Variable Lebesgue Spaces

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Banach space properties of the variable Lebesgue spaces

Outline

Classical Lebesgue spaces

- $L^p(\Omega)$, $1 \le p < \infty$:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < \infty$$

- $L^\infty(\Omega)$: $\|f\|_{L^p(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty$

Hereafter: $\Omega \subset \mathbb{R}^n$ *open set*

Simple problem

On $\mathbb{R}^1$ consider $f(x) = |x|^{-1/2}$

- $f \notin L^p(\mathbb{R})$, $\forall p$, $1 \leq p \leq \infty$
- $f \in L^p([-1,1])$, $1 \leq p < 2$
- $f \in L^p([1,\infty))$, $2 < p \leq \infty$

Question: can we capture behavior w/o splitting domain?

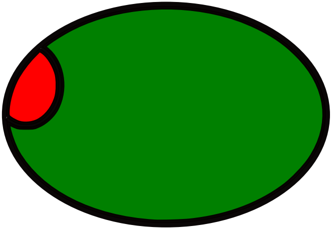
More complicated problem

Theorem (Calderón-Zygmund)

Let $\Omega \subset \mathbb{R}^n$ be bounded, and $f \in L^p(\Omega)$. If u is a solution to

$$\Delta u = f,$$

then $u \in W^{2,p}(\Omega)$.



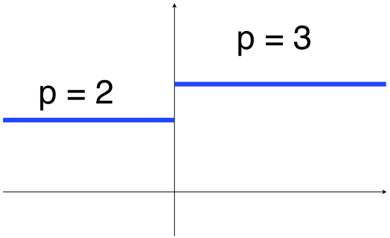
Intuition

Replace constant exponent p by function $p(\cdot)$:

$$L^{p(\cdot)}(\Omega): \quad \int_\Omega |f(x)|^{p(x)}\,dx < \infty$$

Simple example

$$p(x) = \begin{cases} 2 & -5 < x \leq 0 \\ 3 & 0 < x < 5. \end{cases}$$



A simple example (continued)

Variable $L^{p(\cdot)}$ Spaces

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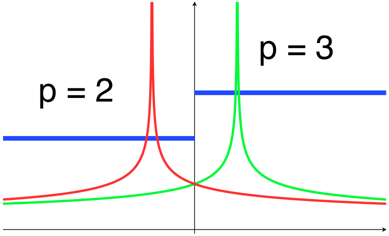
$$|x+1|^{-1/3} \in L^{p(\cdot)}((-5,5))$$
$$|x-1|^{-1/3} \notin L^{p(\cdot)}((-5,5))$$



Original motivations

- Generalized Orlicz spaces: replace

$$\int \Phi(f(x))\,dx \quad \text{with} \quad \int \Phi(f(x),x)\,dx$$

- Calculus of variations: minimize

$$\mathcal{F}(u,\Omega) = \int_\Omega f(u, Du)\,dx$$

where $|z|^{p(x)} \leq f(x,z) \leq L(1+|z|)^{p(x)}$

- Electrorheological fluids:

$$\text{Energy} = \int_\Omega |Du(x)|^{p(x)}\,dx$$

Exponent functions

$$p(\cdot) \in \mathcal{P}(\Omega) \qquad p(\cdot) : \Omega \to [1, \infty]$$

$$\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$$

For $E \subset \Omega$

$$p_-(E) = \operatorname{ess\,inf}\{p(x) : x \in E\}$$
$$p_+(E) = \operatorname{ess\,sup}\{p(x) : x \in E\}$$

Hereafter: $p_- = p_-(\Omega)$, $p_+ = p_+(\Omega)$

The modular & norm

Given $p(\cdot) \in \mathcal{P}(\Omega)$

$$\rho_{p(\cdot)}(f) = \rho(f) = \int_{\Omega \setminus \Omega_\infty} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\Omega_\infty)}$$

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\right\}$$

The space $L^{p(\cdot)}(\Omega)$

Theorem

Given $p(\cdot) \in \mathcal{P}(\Omega)$, $\|\cdot\|_{p(\cdot)}$ is a norm and

$$L^{p(\cdot)}(\Omega) = \{f : \|f\|_{p(\cdot)} < \infty\}$$

is a Banach function space.

