Group stacks in geometry

Behrang Noohi
(notes taken by Christoph Wockel)

June 5, 2009

I 2-group theory

groups = symmetries of objects in a category

higher structures $\rightsquigarrow$ 2-groups = symmetries of objects in a 2-category

Example I.1. For a stack $\mathcal{X}$, the self-equivalences $\mathrm{Aut}(\mathcal{X})$ are a groupoid, together with a monoidal structure, which one can think of as a group objects in groupoids. ■

Definition I.2. A strict 2-group is a

- group objects in groupoids
- groupoid object in groups
- 2-groupoid with one object

In particular, it is a monoidal category. ■

One can weaken

- associativity by introducing "associators"
- identity by introducing "unitators"
- inverses by introducing "invertors"

In the above example, the composition is associative on the nose and the unit is one the nose, but the inverses are only weak.

Example I.3.
- $\mathcal{X}$ a stack $\rightsquigarrow$ $\mathrm{Aut}(\mathcal{X})$
- $\mathcal{P} \to \mathcal{X}$ a (Hitchin gerbe) "principal bundle for a 2-group (in this case BS^1)"
- G reductive algebraic group (over alg. closed field of char. 0), $\widetilde{G} \to (G, G)$ (where $\widetilde{G}$ is the universal cover) then $\widetilde{G} \to G$ is a crossed module $\rightsquigarrow$ $[G/\widetilde{G}]$ is a Picard stack, the *stacky abelianization of* G (is important in the theory of character sheaves, was generalised by M. Kamgarpour) ■

Definition I.4. A morphism $f : \mathbb{G} \to \mathbb{H}$ of 2-groups is a weak monoidal functor (between the underlying monoidal categories), i.e. there exists natural isomorphisms $f(xy) \xrightarrow{\eta_{x,y}} f(x)f(y)$ (satisfying some coherence condition). A 2-morphism $f \Rightarrow g$ is a 2-monoidal transformation between monoidal functors. ■

Fact: Most 2-groups and morphisms appearing in nature are weak.

Good news: Every weak 2-group $\mathbb{G}$ is "equivalent" to a strict 2-group.

Bad news: A weak morphism $f : \mathbb{G} \to \mathbb{H}$ between strict 2-group is in general not *not* equivalent to a strict morphism.

Main Problem (for this talk): Understand the groupoid of weak morphisms $\mathrm{Weak}(\mathbb{G}, \mathbb{H})$ for $\mathbb{G}$ and $\mathbb{H}$ (not necessarily strict 2-groups).

Strategy:

- Find crossed-module modules for $\mathbb{G}$ and $\mathbb{H}$.
- Use butterflies.

Definition I.5. A *crossed module* is a group homomorphism $[G_1 \xrightarrow{\partial} G_0]$, together with a right action $G_1 \curvearrowleft G_0$, such that

- $\partial(\alpha^g) = g^{-1}\alpha g$
- $\alpha^{\partial\beta} = \beta^{-1}\alpha\beta$ ∎

Example I.6.
- G arbitrary $\Rightarrow [1 \to G]$
- A abelian $\Rightarrow [A \to 1]$
- A, B abelian $\Rightarrow [A \to B]$
- G arbitrary $\Rightarrow [G \to \mathrm{Aut}(G)]$
- $G \curvearrowright V \Rightarrow [V \xrightarrow{1} G]$ ∎

I.1 Crossed modules are 2-groups and vice versa

Construction:

- $\mathbb{G}$ a strict 2-group $\curvearrowleft$ set $G_0 = \mathrm{Ob}(\mathbb{G})$ and $G_1 = \mathrm{Mor}(1_{\mathbb{G}}, \cdot)$
- if $[G_1 \to G_0]$ is a crossed module, then we obtain a strict 2-group by setting $G_0 \rtimes G_1$ to me the morphisms and G_0 to be the objects. Then the set pr_1 to be the source map and $(g, \alpha) \mapsto g\partial\alpha$ to be the target map (this is the action groupoid of G_1, acting on G_0 by ∂). Then the group multiplication on objects and morphisms defines a 2-group structures on this groupoid.

