Sobolev, capacitary and isocapacitary inequalities

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I survey my results on applications of isoperimetric and isocapacitary inequalities to the theory of Sobolev spaces.

I began to work on this topic many years ago, when as a fourth year undergraduate student I discovered that Sobolev type inequalities are equivalent to isoperimetric and isocapacitary inequalities [M1], [M2]. It turned out that classes of domains and measures involved in imbedding and compactness theorems could be completely described in terms of length, area and capacity minimizing functions. Moreover, without change of proofs, the same remains true for spaces of functions defined on Riemannian manifolds [M4], [M5], [G]. Nowadays, it is a vast domain of research with applications to nonlinear partial differential equations, geometry, spectral theory, Markov processes, and potential theory.

Most results presented in these lectures can be found in the books [M4] and [MP], where a lot more related information is contained.

1 Classical isoperimetric inequality and its applications

Consider the problem of maximizing the area a of a plane domain Ω with rectifiable boundary of a fixed length l.

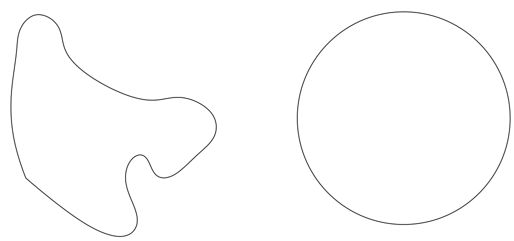

Figure 1: The disc gives the best result.

The maximizing property of the disk can be written as the isoperimetric inequality

$$4\pi a \le l^2. \tag{1}$$

The n-dimensional generalization of (1) is

$$(m_n g)^{\frac{n-1}{n}} \le c_n \mathcal{H}_{n-1}(\partial g), \tag{2}$$

where g is a domain with smooth boundary ∂g and compact closure, and $\mathcal{H}_{n-1}$ is the $(n-1)$-dimensional area. The constant c_n is such that (2) becomes equality for any ball, that is $c_n = n^{-1} v_n^{-1/n}$ with v_n standing for the volume of the unit ball. Inequality (2) holds for arbitrary measurable sets with $\mathcal{H}_{n-1}$ replaced by the so called perimeter in the sense of De Giorgi (1954-1955).

How does this geometric fact concern Sobolev imbedding theorems? The answer is given by the following result [FF], [M1].

Theorem 1.1 *Let $u \in C_0^\infty(\mathbb{R}^n)$. There holds the inequality:*

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx \Big)^{\frac{n-1}{n}} \le C(n) \int_{\mathbb{R}^n} |\nabla u| dx, \tag{3}$$

where the best constant is the same as in the isoperimetric inequality (2).

Proof. First we prove the lower estimate for $C(n)$. Figure 1 shows the graph of the function $u_\varepsilon(|x|)$ to be inserted in (3). (The function is not smooth but

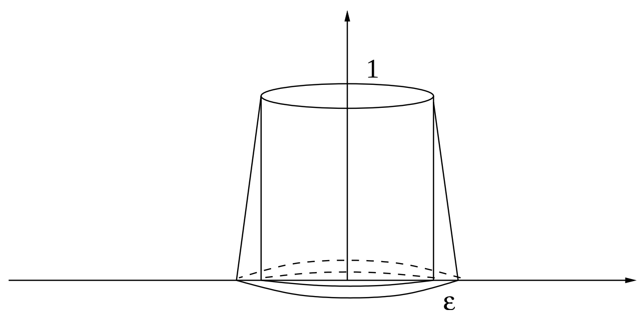

Figure 2: The function u_ε.

it is Lipschitz and can be approximated by smooth functions in the norm $\|\nabla u\|_{L_1(\mathbb{R}^n)}$).

We have

$$\begin{aligned} v_n^{\frac{n-1}{n}} &\leq \Big(\int_{\mathbb{R}^n} |u_\varepsilon|^{\frac{n}{n-1}} dx\Big)^{\frac{n-1}{n}} \leq C(n) \int_{\mathbb{R}^n} |\nabla u_\varepsilon| dx \\ &= C(n) n v_n \int_1^{1+\varepsilon} \Big|\frac{du_\varepsilon}{dr}\Big| r^{n-1} dr = (1+O(\varepsilon)) n v_n C(n). \end{aligned}$$

It follows that

$$n^{-1} v_n^{-1/n} \leq (1+O(\varepsilon)) C(n).$$

and finally

$$C(n) \geq (n v_n^{1/n})^{-1} = c_n.$$

In order to prove (3) we need the coarea formula:

$$\int_{\mathbb{R}^n} |\nabla u| dx = \int_0^\infty \mathcal{H}_{n-1}(E_t) dt, \tag{4}$$

where $E_t = \{x : |u(x)| = t\}$. It is known by Sard's lemma that almost all level sets are C^∞ manifolds. (Note that by Whitney's extension theorem, any multidimensional compact set can be a level set of a C^∞ function and thus the words "almost all" cannot be omitted.)

