The homotopy theory of diffeological spaces

Dan Christensen
Enxin Wu

Lisbon
31 May 2011

Outline:

- Diffeological spaces
- Naive homotopy theory of diffeological spaces
- Towards a model category of diffeological spaces

Motivation

(Smooth) manifolds contain a lot of geometric information: tangent spaces, differential forms, de Rham cohomology, etc.

These can be put to great use in proving results.

But the category of manifolds is not closed under many useful constructions, such as subspaces, quotients and function spaces.

The category Top of topological spaces is closed under these operations, but is missing the geometric information.

Can we have the best of both worlds?

Manifolds

Let M and N be manifolds.

Fact. A map $M \to N$ is smooth iff $U \to M \to N$ is smooth for every n, every open $U \subseteq \mathbb{R}^n$ and every map $U \to M$.

This is clear for manifolds without boundary (use the charts), but is even true for manifolds with boundary. Kriegl-Michor '97

We can use this to make the following definition.

Diffeological spaces

Definition (J. Souriau, 1980)**.** A diffeological space is a set X together with a specified family of maps $U \to X$ (called plots) for each open $U \subseteq \mathbb{R}^n$ and each n such that for every open $U, V \subseteq \mathbb{R}^n$:

(1) Every constant map $U \to X$ is a plot;
(2) If $U \to X$ is a plot and $V \to U$ is smooth, then $V \to U \to X$ is a plot;
(3) If $U = \cup_i U_i$ is an open cover and $U \to X$ is a map such that each restriction $U_i \to X$ is a plot, then $U \to X$ is a plot.

Definition. For diffeological spaces X and Y, a function $X \to Y$ is smooth if for every plot $U \to X$ in X, $U \to X \to Y$ is a plot in Y.

Definition. The category of diffeological spaces and smooth maps is denoted Diff.

Properties of Diff

The category of manifolds is a full subcategory of Diff:

- Each manifold M is a diffeological space: a map $U \to M$ is declared to be a plot iff it is smooth in the usual sense.
- By the Kriegl-Michor result, a function $M \to N$ between manifolds is smooth in the usual sense iff it is smooth as a map of diffeological spaces.

Diff is closed under limits and colimits:

- For example, if Y is a subset of a diffeological space X, we can declare a function $U \to Y$ to be a plot if the composite $U \to Y \to X$ is a plot.
- For Y is a quotient of a diffeological space X, say that $U \to Y$ is a plot if locally it is of the form $U_i \xrightarrow{\text{plot}} X \to Y$.
- If X and Y are diffeological spaces, say that $U \to X \times Y$ is a plot if each of $U \to X$ and $U \to Y$ is.

Properties of Diff, II

The category of manifolds is a full subcategory of Diff.

Diff is closed under limits and colimits.

Diff is cartesian closed: write $\mathrm{Diff}(X, Y)$ for the set of smooth maps from X to Y. This is a diffeological space with a function $U \to \mathrm{Diff}(X, Y)$ defined to be a plot if the natural map $U \times X \to Y$ is smooth. One can show that

$$\mathrm{Diff}(W \times X, Y) \to \mathrm{Diff}(W, \mathrm{Diff}(X, Y))$$

is a diffeomorphism.

Diff is locally presentable. This is a technical result that comes in handy. Note that Top is not locally presentable.

Examples of diffeological spaces

For any diffeological space X, the loop space $\mathrm{Diff}(S^1, X)$ and the path space $\mathrm{Diff}(I, X)$ are diffeological spaces. Such examples are a big motivation for generalizing manifolds.

If X is formed from two squares glued along an edge, a smooth curve $\mathbb{R} \to X$ that crosses the edge must stop for a finite amount of time. We call this a border crossing.

It is also possible for a diffeological space to have a stop sign, in which case a smooth curve need only halt for an instant.

One can put a non-standard diffeology on the unit interval, so that a smooth map to a manifold M must have all derivatives zero at the endpoints. This allows one to compose paths in M as one does in topology, without separately imposing a stationarity condition.

