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COURSE B:

"Weighted Problems for Operators of Harmonic Analysis in some Banach Function Spaces"

In Section 1 the reader can find an enlarged version of Abstracts of the course. Section 2 contains an introductory material for listeners of the course.

1 Contents of lectures 1-3

1.1 *Lectures 1-2*: Singular Integrals and Potentials in Weighted Grand Lebesgue Spaces

Abstracts

In the first two lectures we will discuss the weighted boundedness problems for various integral operators in generalized grand Lebesgue spaces $L_w^{p),\theta}$. Namely, the complete description of weight functions governing one-weight inequalities for the Riesz transform and the Cauchy singular integral will be done. As a corollary of one of the mentioned results we obtain that the Cauchy singular integral on a rectifiable curve Γ is bounded in generalized grand Lebesgue space $L^{p),\theta}$, $1 < p < \infty$, $\theta > 0$, if and only if Γ is a Carleson curve.

Exploring mapping properties of fractional integrals defined on a bounded domain it is proved that for the Riesz potential I_α on a finite interval and pair of spaces $(L^{p),\theta}, L^{q),\theta})$ $1 < p < q < \infty$, $\theta > 0$ with the same positive parameter θ the well-known Sobolev's embedding is fails. Nevertheless, for given θ it is possible to characterize those θ_1 for which the mapping $L^{p),\theta} \to L^{q),\theta_1}$ is saved. Moreover, the complete description of the weight functions w for which the operator $f \to I_\alpha(fw^\alpha)$ is bounded from $L_w^{p),\theta}$ to $L_w^{q),\theta_1}$ is obtained. The similar results for one-sided potentials are also valid.

Finally, we introduce so called grand Morrey space and explore mapping properties of maximal functions, Calderon-Zygmund singular integrals and potentials given on the bounded set of quasi-metric measure space.

1.2 *Lecture 3*: Two-weighted Norm Inequalities in Variable Exponent Lebesgue Spaces

Abstract

The goal of this lecture is to discuss two-weight problems in variable exponent Lebesgue spaces for double Hardy transform and strong maximal functions of variable order. For similar problems in classical Lebesgue spaces we refer the reader to the following books and paper:

[Saw] E. T. Sawyer, Weighted inequalities for two-dimensional Hardy operator, *Studia Math.* **82** (1985), No.1, 1-16.

[Kok-Me-Pe] V. Kokilashvili, A. Meskhi and L. E. Persson. Weighted norm inequalities for integral transforms with product kernels. *Math. Research Dev. Ser. Nova*, 2009.

[We] A. Wedestig. Weighted inequalities of Hardy type and their limiting inequalities, *PhD Thesis, Department of Mathematics, Luleå University of Technology*, 2003.

[We] A. Wedestig. Weighted inequalities for the Sawyer two-dimensional Hardy operator and its limiting geometric mean operator. *J. Inequal. Appl.*, **4**(2005), 387-394.

[Me] A. Meskhi. A note on two-weight inequalities of multiple Hardy-type operators. *J. Funct. Spaces Appl.*, **3**(2005), 223-237.

[Ush] E. Ushakova. Norm inequalities of Hardy and Pólya-Knopp types. *PhD, Department of Mathematicas, Luleå University of Technology*, 2003.

[Kok-Me] V. Kokilashvili and A. Meskhi. Two-weight estimates for strong fractional maximal functions and potentials with multiple kernels. *J. Korean Math. Soc.*, **46**(2009), No. 3, 523-550.

Our interest to the above mentioned problems is stipulated by two circumstances: by needs in various applications to boundary problems in PDE and by the fact that the strong maximal function, unlike of the Hardy-Littlewood maximal function is bounded in $L^{p(\cdot)}$ space if and only if $p(x) \equiv \text{const}$ (see [T. Kopaliani. A note on strong maximal operator in $L^{p(\cdot)}(R^n)$ spaces. Proc. A. Razmadze Math. Inst. 145 (2007), 4346.]). A similar phenomenon occurs for the strong fractional maximal function.

The present lectures are based on the following papers:
[Kok-Me-1] V. Kokilashvili and A. Meskhi, A note on the boundedness of Hilbert transform in weighted grand spaces, Georgian Math. J. 16 (2009), 547-551

[Kok-1] V. Kokilashvili. Boundedness Criterion for the Cauchy Singular Integral Operator in Weighted Grand Lebesgue Spaces and Application to the Riemann Problem, 151 (2009), 129-133

[Me-2] A. Meskhi, Maximal Functions and Singular Integrals in Morrey Spaces Associated with Grand Lebesgue Spaces, 151 (2009), 139-143

[Kok-2] V. Kokilashvili, Singular integrals in generalized grand Lebesgue space with weight. Manuscript (2010)

[Me-1] A. Meskhi, Fractional integrals in generalized grand Lebesgue space. Manuscript (2010)

[Kok-Me-2] V. Kokilashvili and A. Meskhi, Integral operators in some nonstandard Banach Function Spaces. Manuscript (2010)

Some historical remarks

The celebrated classical Hardy inequality states:

Theorem A. *Let $1 < p < \infty$ and let f be a measurable, nonnegative function in $(0, \infty)$. Then*

$$\left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(y) dy \right)^{\frac{1}{p}} \right) \leq \frac{p}{p-1} \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}}.$$

Note at once that weighted Hardy transform in variable exponent Lebesgue spaces was explored in papers [1]-[13].

Two-weighted boundedness criteria for the Hardy transform

$$(H_1 f)(x) = \int_0^x f(y) dy, \quad f \geq 0$$

reads as

Theorem B. *Let $1 < p \leq q < \infty$, and let u and v be weight functions on $\mathbb{R}_+$. Then each of the following conditions are necessary and sufficient for the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(t) dt \right)^q v(x) dx \right)^{\frac{1}{q}} \leq C \left(\int_0^\infty f^p(x) w(x) dx \right)^{\frac{1}{p}} \tag{1.1}$$

to hold for all positive and measurable functions on $\mathbb{R}_+$.

a) The Muckenhoupt condition,

$$A_M := \sup_{x>0} \left(\int_x^\infty v(t) dt \right)^{\frac{1}{q}} \left(\int_0^x w(t)^{1-p'} dt \right)^{\frac{1}{p'}} < \infty. \tag{1.2}$$

Moreover, the best constant C in (1) can be estimated as follows:

$$A_M \leq C \leq \left(1 + \frac{q}{p'} \right)^{\frac{1}{q}} \left(1 + \frac{p'}{q} \right)^{\frac{1}{p'}} A_M.$$

b) The condition of L. E Persson and V. D. Stepanov,

$$A_{PS} := \sup_{x>0} W(x)^{-\frac{1}{p}} \left(\int_0^x v(t) W(t)^q dt \right)^{\frac{1}{q}} < \infty, \quad W(x) := \int_0^x w(t)^{1-p'} dt.$$

Moreover, the best constant C in (1) *satisfies the following estimates:*

$$A_{PS} \le C \le p' A_{PS}.$$

E. T. Sawyer found a characterization of two-weight inequality in terms of four condition for double Hardy transform

$$(H_2 f)(x,y) = \int_0^x \int_0^y f(t,\tau) dt d\tau$$

from L^p_w to L^q_v, $1 < p \le q < \infty$.

The following statements gives two-weight criteria in terms of just one condition when the weight on the right-hand side is a product of two weights of single variables.

Theorem C. *let $1 < p \le q < \infty$ and let $w(x,y) = w_1(x) w_2(y)$. Then the operator H_2 is bounded from L^p_w to L^q_v $(1 < p \le q < \infty)$ if and only if the Muckenhoupt's type condition*

$$\sup_{y_1, y_2 > 0} \left(\int_{y_1}^{\infty} \int_{y_2}^{\infty} v(x_1, x_2) dx_1 dx_2 \right)^{\frac{1}{q}} \left(\int_0^{y_1} \int_0^{y_2} w(x_1, x_2)^{1-p'} dx_1 dx_2 \right)^{\frac{1}{p'}} := A_1 < \infty$$

is fulfilled.