Next we give a plausible argument in favour of the coarea formula. This is not a rigorous proof, as we assume all level sets to be good.

We write $dx = d\mathcal{H}_{n-1} d\nu$ and $|\nabla u(x)| = dt/d\nu$, where $d\nu$ is the element of the trajectory orthogonal to E_t. We obtain

$$\int_{\mathbb{R}^n} |\nabla u(x)| dx = \int_0^\infty \int_{E_t} \frac{dt}{d\nu} d\mathcal{H}_{n-1} d\nu$$

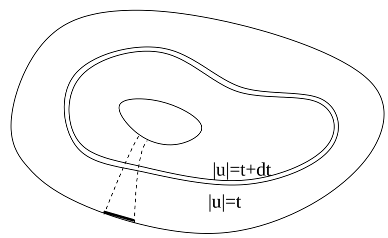

Figure 3: Level surfaces and lines of quickest descent, i.e. the lines orthogonal to the level surfaces.

$$= \int_0^\infty dt \int_{E_t} d\mathcal{H}_{n-1} = \int_0^\infty \mathcal{H}_{n-1}(E_t)dt.$$

A rigorous proof of the coarea formula for smooth functions can be found in [M4], Sect. 1.2.4. This formula was proved for the so called asymptotically differentiable functions of two variables, by A. S. Kronrod (1950). H. Federer obtained a more general result for Lipschitz mappings $\mathbb{R}^n \to \mathbb{R}^m$ (1959). The result was extended to the functions of bounded variation by Fleming and Rishel (1980).

Let us prove (3). By the coarea formula and by the isoperimetric inequality (2),

$$\int_{\mathbb{R}^n} |\nabla u| dx = \int_0^\infty \mathcal{H}_{n-1}(E_t)dx \geq n v_n^{1/n} \int_0^\infty (m_n N_t)^{\frac{n-1}{n}} dt,$$

where $N_t = \{x : |u(x)| \geq t\}$. It follows from the definition of the Lebesgue integral that

$$\Big(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\Big)^{\frac{n-1}{n}} = \Big(\int_0^\infty m_n N_t d(t^{\frac{n}{n-1}})\Big)^{\frac{n-1}{n}}$$

$$= \Big(\frac{n}{n-1}\int_0^\infty (m_n N_t)^{\frac{n-1}{n}} (m_n N_t)^{\frac{1}{n}} t^{\frac{1}{n-1}} dt\Big)^{\frac{n-1}{n}}. \qquad (5)$$

By using the obvious inequality

$$t(m_n N_t)^{\frac{n-1}{n}} \leq \int_0^t (m_n N_\tau)^{\frac{n-1}{n}} d\tau.$$

we conclude that the right-hand side of (5) does not exceed

$$\Big(\frac{n}{n-1}\int_0^\infty (m_n N_t)^{\frac{n-1}{n}} \Big(\int_0^t (m_n N_\tau)^{\frac{n-1}{n}} d\tau\Big)^{\frac{1}{n-1}} dt\Big)^{\frac{n-1}{n}}$$

$$= \int_0^\infty (m_n N_t)^{\frac{n-1}{n}} dt$$

Thus we have obtained (3) with the best possible constant. The proof of Theorem is complete. □

Now, we consider a more general inequality:

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \le C \int_\Omega |\nabla u| dx, \tag{6}$$

where $q \ge 1$, Ω is an open subset of $\mathbb{R}^n$, μ is an arbitrary measure and $u \in C_0^\infty(\Omega)$.

Theorem 1.2 *Inequality* (6) *with* $q \ge 1$ *holds if and only if*

$$\mu(g)^{1/q} \le C\mathcal{H}_{n-1}(\partial g) \tag{7}$$

for every bounded open set g *with smooth boundary,* $\overline{g} \subset \Omega$.

Proof. We start with the necessity. Let $u_\varepsilon \in C_0^\infty(\Omega)$, $u_\varepsilon = 1$ on g, $\overline{g} \subset \Omega$ with smooth ∂g and $u_\varepsilon = 0$ outside of the ε-neighborhood of g. By (6)

$$\mu(g)^{1/q} \le C \int_\Omega |\nabla u_\varepsilon| dx \to C\mathcal{H}_{n-1}(\partial g) \text{ as } \varepsilon \to 0,$$

for all bounded g with smooth ∂g.

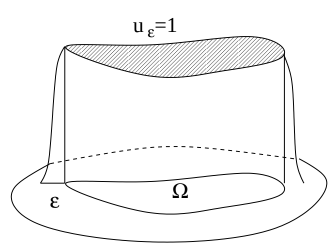

Figure 4: The function u_ε.