Theorem I.7. *The above construction can be extended to give an equivalence between the 2-category of 2-groups and the 2-category of crossed modules (with the appropriate detinition of morphisms and 2-morphisms of crossed modules).*

Notation: $\pi_0([G_1 \xrightarrow{\partial} G_0]) := \mathrm{coker}(\delta)$ (a group), $\pi_1([G_1 \xrightarrow{\partial} G_0]) = \ker(\partial)$ (abelian). If $\mathbb{G}$ is a weak 2-group, then $\pi_0(\mathbb{G})$ is the group of isomorphism classes and $\pi_1(\mathbb{G})$ is $\mathrm{Aut}(1_{\mathbb{G}})$.

Fact: The above equivalence respects π_0 and π_1.
From now on: use the same notation for crossed modules and strict 2-groups (identified via the above theorem).

Definition I.8. A morphism $f : \mathbb{G} \to \mathbb{H}$ is an *equivalence* if the induced maps on π_0 and π_1 are isomorphisms (note that this may not have a strict inverse!). ∎

What is a weak morphism of crossed modules $\mathbb{G} \to \mathbb{H}$? It consists of maps $p_i : G_i \to H_i$ and $G_0 \times G_0 \to H_1$ of pointed sets, that satisfy

- $p_0(\partial\alpha) = \partial p_0(\alpha)$
- $p_1(\alpha\beta) = p_1(\alpha)p_1(\beta)^{\epsilon(\partial\alpha,\partial\beta)}$
- $p_0(xy) = p_0(x)p_0(y)\partial(\epsilon(x,y))$
- cocycle condition: $\epsilon(x,y)^{p_1(z)}\epsilon(xy,z) = \epsilon(y,z)\epsilon(x,yz)$
- equivariance: $\epsilon(x^{-1},x)p_1(\alpha^x) = p_1(\alpha)^{p_1(x)}\epsilon(\partial\alpha,x)\epsilon(x^{-1},\partial x)$

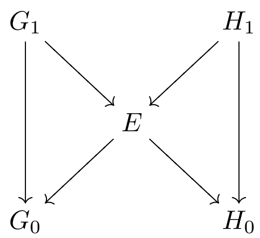
(this is what you get from using the identification of crossed modules with strict 2-groups and rephrase the notion of a weak morphism of 2-groups w.r.t. this identification) This is practically useless (is in particular wrong if $\mathbb{G}$ and $\mathbb{H}$ are topological or Lie group (or algebras))!

II Crossed modules vs. group stacks

Fix a Grothendieck site S and do everything over S (phrase everything in terms of sheaves over S). If $\mathbb{G} = [G_1 \to G_0]$ is a crossed module, then we obtain a quotient stack $[\mathbb{G}] := [G_0/G_1]$ (a group stack over S). If $f : \mathbb{G} \to \mathbb{H}$ is a morphism (over S) of crossed modules, then we obtain a morphism $[f] : [\mathbb{G}] \to [\mathbb{H}]$.

Bad news: Not every morphism $[\mathbb{G}] \to [\mathbb{H}]$ comes from $f : \mathbb{G} \to \mathbb{H}$.

Definition II.1. A *butterfly* $B : \mathbb{G} \to \mathbb{H}$ is a commutative diagam



such that the NE-SW sequence is short exact (+compatibility of actions). A morphism of butterflies is an isomorphism $E \to E'$ commuting with the four maps. ∎

Example II.2. Strict morphisms give rise to butterflies. ∎

Theorem II.3. *A butterfly $B : \mathbb{G} \to \mathbb{H}$ induces a morphism of group stacks $[B] : [\mathbb{G}] \to [\mathbb{H}]$. Furthermore, $B \mapsto [B]$ induces an equivalence of groupoids*

$$\mathrm{Butterfly}(\mathbb{G},\mathbb{H}) \cong \mathrm{Weak}(\mathbb{G},\mathbb{H})$$