One of our goals is to establish two-weight estimates condition for double Hardy transform in variable exponent Lebesgue spaces when the weight on one hand-side is a product of weights of single variables.

From this result as a corollary we deduce a trace inequality criteria for double Hardy transform when the exponent of the initial Lebesgue space is a constant. The other remarkable corollary is that there exists a variable exponent $p(x)$ for which the double average operator is bounded in $L^{p(\cdot)}$.

Recall that the strong Hardy-Littlewood maximal function is bounded in $L^{p(\cdot)}$ if and only if $p(x) \equiv \text{const}$.

Further we focus on two-weighted problem for fractional maximal function of variable order. In particular, we establish a trace inequality criteria for this operator in variable exponent Lebesgue spaces.

In the paper [Kok-Me] [V. Kokilashvili and A. Meskhi. Two-weight estimates for strong fractional maximal functions and potentials with multiple kernels. *J. Korean Math. Soc.*, **46**(2009), $\mathrm{N}_0 3$, 523-550], for constant exponent there was established the trace inequality criteria.

Let

$$(M_{\alpha,\beta}f)(x,y) := \sup_{I\times J\ni(x,y)} \frac{1}{|I|^{1-\alpha}|J|^{1-\beta}} \int_I \int_J |f(t,\tau)|dtd\tau.$$

Theorem D. *Let $1 < p < q < \infty$ and let $0 < \alpha, \beta < 1/p$. Then the following statements are equivalent:*

(i) *$M_{\alpha,\beta}$ is bounded from $L^p(\mathbb{R}^2)$ to $L^q_v(\mathbb{R}^2)$;*

(ii)

$$B_5 := \sup_{I,J} \left(\iint\limits_{I\times J} v(x,y)dxdy \right) |I|^{q(\alpha-1/p)}|J|^{q(\beta-1/p)} < \infty,$$

where I and J are arbitrary bounded intervals in $\mathbb{R}$.

Exploring two-weight problem for the fractional maximal function of variable order, we prove, in particular, an analogue of Theorem D when the exponent of the initial Lebesgue space is constant.

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2 *Introductory material to COURSE B*: Integral Operators In Some Banach Function Spaces

The proposed introductory material is intended for listeners at Summer School HART2010. It consists of two parts. In the first part we recall some definitions and known facts from the Theory of Banach Function Spaces. The main topic of our lectures is Grand Lebesgue Spaces from the standpoint of the boundedness of integral operators in these spaces. We give definitions of these spaces and prove their properties. We further give the proofs of the well known results on the mapping properties of the Hardy transform and Hardy-Littlewood maximal functions in non-weighted Grand Lebesgue spaces.

In the second part we discuss quasi-metric measure spaces and maximal functions defined on these spaces. In particular, spaces of homogeneous type (SHT) are defined and many examples of these spaces are given. Since covering lemmas play an important role in the investigation of metric properties of maximal functions, we give here quite a number of covering lemmas. It should be said that the classical covering lemma of Besikovich, generally speaking, does not hold in SHT.

In conclusion, we give the definition of the Morrey function space on SHT and show the boundedness of maximal functions on these spaces.

We hope that this brief introductory information will help the listeners to get a grasp of the problems we are going to discuss at the first two lectures.

3 Banach Function Spaces

In the sequel, Ω denotes an open subset Ω in R^n. Let M_0 be the set of all measurable function whose values lie in $[-\infty, \infty]$ and are finite a.e. in Ω. Also, let M_0^+ be the class of functions in M_0 whose values lie in $(0, \infty)$.

Definition 3.1. A mapping $\rho : M_0^+ \to [0, \infty]$ is called a Banach function norm if for all f, g, f_n $(n = 1, 2, \dots)$ in M_0^+, for all constants $a \geq 0$ and all measurable subsets

$E \subset \Omega$, the following properties hold:

$$\rho(f) = 0 \Longleftrightarrow f = 0 \quad \text{a.e. in} \quad \Omega$$
$$\rho(af) = a\rho(f)$$
$$\rho(f+g) \le \rho(f) + \rho(g)$$
$$0 \le g \le f \quad \text{a.e. in} \quad \Omega \Longrightarrow \rho(g) \le \rho(f)$$
$$0 \le f_n \uparrow f \quad \text{a.e. in} \quad \Omega \Longrightarrow \rho(f_n) \uparrow \rho(f)$$
$$|E| < +\infty \Longrightarrow \rho(\chi_E) < +\infty$$
$$|E| < +\infty \Longrightarrow \int_E f dx \le C_E \rho(f)$$

for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

Definition 3.2. If ρ is a Banach function norm, the Banach space

$$X = \{f \in M_0 : \rho(|f|) < +\infty\} \tag{3.3}$$

is called a Banach Function Space.

For each $f \in X$ define

$$\|f\|_X = \rho(|f|). \tag{3.4}$$

Definition 3.3. If ρ is a function norm, its associate ρ' is defined on M_0^+ by

$$\rho'(g) = \sup\left\{ \int_\Omega fg dx : f \in M_0^+, \quad \rho(f) \le 1 \right\}. \tag{3.5}$$

Definition 3.4. Let ρ be a function norm and let $X = X(\rho)$ be the Banach Function Space determined by ρ. Let ρ' be the associate norm of ρ. The Banach Function Space $X' = X'(\rho')$ determined by ρ' is called the associate space of X.

A standard example is that one of Orlicz spaces: the associate space of L^ϕ is given by $L^{\widetilde{\phi}}$, where $\widetilde{\phi}$ denotes the complementary function of ϕ, defined by

$$\widetilde{\phi}(t) = \max\{st - \phi(s) : s \ge 0\}.$$

In particular from the definied of $\|f\|_X$ it follows that the norm of a function g in the associate space X' is given by

$$\|g\|_{X'} = \sup\left\{ \int fg dx : f \in M^+, \quad \|f\|_X \le 1 \right\}.$$

Proposition 3.5. *Every Banach Function Space X coincides with its second associate space X''.*

Proposition 3.6. *If X and Y are Banach Function Spaces and $X \subset Y$ (continuous embedding), then $Y' \subset X'$.*

Definition 3.7. A function f in a Banach Function Space X is said to have absolutely continuous norm on X if

$$\lim_{n\to\infty} \|f\chi_{E_n}\| = 0$$

for every sequence $\{E_n\}_{n=1}^{\infty}$ satisfying $E_n \downarrow \varnothing$.

The subspace of functions in X with absolutely continuous norm is denoted by X_a.

If $X = X_a$, then the space X itself is said to have absolutely continuous norm.

Definition 3.8. Let X be a Banach Function Space. The closure in X of the set of bounded functions is denoted by X_b.

Proposition 3.9. *Let X be a Banach Function Space.*
Then

$$X_a \subseteq X_b \subseteq X.$$

Corollary. *If $X_a = X$, then $X_b = X$.*

Proposition 3.10. *The dual Banach Space X^* of a Banach Function Space X is canonically isometric to the associate space X' if and only if X has absolutely continuous norm.*

Proposition 3.11. *A Banach function space is reflexive if and only if both X and its associate space X' have absolutely continuous norm.*

Let m denote the Lebesgue measure.

Definition 3.12. The distribution function m_f of a function f in M_0^+ is given by

$$m_f(\lambda) = m\{x \in \Omega : |f(x)| > \lambda\} \quad (\lambda \geq 0).$$

Two functions are said to be equimeasurable if they have the same distribution functions.

Definition 3.13. Let ρ be a function norm. Then ρ is said to be rearrangement-invariant if $\rho(f) = \rho(g)$ for every pair of equimeasurable functions f and g.

The decreasing rearrangement f^* of f is certainly equimeasurable with f and in some sense f^* may be regarded as the canonical choice of a function equimeasurable with f.

The examples of rearrangement invariant Banach Function Spaces are Lebesgue spaces, Orlicz space, Lorentz spaces etc.

4 Grand and Small Lebesgue Spaces

Let Ω be a bounded set in R^n.