Sufficiency can be established by repeating the previous proof with q instead of $n/(n-1)$, and μ instead of m_n. □

Note that we have obtained a Sobolev type inequality (6) with the best possible constant

$$C = \sup_g \frac{\mu(g)^{1/q}}{\mathcal{H}_{n-1}(\partial g)}.$$

Example 1.3 Consider the inequality of the Hardy-Sobolev type:

$$\Big(\int_{\mathbb{R}^n} |u|^q \frac{dx}{|x|^\alpha}\Big)^{1/q} \leq C \int_{\mathbb{R}^n} |\nabla u| dx, \tag{8}$$

where $q \geq 1$ and $\alpha < n$. We try to find q and the best constant.

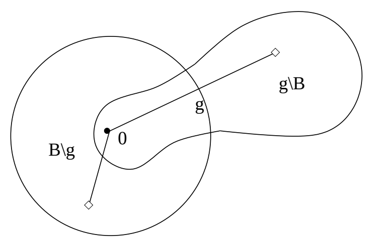

Figure 5: The volumes of B and g are equal.

Let $B = \{x \in \mathbb{R}^n : |x| < R\}$, $m_n B = m_n g$. We write

$$\begin{aligned} \mu(g) &= \int_g \frac{dx}{|x|^\alpha} \leq \int_{g \cap B} \frac{dx}{|x|^\alpha} + R^{-\alpha} m_n (B \setminus g) \\ &\leq \int_B \frac{dx}{|x|^\alpha} = n v_n \int_0^R \rho^{n-1-\alpha} d\rho = \frac{n v_n}{n-\alpha} R^{n-\alpha}, \end{aligned}$$

where $R = v_n^{-1/n} (m_n g)^{1/n}$.

This means that

$$\mu(g) \leq \frac{n v_n^{\alpha/n}}{n-\alpha} (m_n g)^{\frac{n-\alpha}{n}}. \tag{9}$$

Now by (2)

$$\mu(g) \leq \frac{n}{n-\alpha} v_n^{\alpha/n} (n^{-1} v_n^{-1/n})^{\frac{n-\alpha}{n-1}} \mathcal{H}_{n-1}(\partial g)^{\frac{n-\alpha}{n-1}},$$

that is

$$\mu(g)^{\frac{n-1}{n-\alpha}} \leq (n-\alpha)^{(1-n)/(n-\alpha)} (n v_n)^{(\alpha-1)/(n-\alpha)} \mathcal{H}_{n-1}(\partial g).$$

Hence the best constant in (8) with $q = (n-\alpha)/(n-1)$ is given by

$$C = (n-\alpha)^{(1-n)/(n-\alpha)} (n v_n)^{(\alpha-1)/(n-\alpha)}.$$

Example 1.4 Let $u \in C_0^\infty(\mathbb{R}^n)$. Using Theorem 1.2, we obtain the inequality:

$$\int_{\mathbb{R}^{n-1}} |u(x',0)|dx' \le \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u(x)|dx,$$

where $x' = (x_1, \dots, x_{n-1})$, where $1/2$ is the best constant. This example shows that the measure μ in (6) might be not absolutely continuous with respect to the Lebesgue measure.

Remark 1.5 If $\Omega = \mathbb{R}^n$, the inequality

$$\mu(g)^{1/q} \le C\mathcal{H}_{n-1}(\partial g) \tag{10}$$

follows from

$$\mu(B_\rho(x))^{1/q} \le C_1\rho^{n-1}.$$

for all balls $B_\rho(x) = \{y : |y-x| < \rho\}$. (See [M4], p. 56-57.)

Remark 1.6 There is the following simple generalisation of the coarea formula:

$$\int_\Omega \phi(x)|\nabla u(x)|dx = \int_0^\infty dt \int_{E_t} \phi(x)d\mathcal{H}_{n-1}, \tag{11}$$

where ϕ is a Borel function, and $E_t = \{x : |u(x)| \ge t\}$. Moreover, we can write

$$\int_\Omega F(x,\nabla u)dx = \int_0^\infty dt \int_{E_t} F(x,\nu(x))d\mathcal{H}_{n-1}, \tag{12}$$

where $\nu(x) = \frac{\nabla u(x)}{|\nabla u(x)|}$ i.e. the normal unit vectors and F is a continuous positive homogeneous function of degree 1, i.e. $F(x,\alpha y) = |\alpha|F(x,y)$ for all real α.

Now one can easily characterise more general Sobolev type inequalities. Let us consider the inequality

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \le C\int_\Omega F(x,\nabla u)dx, \tag{13}$$

where $q \ge 1$ and μ is an arbitrary measure. Repeating the proof of Theorem 1.2 (with obvious changes), one arrives at the following assertion.