A path not taken

Given $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is:

- an Orlicz-Musielak/Nakano/modular space
- a Banach function space

The problem with direct proofs

With apologies to Lewis Carroll

A meta-theorem

If $p_+ < \infty$ or if $p_+(\Omega \setminus \Omega_\infty) < \infty$, then $L^{p(\cdot)}$ is GOOD and behaves like L^p

If $p_+ = \infty$, then $L^{p(\cdot)}$ is BAD and something interesting happens

Modular vs. norm convergence

Theorem

Given $p(\cdot) \in \mathcal{P}(\Omega)$ the equivalence

$$f \in L^{p(\cdot)}(\Omega) \quad \Longleftrightarrow \quad \rho_{p(\cdot)}(f) < \infty$$

is true if and only if

$$p_+(\Omega \setminus \Omega_\infty) < \infty \quad (\text{or } p_- = \infty).$$

Proof: $p_+(\Omega \setminus \Omega_\infty) < \infty$

$$\begin{aligned}
\rho(f) < \infty \quad & \Rightarrow \quad \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^{p(\cdot)}(\Omega_\infty)} < \infty \\
& \Rightarrow \quad \exists\lambda > 1 : \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1/2 \\
& \qquad\qquad \lambda^{-1}\|f\|_{L^{p(\cdot)}(\Omega_\infty)} \leq 1/2 \\
& \Rightarrow \quad \|f\|_{p(\cdot)} < \infty
\end{aligned}$$

Proof: $p_+(\Omega \setminus \Omega_\infty) < \infty$, continued

$$\|f\|_{p(\cdot)} < \infty$$

$$\Rightarrow \exists \lambda > 1 : \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|f\|_{L^{p(\cdot)}(\Omega_\infty)} \leq 1$$

$$\Rightarrow \lambda^{-p_+(\Omega\setminus\Omega_\infty)} \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)} dx + \lambda^{-1}\|f\|_{L^{p(\cdot)}(\Omega_\infty)} \leq 1$$

$$\Rightarrow \rho(f) < \infty$$

Proof: $p_+(\Omega \setminus \Omega_\infty) = \infty$

Form sets $E_k \subset \Omega \setminus \Omega_\infty$:

- $E_k \subset E_{k+1}$, $|E_k \setminus E_{k+1}| > 0$
- $|E_k| \to 0$
- $p_-(E_k) > k$

$$f(x) = \left(\sum_{k=1}^{\infty} |E_k \setminus E_{k+1}|^{-1} \chi_{E_k \setminus E_{k+1}}(x) \right)^{1/p(x)}$$

$$\lambda > 1 : \rho(f/\lambda) = \sum_{k=1}^{\infty} \fint_{E_k \setminus E_{k+1}} \lambda^{-p(x)}\, dx \leq \sum_{k=1}^{\infty} \lambda^{-k}$$

Embedding in L^p

Theorem

If $|\Omega| < \infty$,

$$c_1\|f\|_{p_-} \leq \|f\|_{p(\cdot)} \leq c_2\|f\|_{p_+}$$

Theorem

$$f \in L^{p(\cdot)}(\Omega) \quad \Rightarrow \quad f = f_1 + f_2 : f_1 \in L^{p_-}(\Omega),\ f_2 \in L^{p_+}(\Omega)$$

Types of convergence

- Norm convergence:

$$\|f - f_k\|_{p(\cdot)} \rightarrow 0$$

- Modular convergence: $\exists \beta > 0$:

$$\rho(\beta|f - f_k|) \rightarrow 0$$

- Convergence in measure: $\forall \epsilon, \ \exists k$:

$$|\{x \in \Omega : |f(x) - f_k(x)| \geq \epsilon\}| < \epsilon$$

Norm Convergence

- Monotone convergence theorem:

$$f_k \nearrow f \quad \Rightarrow \quad \|f_k\|_{p(\cdot)} \nearrow \|f\|_{p(\cdot)}$$

- Fatou's lemma:

$$f_k \to f \quad \Rightarrow \quad \|f\|_{p(\cdot)} \leq \liminf_{k\to\infty} \|f_k\|_{p(\cdot)}$$