The grand Lebesgue space is the set of measurable functions for which

$$\|f\|_{p)} := \sup\left(\frac{\varepsilon}{|\Omega|}\int_\Omega |f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} < \infty.$$

This space is denoted by $L^{p)}(\Omega)$. $L^{p)}$ is a rearrangement invariant Banach function space.

Let us prove that $L^{p)}(0,1)$ is complete. Let $(f_n)_n$ be a Cauchy sequence $L^{p)}$, i.e.

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1 |f_m(x)-f_n(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} = 0.$$

Hence for an arbitrary $\eta>0$ there exists $N\in\mathbb{N}$ such that

$$\left(\varepsilon\int_0^1 |f_m(x)-f_n(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} < \frac{\eta}{3}$$

for an arbitrary ε, $0<\varepsilon<p-1$, when $m>N$, $n>N$.

Consequently $(f_n)_n$ is a Cauchy sequence in $L^{p-\varepsilon}$ for an arbitrary ε, $0<\varepsilon<p-1$ and let f be its limit in $L^{p-\varepsilon}$.

Let $n>N$. According to the definition of the supremum there exists an ε_0 (depending, generally speaking, on n), $0<\varepsilon_0(n)<p-1$, such that

$$\|f-f_n\|_{p)} = \sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1 |f(x)-f_n(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} \le$$
$$\le \left(\varepsilon_0(n)\int_0^1 |f(x)-f_n(x)|^{p-\varepsilon_0(n)}dx\right)^{\frac{1}{p-\varepsilon_0(n)}} + \frac{\eta}{3}.$$

Furthermore, there exists $N_1\in\mathbb{N}$ such that for $m>N_1$

$$\left(\varepsilon_0(n)\int_0^1 |f_m(x)-f(x)|^{p-\varepsilon_0(n)}dx\right)^{\frac{1}{p-\varepsilon_0(n)}} < \frac{\eta}{3}.$$

Therefore

$$\|f-f_n\|_{p)} \le \left(\varepsilon_0(n)\int_0^1 |f_n(x)-f_m(x)|^{p-\varepsilon_0(n)}dx\right)^{\frac{1}{p-\varepsilon_0(n)}} +$$
$$+\left(\varepsilon_0(n)\int_0^1 |f_m(x)-f(x)|^{p-\varepsilon_0(n)}dx\right)^{\frac{1}{p-\varepsilon_0(n)}} + \frac{\eta}{3} \le$$
$$\le \frac{\eta}{3}+\frac{\eta}{3}+\frac{\eta}{3} = \eta.$$

when $n>N$ and $m>N_1$.

Thus

$$\|f-f_n\|_{p)} < \eta$$

for an arbitrary $n>N$.

Example. $\varphi(x) = x^{-\frac{1}{p}}$, $1 < p < \infty$ is an example of a function such that $\varphi \in L^{p)}/L^p$.

The following continuous embeddings

$$L^p \subset L^{p)} \subset L^{p-\varepsilon}, \quad 0 < \varepsilon \leq p-1$$

hold. In the framework of Orlicz spaces we have

$$L^p \log^{-1} L \subset L^{p)} \subset \bigcap_{\alpha<-1} L^p \log^\alpha L.$$

To find an explicit expression of the "best" functional $N_{p'}$, usually called the associate norm of $\|\cdot\|_{p)}$, such that the following Hölder inequality holds

$$\frac{1}{|\Omega|}\int_\Omega fg dx \leq \|f\|_{p)} N_{p'}(g),$$

we introduce the auxiliary Banach space $L^{(p'}$.

Let $1 < p < \infty$, $p' = \frac{p}{p-1}$. By $L^{(p'}$ we denote the set of all functions $g \in M_0$ defined on Ω which can be represented in the form $g(x) = \sum\limits_{k=1}^{\infty} g_k(x)$ (convergence a.e.) and such that the following norm

$$\|g\|_{(p'} = \inf\left\{\sum_{k=1}^{n} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\frac{1}{|\Omega|}\int_\Omega |g_k|^{(p-\varepsilon)'} dx\right)^{\frac{1}{(p-\varepsilon)'}}\right\},$$

is finite, where the final inf is taken with respect to all representations $g(x) = \sum\limits_{k=1}^{\infty} g_k(x)$. Note that the following Hölder type inequality

$$\frac{1}{\Omega}\int\limits_\Omega fg dx \leq \|f\|_{p)}, \|g\|_{(p'}$$

holds.

After the above definition, the small Lebesgue space is defined by

$$L^{p)'}(\Omega) := \{g \in M_0 : \|g\|_{p)'} < +\infty\}$$

where

$$\|g\|_{p)'} = \sup_{\substack{0<\psi\leq|g| \\ \psi\in L^{(p'}(\Omega)}} \|\psi\|_{(p'}.$$

It turns out that $L^{p)'}(\Omega)$ is a Banach function space whose norm satisfies the Fatou property i.e.

$$0 \leq g_n \uparrow g \quad \text{a.e.} \quad \text{in} \quad \Omega \Longrightarrow \|g_n\|_{p)'} \uparrow \|g\|_{p)'}.$$

Proposition 4.14. *Let $1 < p < +\infty$ and $\Omega \subset R^n$, $|\Omega| < +\infty$.*

The following Hölder inequality holds

$$\frac{1}{|\Omega|}\int_\Omega fg dx \le \|f\|_{p)}\|g\|_{p)'}, \quad \forall f\in L^{p)}, \quad g \in L^{p)'}.$$

Proposition 4.15. *The spaces $L^{p)}$ and $L^{p)'}$ are are not reflexive. For the space $L^{p)}$ the non-reflexivity follows from the fact that there exists a function F, for which the norm $\|F\|_{p)}$ is not absolute continuous.*

Indeed such a function F is

$$F(x) = x^{-\frac{1}{p}}, \quad x \in (0,1),$$

for which

$$\lim_{a\to 0}\sup_{\varepsilon>0}\left(\varepsilon\int\limits_0^a x^{-\frac{p-\varepsilon}{p}}dx\right)^{\frac{1}{p-\varepsilon}} \ne 0.$$

Proposition 4.16. *The set C_0^∞ is not dense in $L^{p)}$. Its closure $[L^p]_{p)}$ consists of functions $f\in L^{p)}$ such that*

$$\lim_{\varepsilon\to 0}\varepsilon\int_\Omega |f|^{p-\varepsilon}dx = 0. \tag{4.1}$$

Proof. First we show that if $f \in [L^p]_{p)}$, then (1) holds for f. Indeed, since $f\in [L^p]_{p)}$, we have that there is a sequence of functions $f_n\in L^p$ such that $\|f-f_n\|_{L^{p)}}\to 0$.

Let us take $\delta>0$. Choose n_0 such that $\|f-f_{n_0}\|_{L^{p)}} < \frac{\delta}{2}$ and $f_{n_0}\in L^p$. Now observe that for f_{n_0} we have (by Hölder's inequality)

$$\left(\frac{\varepsilon}{|\Omega|}\int\limits_\Omega |f_{n_0}|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \le \varepsilon^{\frac{1}{p-\varepsilon}}\left(\frac{1}{|\Omega|}\int\limits_\Omega |f_{n_0}|^p\right)^{\frac{1}{p}} \to 0 \ \text{ as } \ \varepsilon\to 0.$$

Hence there is $\varepsilon_0>0$ such that when $\varepsilon<\varepsilon_0$, then

$$\left(\frac{\varepsilon}{|\Omega|}\int\limits_\Omega |f_{n_0}|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} < \frac{\delta}{2}.$$

Finally,

$$\begin{aligned}\left(\frac{\varepsilon}{|\Omega|}\int\limits_\Omega |f|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} &\le \left(\frac{\varepsilon}{|\Omega|}\int\limits_\Omega |f-f_{n_0}|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \\ &+\left(\frac{\varepsilon}{|\Omega|}\int\limits_\Omega |f_{n_0}|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}} \le \|f-f_{n_0}\|_{L^{p)}(\Omega)} + \frac{\delta}{2} \\ &\le \frac{\delta}{2}+\frac{\delta}{2} = \delta\end{aligned}$$

when $\varepsilon < \varepsilon_0$.