Aside: Sheaf-theoretic point of view

Diff can be viewed as the category of concrete sheaves on the concrete site of open subsets of $\mathbb{R}^n$, smooth maps, and open covers, where we think of a diffeological space X as a functor sending U to $X(U) := \{\text{plots } U \to X\}$. Baez-Hoffnung '98

Roughly, this means that all objects have an underlying set, and that $X(U)$ is naturally a subset of the set of functions from U to X.

It follows that Diff is a quasi-topos, which is like a topos except that the subobject classifier classifies the strong monomorphisms.

Aside: Alternative approaches

In his study of iterated integrals and loop spaces, Chen '77 used closed, convex sets instead of open sets. A very similar theory results.

Using a "maps out" approach, Sikorski '72 defined a differentiable space to be a topological space X equipped with a set of functions $X \to \mathbb{R}$ which form a sheaf on X and are closed under post-composition with smooth functions $\mathbb{R}^n \to \mathbb{R}$.

Frölicher '82 defined a smooth space to be a set X along with specified functions $\mathbb{R} \to X$ and $X \to \mathbb{R}$ subject to a compatibility condition.

There are other variations as well, nicely summarized in Stacey '08.

Aside: Geometry and characteristic classes

Tangent spaces, differential forms, de Rham cohomology, connections, curvature, etc. can be defined for diffeological spaces [Laubinger '06, Iglesias-Zemmour] and for the alternative approaches as well.

Mostow '79 shows that if G is a Lie group, then the classifying space BG is naturally a differentiable space.

Moreover, if $E \to M$ is a G-bundle with smooth transition functions w.r.t. a smooth partition of unity, the resulting classifying map $M \to BG$ is smooth.

Finally, Mostow shows that explicit forms representing characteristic classes can be found in $\Omega^*(BG)$ which pull back under $M \to BG$ to ordinary differentiable forms on M. This unifies the Chern-Weil approach to characteristic classes with the homotopy-theoretic approach.

Homotopy theory of diffeological spaces

In his thesis, Iglesias-Zemmour '85 defined the (smooth) homotopy groups $\pi_n^s(X, x_0)$ of a diffeological space X inductively, using loop spaces.

Proposition (C.-Wu)**.** All definitions you might make agree.

(The subtleties involve stationarity conditions.)

As usual, π_0^s is a set, π_1^s is a group, and π_n^s is abelian for $n \geq 2$.

For a manifold, the smooth and topological homotopy groups agree.

Every diffeological space has a natural topology, but we will see that in general the smooth and topological homotopy groups do not agree.

Diffeological bundles

Iglesias-Zemmour '85

Definition. A diffeological bundle is a smooth surjective map $X \to Y$ such that the pullback along any plot in Y is locally trivial.

Theorem. If $X \to Y$ is a diffeological bundle with fibre F, then there is a long exact sequence

$$\cdots \to \pi_n^s(F) \to \pi_n^s(X) \to \pi_n^s(Y) \to \cdots \to \pi_0^s(Y).$$

Theorem. If H is any subgroup of a diffeological group G, then $G \to G/H$ is a diffeological bundle with fibre H.

Example. Let H be a line of irrational slope in the torus T^2 and let $A = T^2/H$. Then $T^2 \to A$ is a diffeological bundle. From the long exact sequence, $\pi_1^s(A) = \mathbb{Z} \oplus \mathbb{Z}$. But $\pi_1(A) = 0$.

All of this and much more is detailed in Iglesias-Zemmour's book in progress, which is freely available online.

Model categories

Quillen '67

A model category is a category $\mathcal{C}$ with specified classes of maps called the weak equivalences, cofibrations and fibrations, such that:

MC1: $\mathcal{C}$ has all limits and colimits.

MC2: If two of f, g and gf are weak equivalences, then so is the third.

MC3: The specified maps are closed under retracts.

MC4: If we are given a solid square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

with i a cofibration, p a fibration, and one of them a weak equivalence, then a lift exists.

MC5: Any map can be factored as pi with i a cofibration and a weak equivalence and p a fibration, and as qj with j a cofibration and q a fibration and a weak equivalence.

Why model categories?

Model categories are a setting in which general techniques from homotopy theory can be used. In fact, they unify ideas from homotopy theory and homological algebra, showing, for example, that projective resolutions and CW approximations are the same concept.