Example II.4.
- $\mathbb{G}$ associated to $1 \to G$, $\mathcal{A}$ associated to $1 \to A$ for A abelian. Then butterflies are the same as central extensions of G by A.
- $\mathbb{H} = [H \to \mathrm{Aut}(H)]$, $\mathbb{G}$ associated to $1 \to G \rightsquigarrow$ butterflies are uniquely determined by the exact NE-SW sequence end thus are the same as arbitrary extensions of G by H ∎

II.1 The bicategory of butterflies

$\mathcal{CM}_S$: bicategory with objects crossed modules, morphisms butterflies and 2-morphisms morphisms of butterflies. Define composition by taking the fiber product of and kill H_1 inside it $(E_1 \times_{H_0} E_2)/H_1$. The homotopy fiber of the butterfly is determined by the NW-SE sequence

Corollary II.5. *A butterfly is an equivalence iff the NW-SE sequence is also exact.*

Theorem II.6 (Aldrovandi-Noohi). *We have biequivalences*

- $\mathcal{CM}_S \cong \mathrm{GrSt}_S$
- $\mathrm{Br}\,\mathcal{CM}_S \cong \mathrm{BrGrSt}_S$
- $\mathrm{Pic}\,\mathcal{CM}_S \cong \mathrm{PicSt}_S$

Theorem II.7. *These are biequivalent to a "localization" of the corresponding strict bicategory w.r.t. equivalences.*

Corollary II.8 (Deligne).

$$D^{[-1,0]}(\mathrm{Ab}_S) \cong \mathrm{PicSt}_S, \quad (A \to B) \mapsto [B/A]$$

II.2 Applications to group stack actions on stacks

Definition II.9. Let Γ be a group stack and $\mathcal{X}$ be a stack. An action of Γ on $\mathcal{X}$ is an equivalence class of a weak morphisms $\Gamma \to \mathrm{Aut}(\mathcal{X})$ (i.e. an object in $\mathrm{Weak}(\Gamma, \mathrm{Aut}(\mathcal{X}))/\sim$). ■

Assume Γ is a group (for simplicity). **Recall:** our strategy was

- Find a crossed module model for $\mathrm{Aut}(\mathcal{X})$
- use butterflies

Example II.10. Let $\mathcal{X} = \mathbb{P}(n_1, ..., n_k)$ be a weighted projective stack. This is the quotient stack of the weighted diagonal action of $\mathbb{G}$ on $\mathbb{A}^n - \{0\}$ via $\lambda \mapsto \lambda^{n_1,\dots,n_k}$ (where $\mathbb{G}$ is the multiplicative group scheme and $\lambda^{n_1,\dots,n_k}$ is the diagonal matrix with $\lambda^{n_1}, ..., \lambda^{n_k}$ on the diagonal). In this case, $\mathrm{Aut}(\mathcal{X})$ can be "computed". Let $G_{n_1,...,n_k} \subseteq \mathrm{GL}(k)$ be the centralizer of the matrices $\lambda^{n_1,\dots,n_k}$. Define $\mathrm{PGL}(n_1, ..., n_k) = [\mathbb{G}_m \to G_{n_1,...,n_k}]$, induced by $\lambda \mapsto \lambda^{n_1,\dots,n_k}$ (trivial action). ■

Theorem II.11 (Behrend–Noohi). $[\mathrm{PGL}(n_1, ..., n_k)] \cong \mathrm{Aut}(\mathbb{P}(n_1, ..., n_k))$

Corollary II.12. *To give an action of* Γ *on* $\mathbb{P}(n_1, ..., n_k)$ *is the same as giving*