Theorem 1.7 *The inequality* (13) *with* $q \ge 1$ *is equivalent the inequality*

$$\mu(g)^{1/q} \le C\int_{\Omega\cap\partial g} F(x,\nu(x))d\mathcal{H}_{n-1},$$

for all sets g *such that* $\overline{g} \subset \Omega$ *and* ∂g *is smooth.*

Remark 1.8 Consider the interpolation inequality

$$\Big(\int_\Omega |u|^q d\mu\Big)^{1/q} \le C\Big(\int_\Omega |\nabla u| dx\Big)^\theta \Big(\int_\Omega |u|^r d\nu\Big)^{(1-\theta)/r},$$

which may be written as

$$\|u\|_{L_q(\mu)} \le C\|\nabla u\|_{L_1}^\theta \|u\|_{L_r(\nu)}^{1-\theta}.$$

This integral inequality is equivalent to

$$\mu(g)^{1/q} \le C\mathcal{H}_{n-1}(\partial g)^\theta \nu(g)^{(1-\theta)/r}$$

The proof is rather similar to that of Theorem 1.2. For more information see [M4], Chapter 2.

2 Sobolev type inequality for functions with unrestricted boundary values

Let us consider the functions, which are not zero on the whole boundary. Let u be a function in $C^\infty(\Omega)$, $u = 0$ on a ball B, $\overline{B} \subset \Omega$. Let us consider the inequality

$$\|u\|_{L_q(\Omega)} \le C\|\nabla u\|_{L_1(\Omega)}. \tag{14}$$

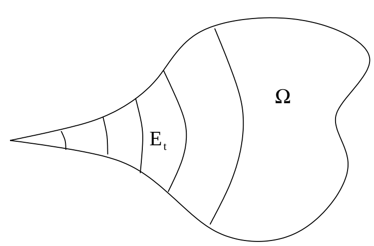

Figure 6: The level sets are not closed if the function u is not constant on $\partial\Omega$.

Let $q \ge 1$. Repeating the above proof of the Gagliardo-Nirenberg inequality (3), we make use of the coarea formula

$$\int_\Omega |\nabla u| dx = \int_0^\infty \mathcal{H}_{n-1}(E_t) dt$$

and we also need the inequality

$$(m_n N_t)^{1/q} \leq C\mathcal{H}_{n-1}(E_t),$$

where as before

$$N_t = \{x \in \Omega : |u(x)| \geq t\}.$$

But now E_t is not the whole boundary of N_t, just a part.

Remark 2.1 If $\partial\Omega$ does not contain inward cusps then it is clear intuitively that

$$\mathcal{H}_{n-1}(\partial\Omega \cap \partial g) \leq c\mathcal{H}_{n-1}(\Omega \cap \partial g) \qquad (15)$$

for all g, $g \cap B = \emptyset$ and we can use the classical isoperimetric inequality (2) in order to obtain (14) with a certain $C = C(\Omega)$.

If we have a cusp, $(n-1)$-measure of the interior part of ∂g is small and we may not apply the isoperimetric inequality (2).

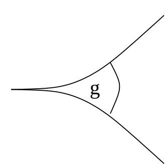

Figure 7: The set g in a domain with cusp. $\mathcal{H}_1(\Omega \cap \partial g) << \mathcal{H}_1(\partial\Omega \cap \partial g)$

Example 2.2 What can we expect for bad domains? Let us consider the curvilinear triangle given in Fig.8.

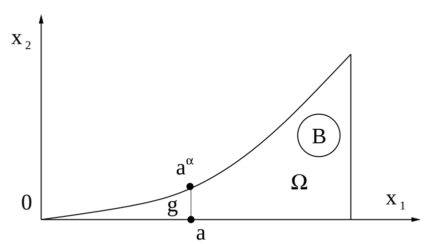

Figure 8: The domain Ω is bounded by lines $x_2 = 0$, $x_1 = 1$, and the curve $x_2 = x_1^\alpha$, $\alpha > 1$.

Here $g = \{x \in \Omega : x_1 < a\}$, $H_1(\Omega \cap \partial g) = a^\alpha$ and $m_2(g) = a^{\alpha+1}/(\alpha+1)$. Hence

$$(m_2 g)^{\alpha/(\alpha+1)} = (\alpha+1)^{-\alpha/(\alpha+1)}\mathcal{H}_1(\Omega \cap \partial g).$$

One can show that there exists a constant C such that

$$(m_2 g)^{\alpha/(\alpha+1)} \le C\mathcal{H}_1(\Omega \cap \partial g)$$

for all $g, \overline{g} \subset \Omega\backslash\overline{B}$. Inequalities of such a form are called *relative isoperimetric inequalities* (Dido's problem). The last inequality enables one to prove the Sobolev type estimate (14) with $q = (\alpha+1)/\alpha$. The proof is the same as that of Theorem 1.2.