Let us now see that there is $f \in L^{p)} \setminus [L^p]_{p)}$. Indeed, let $f(t) = t^{-\frac{1}{p}}$. Then $f \in L^{p)}(0,1)$, but

$$\left(\varepsilon \int\limits_0^1 |f(t)|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}} = \left(\varepsilon \int\limits_0^1 t^{-\frac{p-\varepsilon}{p}} dt\right)^{\frac{1}{p-\varepsilon}} = p^{\frac{1}{p-\varepsilon}} \nrightarrow 0$$

as $\varepsilon \to 0$. □

We give the following known characterization of the grand and small Lebesgue spaces (in the case $\mu\Omega = 1$, for simplicity):

$$\|f\|_{p)} \approx \sup_{0<t<1} (1 - \ln t)^{-\frac{1}{p}} \left(\int_t^1 |f^*(s)|^p ds\right)^{\frac{1}{p}}$$

and

$$\|f\|_{(p} \approx \int_0^1 (1 - \ln t)^{-\frac{1}{p}} \left(\int_0^t |f^*(s)|^p ds\right)^{\frac{1}{p}} \frac{dt}{t},$$

where f^* is a decreasing rearrangement of f defined as

$$f^*(t) = \sup_{|E|=t} \inf_E f, \quad t \in (0,1).$$

5 Hardy's Inequality and Maximal Theorem in Grand Lebesgue Spaces

The classical, celebrated Hardy inequality is stated in the following theorem.

Theorem A. *Let $1 < p < \infty$ and let f be a measurable, nonnegative function in $(0,1)$. Then*

$$\left(\int_0^1 \left(\frac{1}{x}\int_0^x f(y)dy\right)^{\frac{1}{p}}\right) \leq \frac{p}{p-1}\left(\int_0^1 f^p(x)dx\right)^{\frac{1}{p}}.$$

Here we discuss the Hardy inequality in grand and small Lebesgue spaces introduced in Section 2.

Theorem 3.1. [A. Fiorenza, B. Gupta and P. Jain] *Let $1 < p < \infty$. There exists a constant $C(p) > 1$ such that*

$$\left\|\frac{1}{x}\int_0^x f(y)dy\right\|_{p)} \leq C(p)\|f\|_{p)} \tag{5.1}$$

for nonnegative measurable functions f on [0,1].

P r o o f. Let $0<\sigma<p-1$. we have

$$\left\|\frac{1}{x}\int_0^x f(y)dy\right\|_{p)} = \max\left\{\sup_{0<\varepsilon<\sigma}\left(\varepsilon\int_0^1\left(\frac{1}{x}\int_0^x f(y)dy\right)^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}},\right.$$

$$\left.\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1\left(\frac{1}{x}\int_0^x f(y)dy\right)^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}\right\}\leq$$

$$\leq\max\left\{\sup_{0<\varepsilon<\sigma}\left(\varepsilon\int_0^1\left(\frac{1}{x}\int_0^x f(y)dy\right)^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}},\right.$$

$$\left.(\sup_{\sigma\leq\varepsilon<p-1}\varepsilon^{\frac{1}{p-\varepsilon}})\sigma^{-\frac{1}{p-\sigma}}\sigma^{\frac{1}{p-\sigma}}\left(\int_0^1\left(\frac{1}{x}\int_0^x f(y)dy\right)^{p-\sigma}dx\right)^{\frac{1}{p-\sigma}}\right\}\leq$$

$$\leq(p-1)\sigma^{-\frac{1}{p-\sigma}}\left\{\sup_{0<\varepsilon\leq\sigma}\left(\varepsilon\int_0^1\left(\frac{1}{x}\int_0^x f(y)dy\right)^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}\right..$$

Now take $0<\varepsilon\leq\sigma$, so that $p-\varepsilon>1$. Applying the Hardy inequality with the exponent p replaced by $p-\varepsilon$, and multiplying both sides by $\varepsilon^{\frac{1}{p-\varepsilon}}$, we get

$$\left(\varepsilon\int_0^1\left(\frac{1}{x}\int_0^x f(y)dy\right)^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}\leq\frac{p-\varepsilon}{p-\varepsilon-1}\left(\varepsilon\int_0^1 f^{p-\varepsilon}(g)dx\right)^{\frac{1}{p-\varepsilon}}.$$

If we take the sup over $0<\varepsilon\leq\sigma$ on both sides, the previous inequality becomes

$$\sup_{0<\varepsilon\leq\sigma}\left(\varepsilon\int_0^1\left(\frac{1}{x}\int_0^x f(y)dy\right)^{\frac{1}{p-\varepsilon}}dx\right)^{\frac{1}{p-\varepsilon}}\leq\frac{p-\sigma}{p-\sigma-1}\sup_{0<\varepsilon<\sigma}\left(\varepsilon\int_0^1 f^{p-\varepsilon}(x)dx\right)^{\frac{1}{p-\varepsilon}}.$$

and therefore

$$\left\|\frac{1}{x}\int_0^x f(y)dy\right\|_{p)}\leq(p-1)\sigma^{-\frac{1}{p-\sigma}}\frac{p-\sigma}{p-\sigma-1}\sup_{0<\varepsilon<p-1}\left(\varepsilon\int_0^1 f^{p-\varepsilon}(y)dy\right)^{\frac{1}{p-\varepsilon}}.$$

Letting

$$C(p):=\inf_{0<\sigma<p-1}(p-1)\sigma^{-\frac{1}{p-\sigma}}\frac{p-\sigma}{p-\sigma-1}>1,$$

we get the desired inequality (1). □

Theorem 3.2. [Fi-Ju-I] *Let $1<p<\infty$. There exists a constant $C(p)>1$ such that*

$$\left\|\frac{1}{x}\int_0^x f(y)dy\right\|_{(p}\leq C(p)\|f\|_{(p}\tag{5.2}$$

for all nonnegative measurable functions f on $[0,1]$.

P r o o f. The inequality (2) can be easily deduced from the expression of norm (see Section 2), using the rearrangement-invariance and standard Hardy inequality

$$\left\|\frac{1}{x}\int_0^x f(y)dy\right\|_{(p}\leq C(p)\int_0^1(1-\log t)^{-\frac{1}{p}}\left(\int_0^t\left(\frac{1}{x}\int_0^x f(y)dy\right)^p dx\right)^{\frac{1}{p}}\frac{dt}{t}\leq$$

$$\leq\frac{C(p)p}{p-1}\|f\|_{(p}.$$

To give the proof of boundedness of Hardy-Littlewood maximal function we use an important relation between rearrangements and maximal operator

$$Mf(x) = \sup_{(0,1)\supset I\ni x} \frac{1}{|I|} \int_{I(x\in(0,1))} f(y)dy$$

given by the well-known Herz theorem (see. e.g. [Ben-Sha], Theorem 3. (10, p.125)) through the notion of decreasing rearrangement f^* of f.

Let

$$f^{**}(t) = \frac{1}{t} \int_0^t f^*(s)ds, \quad t \in [0,1].$$

□

Theorem [He]. *There are absolute constants c and c' such that for all $f \in L^1(0,1)$,*

$$c(Mf)^*(t) \le f^{**}(t) \le c'(Mf)^*(t), \quad t \in (0,1). \tag{5.3}$$

Corollary 5.17. *Let $1 < p < \infty$. There exists a constant $C(p) > 1$ such that*

$$\|Mf\|_{p)} \le C(p)\|f\|_{p)}$$

for all $f \in L^1(0,1)$.

Proof. Since

$$\|f\|_p = \|f^*\|_p,$$

from (3) and Theorem 3.2 applied to f^* we get

$$\|Mf\|_{p)} = \|(Mf)^*\|_{p)} \le C\|f^{**}\|_{p)} = C\|f\|_{p)}$$

from which the assertion follows. □

We remark that the general result by Lorentz and Shimogaki on the characterization of the rearrangement-invariant spaces on which the Hardy-Littlewood maximal operator is bounded (see e.g. [Ben-Sha]) cannot be appliced, due to the fact that the underlying measure space is finite.

