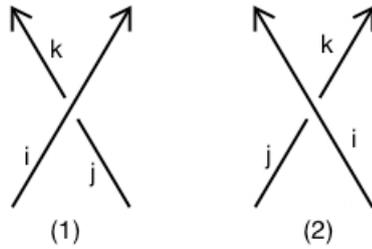
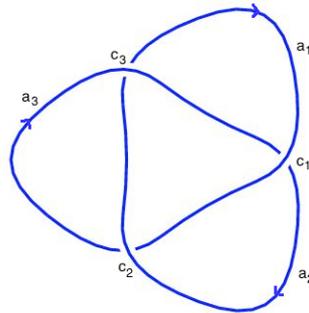


Figure 6.5

Theorem 5. *If the Alexander polynomial for a knot is computed using two different sets of choices for diagrams and labelings, the two polynomials will differ by a multiple of $\pm t^k$, for some integer k .*

Example 13 (The Alexander polynomial of the trefoil computed by the Alexander matrix).



The corresponding matrix is

$$\begin{pmatrix} 1-t & t & -1 \\ -1 & 1-t & t \\ t & -1 & 1-t \end{pmatrix}$$

The determinant of the matrix resulting from deleting the bottom row and the last column is the wanted polynomial: $\Delta(\text{trefoil}) = t^2 - t + 1$.

Example 14 (The Alexander polynomial of the trefoil computed by skein relation). Taking into account figure 6.1, we have

$$\begin{aligned} \Delta(\text{trefoil}) - \Delta(K_1) + (t^{1/2} - t^{-1/2})\Delta(K_1) &= 0 \\ \Delta(\text{trefoil}) &= 1 - (t^{1/2} - t^{-1/2})(\Delta(K_{21}) - (t^{1/2} - t^{-1/2})\Delta(K_{22})) \\ \Delta(\text{trefoil}) &= 1 - (t^{1/2} - t^{-1/2})(0 - (t^{1/2} - t^{-1/2})) = -1 + t + t^{-1} \end{aligned}$$

This last polynomial differs by a multiple of $+t$ from the polynomial computed in example 13: $t^2 - t + 1 = (+t)(-1 + t + t^{-1})$.

References

- [1] Adams, Colin C. 2004. *The Knot Book*. Providence, Rhode Island: American Mathematical Society.
- [2] Livingston, C. 1993. *Knot Theory*. Washington, DC: The Mathematical Association of America.
- [3] Aneziris, Charilaos. 1997 *Computer Programs for Knot Tabulation*, <http://arxiv.org/abs/q-alg/9701006v1>.