Definition 2.3 We introduce the *area minimizing function*

$$\lambda(s) = \inf \mathcal{H}_{n-1}(\Omega \cap \partial g)$$

where infimum is extended over all admissible sets g with $m_n g \ge s$. In case $n = 2$ it is more appropriate to speak about the *length minimizing function* but we shall not mention this any more.

The following visible technical assertion was proved in [M4], Sect. 3.2.2. In its formulation and in the sequel we call an open subset g of Ω *admissible* if $\Omega \cap \partial g$ is a smooth surface.

Lemma 2.4 *Let g be an admissible subset of Ω such that $\mathcal{H}_{n-1}(\Omega \cap \partial g) < \infty$. Then there exists a sequence of functions $\{w_m\}_{m\ge1}$ with the properties:*

1) *w_m is locally Lipschitz in Ω;*

2) *$w_m(x) = 0$ in $\Omega \setminus g$,*

3) *$w_m(x) \in [0,1]$ in Ω,*

4) *for any compactum $K \subset g$ there exists an integer $N(e)$, such that $w_m(x) = 1$ for $x \in K$ and $m \ge N(e)$,*

5) $\limsup_{m\to\infty} \int_\Omega |\nabla w_m(x)| dx = \mathcal{H}_{n-1}(\Omega \cap \partial g)$.

Now we are in a position to obtain a necessary and sufficient condition for the Sobolev type inequality (14) to hold for all functions $u \in C^\infty(\Omega)$, $u = 0$ on a ball B, $\overline{B} \subset \Omega$.

Theorem 2.5 *The best constant in* (14) *with $q \ge 1$ is given by*

$$C = \sup_g \frac{m_n(g)^{1/q}}{\mathcal{H}_{n-1}(\Omega \cap \partial g)},$$

where the supremum is extended over all admissible sets g.

The proof of sufficiency is the same as that in Theorem 1.2. Necessity follows by setting the functions w_m from Lemma 2.4 into (14).

Remark 2.6 This theorem means that (14) holds if and only if the last supremum is finite which is equivalent to the inequality

$$\liminf_{s\to 0} s^{-1/q}\lambda(s) > 0. \tag{16}$$

Example 2.7 Consider the union Ω of the squares

$$Q_m = \{(x,y) : 2^{-m-1} \le x \le 3\cdot 2^{-m-2},\ 0 < y < 2^{-m-2}\}$$

and the rectangles

$$R_m = \{(x,y) : 3\cdot 2^{-m-2} \le x \le 2^{-m},\ 0 < y < 1\}$$

where $m = 0, 1, \dots$ (Fig. 9). One can show that there exist constants c_1 and c_2 such that

$$c_1 s \le \lambda(s) \le c_2 s$$

(see [M4], p. 171).

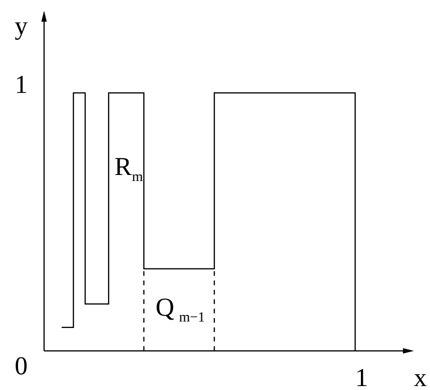

Figure 9: For this domain the inequality (14) holds for $q = 1$ and does not hold for any $q > 1$.

Example 2.8 Let Ω be an n-dimensional "whirlpool"

$$\{x = (x', x_n),\ |x'| < \varphi(x_n),\ 0 < x_n < 1\}, \tag{17}$$

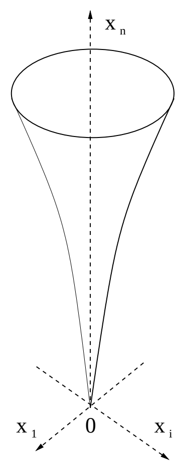

Figure 10: For the β-cusp, inequality (14) holds if and only if $q \leq 1+1/\beta(n-1)$.

where φ is a continuously differentiable convex function on $[0,1]$, $\varphi(0)=0$ (see Fig. 10). The area minimizing function satisfies

$$c\,[\varphi(t)]^{n-1} \leq \lambda\Big(v_{n-1}\int_0^1 [\varphi(t)]^{n-1} d\tau\Big) \leq [\varphi(t)]^{n-1} \qquad (18)$$

for sufficiently small t. (See [M4], p. 175-176).