6 Generalized Grand and Small Lebesgue Spaces

In this section we deal with a generalization of the grand and small Lebesgue spaces, namely the spaces $L^{p),\theta}(\Omega)$, $\theta > 0$, defined by

$$\|f\|_{p),\theta} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^\theta}{|\Omega|} \int_\Omega |f|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}$$

and the spaces $L^{(p',\theta}(\Omega)$, $\theta > 0$, defined by

$$\|g\|_{(p',\theta} = \inf_{g=\sum\limits_{k=1}^{\infty} g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_\Omega |g_k|^{(p-\varepsilon)'} dx \right) \right\}^{\frac{1}{(p-\varepsilon)'}}.$$

For $\theta = 0$ we have $\|f\|_{p),0} = \|f\|_p$ (this follows from the classical Hölder's inequality and Theorem 194 p. 142 of [Hardy, Littlewood and Polya], while $\|f\|_{(p',0} = \|f\|_{p'}$ (this follows from the classical Hölder's and Minkowski's inequalities). For $\theta = 1$ such spaces reduce obviously to the spaces $L^{p)}(\Omega)$ and $L^{(p}(\Omega)$, respectively.

For each $g \in M_0$ we set

$$\|g\|_{p)',\theta} = \sup_{0\le\psi\le|g|} \|\psi\|_{(p',\theta}.$$

Proposition 6.18. *The space defined by*

$$L^{p)',\theta}(\Omega) = \{g \in M_0 : \|g\|_{p),'\theta} < +\infty\}$$

is a Banach Function Spaces, and is associate space of $L^{p),\theta}(\Omega)$*.*

Proposition 6.19. *Let* (f_m) *be a monotone decreasing sequence (i.e.* $f_m \le f_{m+1}$*,* $m \in N$*) such that* $\sup\limits_m \|f_m\|_{(p',\theta} = M < \infty$*. Then the function* $f = \sup\limits_m f_m$ *is such that* $f \in L^{(p',\theta}(\Omega)$*,* $f_m \nearrow f$ *a.e. and* $f_m \to f$ *in* $L^{(p',\theta}(\Omega)$*.*

Proposition 6.20. *The space* $L^{(p',\theta}(\Omega)$ *is a Banach Function Space and*

$$L^{p)',\theta}(\Omega) \equiv L^{(p',\theta}(\Omega)$$

and therefore

$$\|g\|_{p)',\theta} = \inf_{g=\sum g_k} \left\{ \sum_{k=1}^{\infty} \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{\theta}{p-\varepsilon}} \left(\frac{1}{|\Omega|} \int_\Omega |g_k|^{(p-\varepsilon)'} dx \right)^{\frac{1}{(p-\varepsilon)'}} \right\}.$$

Proof. Let us observe that since $\|g\|_{(p',\theta}$ is an order preserving norm, then

$$\|\psi\|_{(p',\theta} \le \|f\|_{(p',\theta} \quad \forall \psi \in L^{(p',\theta}(\Omega), \quad \psi \le |f|$$

hence

$$\|f\|_{p)',\theta} \le \|f\|_{(p',\theta}$$

Let us prove the opposite inequality. To this aim, let f be such that $\|f\|_{p)',\theta} < \infty$ otherwise there is nothing to prove. Let

$$f_n = \min\{|f|, n\}.$$

The sequence (f_n) verifies the hypothesis of Proposition 4.2, so for $|f| = \sup_n f_n$ we have $|f| \in L^{(p',\theta}(\Omega)$ and replacing ψ by $|f|$ in the definition of $\|f\|_{p)',\theta}$, we obtain

$$\|f\|_{p)',\theta} \ge \|\,|f|\,\|_{(p',\theta} = \|f\|_{(p',\theta}$$

The proof is over. $\square$

As a consequence of Propositions 4.1 and 4.3, we get

Corollary 6.21. *The following Hölder inequality holds*

$$\frac{1}{|\Omega|}\int_\Omega fg dx \le \|f\|_{p),\theta}\|g\|_{(p',\theta}$$

and $\|\cdot\|_{(p',\theta}$ *is the smallest norm for which the last inequality holds.*

Lemma 6.22. *Let* $F_n \subset \Omega$, $n \in \mathbb{N}$, *be such that* $\chi_{F_n} \downarrow 0$ *a.e. in* Ω *and let* g *be any function in* $L^{(p',\theta}(\Omega)$. *Then*

$$\|g\chi_{F_n}\|_{(p',\theta} \to 0.$$

Corollary 6.23. *The Banach Function Space* $L^{(p',\theta}$ *has absolutely continuous norm and therefore the set of bounded functions is dense in* $L^{(p',\theta)}(\Omega)$.

Proposition 6.24. *The dual of* $L^{(p',\theta}(\Omega)$ *is canonically isometric to the associate space of* $L^{(p',\theta}(\Omega)$ *and the following relation holds*

$$(L^{(p',\theta}(\Omega))^* = (L^{(p',\theta}(\Omega))^* = L^{p),\theta}(\Omega).$$

Proposition 6.25. *The subspace* $C_0^\infty(\Omega, R^n)$ *is not dense in* $f \in L^{p),\theta}(\Omega)$. *Its closure consists of functions* $f \in L^{p),\theta}(\Omega)$ *such that*

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p}}\|f\|_{p-\varepsilon} = 0.$$

The proof is based on the same arguments as the proof of Proposition 2.3.

7 Spaces of Homogeneous Type

Definition 7.26. A space of homogeneous type (SHT in the following) (X, d, μ) is a topological space endowed with a measure μ such that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$ and there exists a non-negative real-valued function $d : X \times X \to \mathbb{R}^1$ satisfying

(i) $d(x,x) = 0$ for all $x \in X$.

(ii) $d(x,y) > 0$ for all $x \ne y, x, y \in X$.

(iii) There is a constant $a_0 > 0$ such that $d(x,y) \le a_0 d(y,x)$ for all $x, y \in X$.

(iv) There is a constant $a_1 > 0$ such that $d(x,y) \le a_1(d(x,z) + d(z,y))$

$$\text{for all } x,y,z \in X.$$

(v) For every neighbourhood V of x in X there is $r>0$ such that the ball

$$B(x,r)=\{y\in X;\ d(x,y)<r\} \text{ is contained in } V.$$

(vi) Balls $B(x,r)$ are measurable for every $x\in X$ and every $r>0$.

(vii) There is a constant $b>0$ such that $\mu B(x,2r)\le b\mu B(x,r)<\infty$

$$\text{for every } x\in X \text{ and every } 0<r<\infty.$$

Let us observe that a standard definition of an SHT requires that X is a metric space. However, a space X with the above properties is always metrizable (see later in this chapter).

A classical example of an SHT is of course the space $\mathbb{R}^n$ with the Euclidean distance and the Lebesgue measure. Some of the familiar facts, valid in $\mathbb{R}^n$, do not remain valid when passing to a general SHT. This causes technical difficulties, for instance, considering problems with weights.

Several examples of spaces of homogeneous type follow:

(1) Let $X=\mathbb{R}^n$, and let d be the Euclidean distance. Put $d\mu(x)=|x|^\alpha dx$, $\alpha\ge 0$.

We see that several Borel measures can be associated to a quasidistance in order to get an SHT.

(2) Let $X=\mathbb{R}^n$, $d(x,y)=\sum\limits_{j=1}^{n}|x_j-y_j|^{\alpha_j}$ with positive $\alpha_1,\dots,\alpha_n$, and let μ be the Lebesgue measure. Generally, with α_j's not all equal, the last formula defines an anisotropic distance. This shows that several quasidistances can be associated to a Borel measure μ, getting an SHT.

(3) $X=(0,1)$, $d(x,y)$ being the length of a smallest dyadic interval containing x and y.

(4) $X=[-1,1]$, d the usual distance, and $d\mu(x)=(1-x)^\alpha(1+x)^\beta dx$, where $\alpha,\beta>-1$.

(5) $X=[0,\infty)$, $d(x,y)=|x^r-y^r|$, the measure $d\mu=xdx$.