- *a central extension* $1 \to \mathbb{C}^\times \to E \to \Gamma \to 1$
- *a linear representation* $E \xrightarrow{\rho} \mathrm{GL}(k)$ *such that* $\rho|_{\mathbb{G}_m} = \partial$ *and images of* ρ *and* ∂ *commute*

up to an appropriate equivalence relation (conjugation on ρ*).*

II.3 Application to group cohomology with coefficients in a crossed module

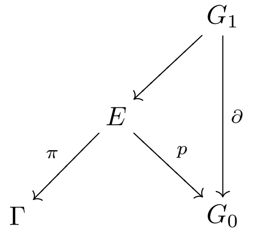
(Dedecker, Breen, Borovoi, Granada school)
$\mathbb{G} = [G_1 \to G_0]$ a crossed module, T a discrete group, acting strictly on $\mathbb{G}$. $\rightsquigarrow$ definition of H^{-1}, H_0, H^1 is always possible, for H^2 only works for a braiding,...

Borovoi's definition (of H^1): of $H^1(\Gamma, \mathbb{G}) = \{1-\text{cocyles}\}/\sim$, where a 1-cocycle is a pair (p, ε) such that

- $p : \Gamma \to G_0$ and $\varepsilon : \Gamma \times \Gamma \to G_1$ are pointed set maps
- $p(\sigma\tau) \cdot \partial\varepsilon(\sigma, \tau) = p(\sigma) \cdot {}^\sigma p(\tau)$
- $\varepsilon(\sigma, \tau\nu) \cdot {}^\sigma \varepsilon(\tau, \nu) = \varepsilon(\sigma\tau, \nu) \cdot \varepsilon(\sigma, \tau)^{{}^{\sigma\tau}p(\nu)}$

and $\sim$ is some cumbersome equivalence relation.

The butterfly definition: A 1-cocycle is a diagram



such that p is a set map such that $p(xy) = p(x) \cdot^{\pi(x)} p(y)$ (a crossed homomorphism) and p respects the actions. Two such diagrams are equivalent if one is isomorphic to the conjugate of the other by some $g \in G_0$.

II.4 Application to classification of forms of algebraic stacks

Let $\mathcal{X}$ be an algebraic stack over a field K. Assume $\mathcal{X}$ is nice (so that we have descent theory). We want to classify all $\mathcal{Y}$ such that $\mathcal{Y} \times_K \bar{K} \cong \mathcal{X} \times_K \bar{K}$.

Required input: as before, we need a crossed module model $\mathbb{G}$ for $\mathrm{Aut}(\mathcal{X})$. Then, $\mathbb{G}(\bar{K})$ will be a $\mathrm{Gal}(\bar{K}/K)$-equivariant crossed module. We have

$$\{\text{forms of } \mathcal{X}\} \cong H^1(\mathrm{Gal}(\bar{K}/K), \mathbb{G}(\bar{K})).$$

The right hand side can be computed using butterfly 1-cocycles.

Example II.13 (Brauer-Severi stacks). By definition, these are forms of $\mathbb{P}(n_1, ..., n_k)$. We have

$$\{\text{forms of } \mathbb{P}(n_1, ..., n_k)_K \text{ trivialized over } F\} \cong H^1(\mathrm{Gal}(F/K), \mathrm{PGL}(n_1, ..., n_k)(F)).$$

So, a Brauer-Severi stack is determined by the following data:

- a central extension $1 \to F^* \to E \xrightarrow{\pi} \mathrm{Gal}(F/K) \to 1$ such that the conjugation action of E on F^* is compatible with the action of $\mathrm{Gal}(F/K)$ on F^*
- a crossed homomorphism $\rho : E \to \mathrm{GL}(k, F)$ whose image commutes with matrices $\lambda^{n_1,\dots,n_k}$ (note that E acts on $GL(k, F)$ via π).

Two such data give rise to the same Brauer-Severi stack iff they are "conjugate" via some $g \in \mathrm{GL}(k, F)$, that is, $\rho'(x) = g^{-1}\rho(x) \cdot^{\pi(x)} g$. ■