In particular, for the β-cusp

$$\Omega = \Big\{x : \sum_{i=1}^{n-1} x_i^2 < x_n^{2\beta},\ \ 0 < x_n < 1\Big\}\ \ (\beta > 1)$$

one has

$$c_1 s^\alpha \leq \lambda(s) \leq c_2 s^\alpha, \quad \alpha = \frac{\beta(n-1)}{\beta(n-1)+1}.$$

Example 2.9 Let us consider a tube of finite volume narrowing at infinity $\Omega = \{x = (x', x_n),\ |x'| < \varphi(x_n)\}$, where φ is a convex continuously differentiable function on $[0,\infty]$ (see Fig. 11). One can show that for sufficiently large t the area minimizing function satisfies

$$c\,[\varphi(t)]^{n-1} < \lambda\Big(v_{n-1}\int_t^\infty |\varphi(t)]^{n-1} d\tau\Big) \leq [\varphi(t)]^{n-1}.$$

(see [M4], p. 176-178).

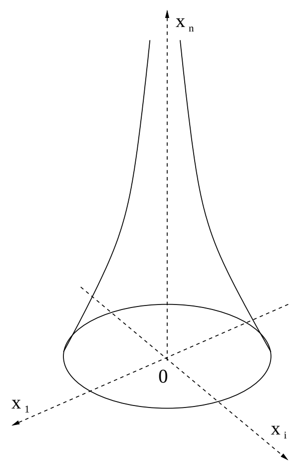

Figure 11: For the "β-tube" inequality (14) holds if and only if $q < 1 - 1/\beta(n-1)$.

In particular, the area minimizing function of the "β-tube" of finite volume

$$\Omega = \{x : |x'| < (1 + x_n)^{-\beta},\ \ 0 < x_n < \infty\},\ \ \beta(n-1) > 1,$$

is subject to the inequalities

$$c_1 s^\alpha \le \lambda(s) \le c_2 s^\alpha, \quad \alpha = \frac{\beta(n-1)}{\beta(n-1) - 1}.$$

3 Compactness criterion

The following theorem was proved by Sobolev's student Kondrashov (1938). An earlier compactness result of the same nature which concerns the imbedding of $L_2^1(\Omega)$ into $L_2(\Omega)$ is called Rellich's lemma.

Theorem 3.1 *Let Ω be bounded and satisfy the cone property. Then*

(1) *$W_p^l(\Omega)$ is compactly imbedded in $L_\infty(\Omega)$ if $pl > n$.*

(2) *$W_p^l(\Omega)$ is compactly imbedded into $L_q(\Omega)$ if $q < \frac{pn}{n-pl}$, $n \ge pl$.*

In the following compactness theorem, Ω is an arbitrary open set of finite volume. Compare this result with Remark 2.6 where the boundedness criterion for the same imbedding operator is formulated.

Theorem 3.2 *The ball*

$$\{u \in L_1^1(\Omega) : \|u\|_{L_1^1(\Omega)} \leq 1\}$$

is precompact in $L_q(\Omega)$, $n/(n-1) > q \geq 1$ *if and only if*

$$\lim_{s\to 0} \frac{\lambda(s)}{s^{1/q}} = \infty. \tag{19}$$

Proof. (Sufficiency) Consider the domain Ω. Take a subdomain ω such that $\overline{\omega} \subset \Omega$ and $m_n(\Omega \setminus \omega) < s$, where $s < m_n\Omega$.

Now we take another domain ω_1 such that

$$\overline{\omega} \subset \omega_1 \subset \overline{\omega_1} \subset \Omega.$$

Define a smooth function η with $\eta = 1$ on ω and $\eta = 0$ on $\Omega \setminus \omega_1$.

Now for all $u \in L_1^1(\Omega)$

$$\begin{aligned}
\|u\|_{L_q(\Omega)} &\leq \|(1-\eta)u\|_{L_q(\Omega)} + \|\eta u\|_{L_q(\Omega)} \\
&\leq \sup_{g\subset\Omega\setminus\omega} \frac{(m_n g)^{1/q}}{\mathcal{H}_{n-1}(\Omega\cap\partial g)} \|\nabla((1-\eta)u)\|_{L_q(\Omega)} + \|u\|_{L_q(\omega_1)} \\
&\leq \frac{s^{1/q}}{\lambda(s)} \|\nabla((1-\eta)u)\|_{L_1(\Omega)} + \|u\|_{L_q(\omega_1)} \\
&\leq \frac{s^{1/q}}{\lambda(s)} (\|\nabla u\|_{L_1(\Omega)} + \max|\nabla\eta| \|u\|_{L_1(\omega)}) + \|u\|_{L_q(\omega_1)} \\
&\leq \frac{s^{1/q}}{\lambda(s)} \|\nabla u\|_{L_1(\Omega)} + C(s)\|u\|_{L_q(\omega_1)}.
\end{aligned}$$