(6) Let $\Gamma\subset\mathbb{C}$ be a connected rectifiable curve and ν be an arclength measure on Γ. By definition, Γ is regular if

$$\nu(\Gamma\cap B(z,r))\le cr$$

for every $z\in\mathbb{C}$ and $r>0$.

For r smaller than half the diameter of Γ, the reverse inequality

$$\nu(\Gamma\cap B(z,r))\ge r$$

holds for all $z\in\Gamma$. Equipped with ν and the Euclidean metric, the regular curve Γ becomes an SHT.

(7) A piecewise smooth Lipschitz manifold $X\subset\mathbb{R}^n$ with an induced metric and a Hausdorff measure of an appropriate dimension.

(8) The boundary of a Lipschitz bounded domain in $\mathbb{R}^n$ with the corresponding harmonic measure is an SHT.

(9) It is known (see [Volberg and Konyagin., *Dokl. Akad. Nauk SSSR* 278, 3(1984), 783-786. English transl.: *Soviet Math. Dokl.* 30(1984), 453-456]; and [Volberg and Konyagin, On measures with the doubling condition. *Izv. Akad. Nauk SSSR, Ser. Mat.* 51(1987), N 3, 666-675. English transl.: *Math. USSR Izv.* 30(1988), N 3, 629-638]) that on an arbitrary closed subset of $\mathbb{R}^n$ there exists a measure satisfying property (vii) from the definition of an SHT (the doubling condition). Hence, any closed subset of $\mathbb{R}^n$ with the Euclidean metric and such a measure is an example of an SHT.

(10) Every compact Riemann manifold with the usual metric and measure is an SHT.

(11) The boundary of a strictly pseudoconvex domain in C^n with Lebesgue measure and anisotropic distance associated with the complex structure. For instance, take the unit sphere $\sum\limits_{j=1}^{n} |z_j|^2 = 1$, $d(z,w) = |1 - z\overline{w}|^{1/2} = \left|1 - \sum\limits_{j=1}^{n} z_j \overline{w_j}\right|^{1/2}$.

(12) $X = [0, \infty)$ with the measure $r^{n-1}dr$ and the usual Euclidean distance.

(13) $X = \sum_{n-1} = \{x \in \mathbb{R}^n;\ |x| = 1\}$, and μ the (unique) rotation invariant measure on $\sum_{n-1}$ such that $\mu \sum_{n-1} = 1$, and $d(x,y) = |1 - xy|^{\alpha}$, $\alpha > 0$, $x \cdot y = \sum\limits_{j=1}^{n} x_j y_j$.

In the following we shall need several facts about the geometry of SHTs. The first of them is the following principle (see e. g. the introductory Lemma 1 in Strömberg and Torchinsky. Weights, sharp maximal functions and Hardy spaces. Bull. Amer. Math. Soc. 3(1980), 1053-1056.):

Proposition 7.27. *Given $c > 0$, there is an a_2 such that if $B(x,r) \cap B(y,r') \neq \emptyset$ and $r \leq r'$, then $B(x,r) \subset B(y, a_2 r')$. Moreover, $a_2 = a_1(1 + ca_1(1 + a_0))$.*

Proof. Let $z \in B(x,r)$, $z_1 \in B(x,r) \cap B(y,r')$. Then we have

$$\begin{aligned} d(y,z) &\leq a_1(d(y,z_1) + d(z_1,z)) \leq a_1(d(y,z_1) + a_1(d(z_1,x) + d(x,z))) \\ &\leq a_1(r' + a_1(a_0 r + r)) \leq a_1(r' + ca_1(a_0+1)r') \\ &= a_1(1 + ca_1(a_0+1))r'. \end{aligned}$$

□

Proposition 7.28. *Let (X,d,μ) be an SHT. Then there is a constant $h = h(b, a_1)$ such that every ball $B(x,r)$ cannot contain more than h^n points $\{x_j\}$ for which $d(x_i,x_j) > r2^{-n}$ for $i \neq j$.*

The proof of this proposition can be found in R.Coifmann and G.Weiss "Analyse harmonique non-commutative sur certains espaces homogénes" (see p. 68), *Lecture Notes in Math.*, Vol. 242, Springer-Verlag, Berlin, 1971.

8 Homogeneous Groups

We shall describe the particularly important examples of an SHT having a group structure. A. homogeneous group is a simply connected nilpotent Lie group G on a Lie algebra g with a one-parameter group of transformations $d_t = \exp(A \log t)$, $t > 0$, where A is a diagonalized linear operator on g with positive eigenvalues. In a homogeneous group the mappings $\exp \circ \delta_t \circ \exp^{-1}$, $t > 0$, are automorphisms in G, which will be again denoted by δ_t. The number $Q = \mathrm{tr} A$ is the dimension of G. The symbol e will stand for the neutral element in G.

It turns out that it is possible to equip G with a homogeneous norm $r : G \to [0, \infty)$, continuous in G and smooth in $G \backslash \{e\}$ and satisfying

$$\text{(i) } r(x) = r(x^{-1}) \text{ for every } x \in G;$$

$$\text{(ii) } r(\delta_t x) = \mathrm{tr}(x) \text{ for every } x \in G \text{ and } t > 0;$$

$$\text{(iii) } r(x) = 0 \text{ iff } x = e;$$

$$\text{(iv) there exist } c_0 > 0 \text{ such that}$$

$$r(xy) \leq c_0(r(x) + r(y)), \quad x, y \in G.$$

A ball in G, centered at x and with radius ρ, is defined as

$$B(x, \rho) = \{y \in G;\ r(xy^{-1}) < \rho\}.$$

Observe that $\delta_\rho B(e, 1) = B(e, \rho)$.

Let us fix a Haar measure $|\cdot|$ in G such that $|B(0, 1)| = 1$. Then $|\delta_t E| = t^Q |E|$, in particular, $|B(x, r)| = r^Q$, $x \in G$, $r > 0$. (For details see Folland and Stein. Hardy spaces on homogeneous groups. Math. Notes, Vol. 28, Princeton University Press; University of Tokyo Press XII, Princeton, New Jersey 1982, p.5)

We shall present several typical examples.

1. The Euclidean n-dimensional space $\mathbb{R}^n$ with the operation of addition becomes a Lie group. Fix a diagonal matrix $A = (\alpha_{ij})$, $i, j = 1, \ldots, n$, and consider the one-parameter group of transformations $\delta_t = \exp(A \log t) : \mathbb{R}^n \to \mathbb{R}^n$. In this way we obtain a homogeneous group. If $A = E$, the unit matrix, then the group of transformations coincides with the group of Euclidean homothetic transformations. In this case we obtain a homogeneous norm and $Q = n$.

2. The Heisenberg group is the set of matrices of the form

$$[x] = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & \tau \\ 0 & 1 & 0 & \cdots & 0 & x_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & x_{2n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \tag{8.1}$$

where

$$\tau = x_0 + 2^{-1} x' x'', \quad x' = (x_1, \ldots, x_n), \quad x'' = (x_{n+1}, \ldots, x_{2n}),$$

$$x' x'' = \sum_{i=1}^{n} x_i x_{n+i},$$

equipped with the operation of matrix multiplication. We shall denote this group by Heis. The dimension of the group is $2n+1$ and due to its special form one can describe it as a manifold with an atlas consisting of a single map. Indeed, let us consider $\mathbb{R}^{2n+1}$ with elements denoted by $x = (x_0, x', x'')$, where $x_0 = \tau - 2^{-1}x'x''$. When passing to this pointwise interpretation the original group multiplication transforms into an operation $\otimes$ defined as follows. If $x = (x_0, x', x'')$ and $y = (y_0, y', y'')$, then

$$z = (z_0, z', z'') = x \otimes y = (x_0 + y_0 + 2^{-1}(x'y'' - x''y'), x' + y', x'' + y''). \tag{8.2}$$

Let us denote $\mathbb{R}^{2n+1}$ with the operation $\otimes$ by H^n and, for brevity, $x \otimes y$ simply by xy. In this way an isomorphism is constructed between the Heisenberg group Heis and H^n, thus H^n and Heis can be identified. Observe that the neutral element $[e]$ of Heis, that is, the unit matrix, corresponds to the point $e = \underbrace{(0, \dots, 0)}_{(2n+1)-\text{times}}$ and the inverse of $x = (x_0, x', x'') \in H^n$ is $-x = (-x_0, -x', -x'') \in H^n$. The operator of dilatation

$$\delta_t x = \delta_t(x_0, x', x'') = (t_2 x_0, tx', tx'')$$

is an automorphism in H^n. The matrix A_t corresponding to δ_t has the form

$$A_t = \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

The automorphism δ_t is a certain analogue of a homothety in Euclidean space.