Every model category $\mathcal{C}$ has a homotopy category $\mathrm{ho}(\mathcal{C})$ in which the weak equivalences have been inverted. In some cases, just the fact that the homotopy category exists is reason enough to construct a model category.

Model categories also allow one to define derived functors, and these derived functors can be used to construct equivalences of the associated homotopy categories. This is why they were first introduced by Quillen, for his work on rational homotopy theory.

In general, it is difficult to show that the axioms hold.

Why a model category of diffeological spaces?

A model category of diffeological spaces would allow the standard techniques of homotopy theory to be applied to these generalized spaces, while still allowing the use of smooth methods such as are used in differential topology.

For example, the long exact sequence in homotopy groups for a diffeological bundle should follow automatically from the fact that we have a model category.

Moreover, even for manifolds, the hope is that ad hoc techniques such as using stationary paths or putting submanifolds into general position would fall out naturally from the model category point of view.

A possible model structure on Diff

[Work in progress with my student, Enxin Wu. Many variations are being considered as well.]

Let $\mathbb{A}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1\}$, with the subdiffeology. Then $\mathbb{A}^n \cong \mathbb{R}^n$, but the $\mathbb{A}^n$'s behave formally like the simplices Δ^n: they form a cosimplicial object.

For formal reasons, we get an adjoint pair

$$|-| : \mathrm{sSet} \leftrightarrows \mathrm{Diff} : S$$

called geometric realization and the (smooth) singular simplicial set.

We have $|\Delta^n| = \mathbb{A}^n$, that $|-|$ preserves colimits and that $S(X)_n = \mathrm{Diff}(\mathbb{A}^n, X)$.

A possible model structure on Diff

We define a map $X \to Y$ in Diff to be a weak equivalence (fibration) if $S(X) \to S(Y)$ is a weak equivalence (fibration) in sSet.

A map $X \to Y$ is a cofibration if it has the left lifting property with respect to the maps which are both weak equivalences and fibrations.

We haven't yet completed the proof that these form a model category, but we have some partial results.

Fibrant objects

An important first step is to study the fibrant objects. By definition, these are the diffeological spaces such that $S(X)$ is a fibrant simplicial set, i.e. a Kan complex.

Why is this important?

Proposition (E. Wu)**.** If X is fibrant, then $\pi_*^s(X) \cong \pi_*(SX)$.

Corollary. If X and Y are fibrant, then a map $X \to Y$ is a weak equivalence iff $\pi_*^s(X) \to \pi_*^s(Y)$ are isomorphisms for all basepoints.

Unfortunately, not all objects are fibrant. For example, if X is built from two copies of $\mathbb{R}$ with the origins identified, then X is not fibrant.

Fibrant objects, II

Since a simplicial group is always a fibrant simplicial set, it follows that a diffeological group is always fibrant.

Lemma (E. Wu)**.** A diffeological bundle with fibrant fibre is a fibration.

Proposition (E. Wu)**.** If $X \to Y$ is a diffeological bundle with X fibrant, then Y is fibrant.

Corollary (E. Wu)**.** For H any subgroup of a diffeological group G, the homogeneous space G/H is fibrant.

Corollary (E. Wu, P. Iglesias-Zemmour)**.** Any manifold M without boundary is fibrant.

Proof. The diffeological group $\mathrm{Diff}(M, M)$ acts transitively on M, so M is a homogeneous space. (Fix $x_0 \in M$. Then $M \cong \mathrm{Diff}(M, M)/\mathrm{Diff}_{x_0}(M, M)$.) □

Discuss. . .

Questions

We don't know whether manifolds with boundary are fibrant. We aren't even sure about the unit interval.

The geometric realization $|K|$ of any simplicial set is cofibrant, but we don't know whether other diffeological spaces (such as manifolds) are cofibrant.

We have an outline of a proof that Diff is a model category, but the details are extremely technical.

Conclusions

Diff is a rich category. By working in a category closed under constructions like loop spaces, automorphism groups, etc., one can often simplify and unify arguments.

Many geometric and homotopical properties of Diff are understood, but a framework for these results is not yet in place.