- Dominated convergence theorem: Iff $p_+ < \infty$,

$$f_k \to f, \ |f_k| \leq g \in L^{p(\cdot)}(\Omega) \quad \Rightarrow \quad \|f - f_k\|_{p(\cdot)} \to 0$$

Modular convergence

Iff $p_+(\Omega \setminus \Omega_\infty) < \infty$ (or $p_- = \infty$)

$$f_k \to f \text{ in modular} \quad \Longleftrightarrow \quad f_k \to f \text{ in norm}$$

Key ingredient of proof: nested sets $\{E_k\}$ and functions

$$f(x) = \left(\sum_{k=1}^{\infty} 2^{-k} |E_k \setminus E_{k+1}|^{-1} \chi_{E_k \setminus E_{k+1}}(x) \right)^{1/p(x)}$$

$$f_k(x) = f(x)\chi_{E_k}(x)$$

Convergence in measure

Theorem

If $p_+ < \infty$, TFAE:

- *Convergence in norm*
- *Convergence in modular*
- *Convergence in measure and* $\exists \lambda > 0 : \rho(\lambda f_k) \to \rho(\lambda f)$

Convergence in measure

Theorem

If $p_+ = \infty$*, TFAE*

- $f_k \to f$ *in modular* $\Rightarrow$ $f_k \to f$ *in measure &* $\exists \lambda > 0 : \rho(\lambda f_k) \to \rho(\lambda f)$
- $|\{x : p(x) > N\}| \to 0$ *as* $N \to \infty$.

Theorem

If $p_+ = \infty$ *&* $|\Omega_\infty| = 0$, $f_k \to f$ *in measure &* $\exists 0 < \lambda < 1 : \rho(\lambda f) < \infty$ *&* $\rho(\lambda f_k/3) \to \rho(\lambda f/3)$

$\Rightarrow$ $f_k \to f$ *in modular*

Density

Theorem

TFAE:

- $p_+ < \infty$
- *Bounded functions of compact support are dense in* $L^{p(\cdot)}(\Omega)$

Simple example

Let $\Omega = \mathbb{R}$, $p(x) = |x| + 1$

$$f(x) \equiv 1 \in L^{p(\cdot)}(\mathbb{R}) : \rho(f/2) = \int_{\mathbb{R}} 2^{-(1+|x|)}\, dx < \infty$$

$$\operatorname{supp}(g) \subset [-N, N]$$

$$\Rightarrow \quad \int_{|x|>N} |f(x) - g(x)|^{1+|x|}\, dx = \infty$$

$$\Rightarrow \quad \|f - g\|_{p(\cdot)} \geq 1.$$

Separability

Theorem

TFAE:

- $p_+ < \infty$
- $L^{p(\cdot)}(\Omega)$ *is separable: has countable dense subset*

Key ingredients of proof: nested sets $\{E_k\}$, *function f and associate norm*

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Hölder's inequality

$$\int_\Omega |f(x)||g(x)|\,dx \leq K_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}$$

$$K_{p(\cdot)} = \frac{1}{p_-} - \frac{1}{p_+} + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty$$

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

Associate norm

$$\|f\|'_{p(\cdot)} = \sup_{\|g\|_{p'(\cdot)}\leq 1} \int_\Omega f(x)g(x)\,dx$$

Theorem

$$k_{p(\cdot)}\|f\|_{p(\cdot)} \leq \|f\|'_{p(\cdot)} \leq K_{p(\cdot)}\|f\|_{p(\cdot)}$$

$$k_{p(\cdot)}^{-1} = \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty$$

Linear functionals

Given $g \in L^{p'(\cdot)}(\Omega)$, $\Phi_g : L^{p(\cdot)}(\Omega) \to \mathbb{R}$,

$$\Phi_g(f) = \int_\Omega f(x)g(x)\,dx$$

is a bounded linear functional: $g \in L^{p(\cdot)}(\Omega)^*$.

Duality

Theorem

TFAE:

- $p_+ < \infty$
- $L^{p(\cdot)}(\Omega)^* \cong L^{p'(\cdot)}(\Omega)$

Key ingredient in proof: nested sets $\{E_k\}$ and f defined above

The dual space

Characterize $L^{p(\cdot)}(\Omega)^*$ if $p_+ = \infty$.

Conjecture: Depends on whether or not $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$

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