The space H^n can be equipped with a homogeneous norm

$$|x|_{H^n} = \left(x_0^2 + \left(\sum_{j=1}^{2n} x_j^2 \right)^2 \right)^{1/4};$$

it is easy to show that

$$|\delta_t x|_{H^n} = t|x|_{H^n}$$

and that

$$|x + y|_{H^n} \le |x|_{H^n} + |y|_{H^n}, \quad x, y \in H^n.$$

Now one can introduce an H-metric ρ, putting

$$\rho(x, y) = |y^{-1}x|_{H^n} = (|x_0 - y_0 - 2^{-1}(x'y'' - x''y')|^2 + |x''' - y'''|^4)^{1/4},$$

where $x''' = (x', x'')$, $y''' = (y', y'')$. Observe that ρ is invariant with respect to the group operation, that is,

$$\rho(ux, uy) = \rho(x, y), \quad u, x, y \in H^n.$$

9 Some Covering Lemmas

As is well known, and SHT does not generally satisfy the Besicovitch property. A covering lemma of this type typically fails if the family of balls in question become eccentric as they shrink and tilt as their centers move (see Sawyer and Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous type spaces. *Amer. J. Math.* 114(1992), p.863).

Below we shall use the following covering lemma, whose proof is taken from Strömberg and Torchinsky. Weighted Hardy spaces. *Lecture Notes in Math.*, Vol. 1381, Springer-Verlag, Berlin 1989, p.2.

Lemma 9.29. *Let $\mathcal{F} = \{B_\alpha\}_{\alpha\in A}$ be a family of balls which covers a set E.*

(1) *If $\sup\{radB_\alpha;\ \alpha \in A\} < \infty$, then there is a countable subfamily of pairwise disjoint balls $\{B_J\}$ such that $E \subset \cup NB_j$, where $N = a_1(1 + 2a_1(1 + a_0))$ and if $B_j = B(x_j, r_j)$, then NB_j denotes the ball $B(x_j, Nr_j)$. Moreover, each ball B in $\mathcal{F}$ is contained in one of the balls NB_j and $radB \le 2radB_j$.*

(2) *If $\sup\{radB_\alpha;\ \alpha \in A\} = \infty$ and $\mu E < \infty$, then E is contained in some B_{α_0}.*

Proof. First let us prove (1). Let $M = \sup\{\mathrm{rad}B_\alpha;\ \alpha \in A\}$. For each $k = 0, 1, 2, \dots$ we construct recursively a family $\{B(x_{i,k}, r_{i,k})\}$ of balls with the following property:

$$\{B(x_{i,k}, r_{i,k})\} \in \mathcal{F}, \qquad 2^{-k-1}M < r_{i,k} \le 2^{-k}M,$$

and the balls $\{B(x_{i,n}, r_{i,n})\}$ are pairwise disjoint for $0 \le n \le k$. We also suppose that for each k the family is maximal with respect to this property.

If now $B = B(x, r) \in \mathcal{F}$, then $2^{-k-1}M < r \le 2^{-k}M$ for some k and $B(x, r)$ intersects with one of the balls $B(x_{i,n}, r_{i,n})$ with $n \le k$. In this case we have $r < 2r_{i,n}$ and by Proposition 1.1.4, $B(x, r) \subset B(x_{i,n}, Nr_{i,n})$. Consequently, $E \subset \cup NB_J$.

For (2) let us note that E is bounded. Therefore there is a ball B_0 such that $E \subset B_0$ and as $\sup\{\mathrm{rad}B_\alpha;\ \alpha \in A\} = \infty$, then there exists a ball B_{α_0} such that $B_{\alpha_0} \cap B_0$ is not empty and $\mathrm{rad}B_{\alpha_0} > \frac{\mathrm{rad}B_0}{2}$. Then by Proposition 1.1.4, $E \subset B_0 \subset NB_{\alpha_0}$. □

Lemma 1.3′. *Suppose E is a bounded set (i.e. contained in a ball) in X such that for each $x \in X$ there is given a ball $B(x, r(x))$. Then there is a (finite or infinite) sequence of points $x_j \in E$ such that $\{B(x_j, r(x_j))\}$ is a disjoint family of balls and $\{B(x_j, a_1(1 + 2a_0)r(x_j))\}$ is a covering of E.*

Proof. As the set E is bounded, we may assume that $\sup\{r(x);\ x \in E\} < \infty$, otherwise there is a point $x \in E$ such that $E \subset B(x, r(x))$ and we are done. Now let us choose a point $x_1 \in E$ such that

$$r(x_1) > \frac{1}{2}\sup\{r(x);\ x \in E\}.$$

Suppose that we have chosen $x_1, x_2, \dots, x_{n-1}$ and take x_n which belongs to

$$E_n = E \setminus \bigcup_{j=1}^{n-1} B(x_j, a_1(1 + 2a_0)r(x_j))$$

and such that

$$r(x_n) > \frac{1}{2}\sup\{r(x);\ x \in E_n\}.$$

Then we have that

$$B(x_n, r(x_n)) \cap B(x_j, r(x_j)) = \emptyset, \quad j < n,$$

otherwise there is a point $y \in B(x_n, r(x_n)) \cap B(x_j, r(x_j))$ and

$$\begin{aligned} d(x_j, x_n) &\le a_1(d(x_j, y) + d(y, x_n)) \le a_1(d(x_j, y) + a_0 d(x_n, y)) \\ &< a_1(r(x_j) + a_0 r(x_n)) < a_1(r(x_j) + 2a_0 r(x_j)) \\ &= a_1(1 + 2a_0) r(x_j) \end{aligned}$$

but by our assumption x_n does not belong to

$$B(x_j, a_1(a_1 + 2a_0) r(x_j))$$

when $j < n$.

There are two possible cases:

a) If $E_{n+1} = \emptyset$ for some n, then the family $\{B(x_j, a_1(a_1+2a_0)r(x_j))\}$, $j \le n$, covers E and there is nothing to prove.

b) If this process continues ad infinitum, then $r(x_n)$ tend to zero. If this is not the case, then there exists a positive number ε such that $r(x_n) > \varepsilon$. Since the points x_n, $n \ge 1$, belong to E and the balls $B(x_n, r(x_n))$ are disjoint, then there are infinitely many points x_n in a ball B containing E and

$$d(x_i, x_j) > \min(r(x_i, r(x_j)) > \varepsilon > 0,$$

which contradicts Proposition 1.1.5.

Suppose now that there exists a point x in $E \setminus \bigcup_{n=1}^{\infty} B(x_n, a_1(a_1 + 2a_0) r(x_n))$; then there is an integer n_0 such that $r(x) > 2r(x_{n_0}$, but we have

$$r(x) \le \sup\{r(x);\ x \in E_{n_0}\} < 2r(x_{n_0},$$

which proves the lemma. □

By analogy with the classical situation, lemmas of this type can be used for the proof of a weak type estimate of maximal function in an SHT.

Given a function $f : X \to \mathbb{R}^1$, locally integrable with respect to the measure μ, we define the maximal function

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu B} \int_B |f(y)| d\mu,$$

where the supremum is taken over all balls B containing x.