Let $\{u_k\}_{k\geq 1}$ be a sequence satisfying

$$\|\nabla u_k\|_{L_1(\Omega)} + \|u_k\|_{L_1(\omega_1)} \leq 1.$$

Since the boundary of ω_1 is smooth, the imbedding operator $L_1^1(\omega_1) \to L_q(\omega_1)$ is compact and we may suppose that $\{u_k\}_{k\geq 1}$ is a Cauchy sequence in $L_q(\omega_1)$. We have

$$\|u_m - u_l\|_{L_q(\Omega)} \leq \frac{2s^{1/q}}{\lambda(s)} + C(s)\|u_m - u_l\|_{L_q(\omega_1)}$$

and hence

$$\limsup_{m,l\to\infty} \|u_m - u_l\|_{L_q(\Omega)} \leq \frac{s^{1/q}}{\lambda(s)}.$$

It remains to pass to the limit in the right-hand side as $s \to 0$ and take (19) into account.

(Necessity) Let the imbedding $L_1^1(\Omega) \subset L_q(\Omega)$ be compact. Then $L_1^1(\Omega) \subset L_q(\Omega)$ and the elements of a unit ball in $W_1^1(\Omega)$ have absolutely equicontinuous norms in $L_q(\Omega)$. Hence, for all $u \in L_1^1(\Omega)$

$$\Big(\int_g |u|^q dx\Big)^{1/q} \le \varepsilon(s) \int_\Omega (|\nabla u| + |u|)dx, \tag{20}$$

where g is an arbitrary admissible subset of Ω whose measure does not exceed s and $\varepsilon(s)$ tends to zero as $s \to +\infty$.

We insert the sequence $\{w_m\}$ from Lemma 2.4 into (20). Then for any compactum $K \subset g$

$$m_n(K)^{1/q} \le c\varepsilon(s)(\mathcal{H}_{n-1}(\Omega \cap \partial g) + m_n(g))$$

and hence

$$m_n(g)^{1/q} \le c_1\varepsilon(s)\mathcal{H}_{n-1}(\Omega \cap \partial g).$$

The theorem is proved.

Example 3.3 The compactness condition (19) for the whirlpool domain in Example 2.8 is equivalent to

$$\lim_{x\to 0} \Big(\int_0^x [f(\tau)]^{n-1} d\tau\Big)^{1/q} [f(x)]^{1-n} = 0.$$

Let, in particular, $f(x) = x^\beta$, $\beta > 1$. Then $L_1^1(\Omega)$ is compactly imbedded into $L_q(\Omega)$ if and only if

$$q < \frac{\beta(n-1)+1}{\beta(n-1)}$$

Example 3.4 For the domain shown in Fig.12, the imbedding operator from $L_1^1(\Omega)$ into $L_1(\Omega)$ is compact for $\alpha < 2$, bounded and noncompact for $\alpha = 2$ and unbounded for $\alpha > 2$ (see [M4], Sect. 4.10.3).

4 The case $p = 1$, $q < 1$ in the Sobolev type inequality (14)

Let u be a function in Ω measurable with respect to the Lebesgue measure m_n. We associate with u its nonincreasing rearrangement u^* on $(0, \infty)$ which is introduced by

$$u^*(t) = \inf\{s > 0 : m_n(M_s) \le t\}, \tag{21}$$

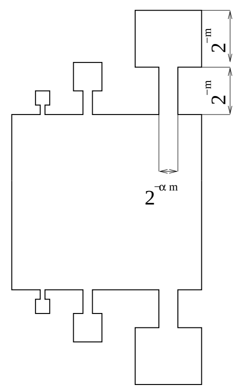

Figure 12: This domain is borrowed from vol.2 of Courant-Hilbert

where $M_s = \{x \in \Omega : |u(x)| > s\}$.

Clearly u^* is nonnegative and nonincreasing on $(0, \infty)$; $u^*(t) = 0$ for $t \geq m_n(\Omega)$. Furthermore, it follows from the definition of u^* that

$$u^*(m_n(M_s)) \leq s \tag{22}$$

and

$$m_n(M_{u^*(t)}) \leq t, \tag{23}$$

the last because the function $s \to m_n(M_s)$ is continuous from the right.

The nonincreasing rearrangement of a function has the following important property.