Proposition 9.30. *There is a constant $c > 0$ such that*

$$\mu\{x \in X;\ Mf(x) > \lambda\} \le \frac{c}{\lambda} \int_{\{|f(x)| < \lambda/2\}} |f(y)| d\mu.$$

We omit the proof as its principle is well known (cf. e. g. J. O. Strömberg and A. Torchinsky, Weighted Hardy spaces, *Lecture Notes in Math.*, Vol. 1381, Springer-Verlag, Berlin 1989, p. 3 and R. R. Coifman and G. Weiss. Analyse harmonique non-commutative sur certains espaces homogénes, *Lecture Notes in Math.*, Vol. 242, Springer-Verlag, Berlin 1971, p.71).

Since the set of compactly supported continuous functions is dense in $L^1(X,\mu)$ one can prove the Lebesgue Differentiation Theorem in a standard way, namely: Given $f \in L^1(X,\mu)$, then

$$\lim_{r\to 0}(\mu(B(x,r)))^{-1}\int\limits_{B(x,r)} f(y)d\mu = f(x)$$

μ-almost everywhere in X.

Proposition 9.31. *Let $1 < p < \infty$. Then there is a positive constant c_0 non-depending on p such that*

$$\|Mf\|_{L^p(X,\mu)} \le c_0 \left[\frac{p}{p-1}\right]^{\frac{1}{p}} \|f\|_{L^p(X,\mu)} \tag{9.1}$$

Proof. It is obvious that operator M is of strong type (∞,∞). By Proposition 3.1 this operator is of weak type (1.1). The proof Proposition 3.2 follows directly from the Marcinkiewicz interpolation theorem. The constant c_0 arises from the appropriate covering Lemma 3.1. □

Let us denote by $L^{p,\lambda}(X,\mu)$ the classical Morrey space, where $1 < p < \infty$ and $0 \le \lambda < 1$, which is the class of all μ-measurable functions f for which the norm

$$\|f\|_{L^{p,\lambda}(X,\mu)} = \sup_{\substack{x\in X\\ 0\le r<d}} \left[\frac{1}{(\mu B(x,r))^{\lambda}}\int\limits_{B(x,r)} |f(y)|^p d\mu(y)\right]^{\frac{1}{p}}$$

is finite. If $\lambda = 0$, then $L^{p,\lambda}(X,\mu) = L^p(X,\mu)$.

Proposition 9.32. *Let $1 < p < \infty$ and let $0 \le \lambda < 1$. Then*

$$\|Mf\|_{L^{p,\lambda}(X,\mu)} \le \left(b^{\lambda/p} c_0 \left(\frac{p}{p-1}\right)^{\frac{1}{p}} + 1\right) \|f\|_{L^{p,\lambda}(X,\mu)}$$

holds, where the positive constant b arises in the doubling condition for μ and c_0 is the constant from (1).

Proof. Let r be a small positive number and let us represent f as follows:

$$f = f_1 + f_2,$$

where $f_1 = f \cdot \chi_{B(x,\bar{a}r)}$, $f_2 = f - f_1$ and a is the positive constant given by $\overline{a} = a_1(a_1(a_0+1)+1)$ (here a_0 and a_1 are constants arisen in the triangle inequality for the quasimetric ρ).

We have that

$$\left[\frac{1}{(\mu B(x,r))^{\lambda}} \int\limits_{B(x,r)} (Mf)^p(a)d\mu(a)\right]^{\frac{1}{p}} \leq \left(\frac{1}{(\mu B(x,r))^{\lambda}} \int\limits_{B(x,r)} (Mf_1)^p(y)d\mu(y)\right)^{\frac{1}{p}}$$

$$+\left(\frac{1}{(\mu B(x,r))^{\lambda}} \int\limits_{B(x,r)} (Mf_2)^p(y)d\mu(y)\right)^{1/p} =: J_1(x,r)+J_2(x,r)$$

By applying Proposition 3.2 we have that

$$J_1(x,r) \leq \frac{1}{(\mu B(x,r))^{\lambda/p}} \left(\int\limits_X (Mf_1(y))^p d\mu(a)\right)^{1/p}$$

$$\leq c_0(p')^{\frac{1}{p}}(\mu B(x,r))^{-\lambda/p} \left(\int\limits_{B(x,\overline{a}r)} |f(y)|^p d\mu(y)\right)^{1/p}$$

$$\leq c_0 b^{\frac{\lambda}{p}}(p')^{\frac{1}{p}}\|f\|_{L^{p,\lambda}(X,\mu)},$$

where $p' = p/(p-1)$, c_0 is the constant from (1) and b arises from the doubling condition:

$$\mu B(x,\overline{a}r) \leq b\mu B(x,r).$$

Further, observe that (see also [Kokilashvili-Meskhi, Armen. J. Math. 1 (2008), No. 1, 18-28]) $B(x,r) \subset B(x,a_1(a_0+1)r) \subset B(x,\overline{a},r)$. Hence, if $y \in B(x,r)$, then

$$Mf_2(y) \leq \sup_{B(x,r)\subset B} \frac{1}{\mu B} \int\limits_B |f|.$$

Consequently,

$$J_2(x,r) \leq \frac{1}{(\mu B(x,r))^{\lambda/p}}(\mu B(x,r))^{\frac{1}{p}} \sup_{B(x,r)\subset B}\left(\frac{1}{\mu B}\int\limits_B |f|\right)$$

$$= \mu(B(x,r))^{\frac{1-\lambda}{p}} \sup_{B(x,r)\subset B}\left(\frac{1}{\mu B}\int\limits_B |f|^p\right)^{1/p}$$

$$\leq \sup_B(\mu B)^{-\lambda/p}\left(\int\limits_B |f|^p\right)^{1/p} = \|f\|_{L^{p,\lambda}(X,\mu)}.$$

Taking into account the estimates for $J_1(x,r)$ and $J_2(x,r)$ we conclude that

$$\left(\frac{1}{B(x,r)^{\lambda}}\int\limits_{B(x,r)}(Mf(y))^p d\mu(y)\right)^{1/p}$$

$$\le \left(c_0 b^{\lambda/p}(p')^{1/p}+1\right)\|f\|_{L^{p,\lambda}(X,\mu)}.$$

□

Finally we note that in our exposition of Introductory material to Course B we followed freely the following books and papers:

Section 1

[Be-Sha] C. Bennett and R. Sharpley. Interpolation of operators. *Akademic Press, INC. Boston, San Diego, New York* 1988.

Section 2-4

[Iw-Sb] T. Iwaniec and C. Sbordone, On the integrability of the Jacobian under minimal hypotheses, *Arch. Rational Mech. Anal.* **119** (1992) 129–143.

[Gr-Iw-Sb] L. Greco, T. Iwaniec and C. Sbordone, Inverting the p-Harmonic Operator, *Manuscripta Math.* **92**(1997), 249–258.

[Fi1] A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces, *Collect. Math.* **51 (2)** (2000), 131-148.

[Fi-Gu-Ja] A. Fiorenza, B. Gupta and P. Jain, The maximal theorem in weighted grand Lebesgue spaces, *Studia Math.* **188(2)**(2008), 123-133.

Section 5-7

[Co-We] R. R. Coifman and G. Weiss. Analyse harmonique noncommutative sur certains espaces homogénes. *Lecture Notes in Math.* V. 242, Springer Verlag, Berlin, 1971.

[Fo-St] G. B. Folland and E. M. Stein. Hardy spaces on homogeneous groups. *Math. Notes.* V. 28, Princeton Univ. Press; Univ. of Tokyo Press XII, Princeton, New Jersey, 1982.

[St-To] J. O. Strömberg and A. Torchinsky. Weights, Sharp Maximal functions and Hardy spaces. *Bull. Amer. Math. Soc.*. 3(1980), 1053-1056.

[Ed-Ko-Me] D. E. Edmunds, V. Kokilashvili, and A. Meskhi, Bounded and compact integral operators. Mathematics and its Applications, 543. Kluwer Academic Publishers, Dordrecht, 2002.

[Ge-Go-Ko-Kr] I. Genebashvili, A. Gogatishvili, V. Kokilashvili, and M. Krbec, Weight Theory for Integral Transforms on Spaces of Homogeneous Type. Pitman Monographs and Surveys in Pure and Applied Mathematics. Addision Wesley Longman 1998, 234 pages.