Lemma 4.1 *If $p \in (0, \infty)$, then*

$$\int_\Omega |u(x)|^p dx = \int_0^\infty (u^*(t))^p dt.$$

Proof. The required equality if a consequence of the formula

$$\int_\Omega |u(x)|^p dx = \int_0^\infty m_n(M_t) d(t^p)$$

and the identity

$$m_1(M_s^*) = m_n(M_s), \quad s \in (0, \infty), \tag{24}$$

in which $M_s^* = \{t > 0 : u^*(t) > s\}$. To check (24), we first note that

$$m_1(M_s^*) = \sup\{t > 0 : u^*(t) > s\} \tag{25}$$

by the monotonicity of u^*. Hence, (22) yields

$$m_1(M_s^*) \le m_n(M_s).$$

For the inverse inequality, let $\varepsilon > 0$ and $t = m_n(M_s^*) + \varepsilon$. Then (25) implies $u^*(t) \le s$ and therefore

$$m_n(M_s^*) \le m_n(M_{u^*(t)}) \le t$$

by (23). Thus $m_n(M_s) \le m_1(M_s^*)$ and (24) follows.

Let $u \in C^\infty(\Omega)$, $u = 0$ on a ball B, $\overline{B} \subset \Omega$, as in Section 4. Here we show that the case $q < 1$ in inequality (14), also admits a complete solution (see [M4], Sect. 4.4 and [MN]). Let, as before, $\lambda(s)$ be the area minimizing function.

Theorem 4.2 *Let Ω be a domain in $\mathbb{R}^n$, B is an open ball, $\overline{B} \subset \Omega$ and $0 < q < 1$.*

(i)(Sufficiency) If

$$D := \int_0^{m_n(\Omega)} \Big(\frac{s^{1/q}}{\lambda(s)}\Big)^{\frac{q}{1-q}} \frac{ds}{s} < \infty, \tag{26}$$

then (14) holds for all $u \in C^\infty(\Omega)$, $u = 0$ on B. The constant C satisfies $C \le c_1(q) D^{(1-q)/q}$.

(ii)(Necessity) If there is a constant $C > 0$ such that (14) holds for all $u \in C^\infty(\Omega)$, $u|_B = 0$, then (26) holds and $C \ge c_2(q) D^{(1-q)/q}$.

Proof. (Sufficiency) Note that (26) implies $m_n(\Omega) < \infty$ and that λ is a positive function. By monotonicity of $m_n(N_t)$, one obtains

$$\begin{aligned}\int_\Omega |u|^q dx &= \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} m_n(N_t) d(t^q) \\ &\le \sum_{j=-\infty}^{\infty} \mu_j (2^{q(j+1)} - 2^{qj}),\end{aligned}$$

where $\mu_j = m_n(N_{2^j})$. We claim that the estimate

$$\sum_{j=r}^{m} \mu_j (2^{q(j+1)} - 2^{qj}) \le c D^{1-q} \|\nabla u\|^q_{L_1(\Omega)} \tag{27}$$

is true for any integers r, m, $r < m$. Once (27) has been proved, (26) follows by letting $m \to \infty$ and $r \to -\infty$ in (27). Clearly, the sum on the left in (27) is not greater than

$$\mu_m 2^{q(m+1)} + \sum_{j=1+r}^{m} (\mu_{j-1} - \mu_j) 2^{jq}. \tag{28}$$

Let $S_{r,m}$ denote the sum over $1 + r \le j \le m$. Hölder's inequality implies

$$S_{r,m} \le \Big[\sum_{j=1+r}^{m} 2^j \lambda(\mu_{j-1}) \Big]^q \Big\{ \sum_{j=1+r}^{m} \frac{(\mu_{j-1} - \mu_j)^{1/(1-q)}}{\lambda(\mu_{j-1})^{1/(1-q)}} \Big\}^{1-q}. \tag{29}$$

We have

$$(\mu_{j-1} - \mu_j)^{1/(1-q)} \le \mu_{j-1}^{1/(1-q)} - \mu_j^{1/(1-q)}.$$

Hence, by the monotonicity of λ, the sum in curly braces is dominated by

$$\sum_{j=1+r}^{m} \int_{\mu_j}^{\mu_{j-1}} \lambda(t)^{q/(q-1)} d(t^{1/(1-q)}),$$

which does not exceed $D/(1-q)$. By the coarea formula the sum in square brackets in (29) is not greater than

$$2 \sum_{j=-\infty}^{\infty} \int_{N_{2^{j-1}} \setminus N_{2^j}} |\nabla u| dx.$$

Thus

$$\sum_{j=1+r}^{m} (\mu_{j-1} - \mu_j) 2^{qj} \le c D^{1-q} \|\nabla u\|^q_{L_1(\Omega)}.$$

To conclude the proof of (27), we show that the first term in (28) is also dominated by the right part of (27). Indeed, if $\mu_m > 0$, then

$$\begin{aligned} \mu_m 2^{mq} &\le (2^m \lambda(\mu_m))^q (\mu_m / \lambda(\mu_m))^{q/(1-q)} \mu_m)^{1-q} \\ &\le c \|\nabla u\|^q_{L_1(\Omega)} \Big(\int_0^{\mu_m} \Big(\frac{t}{\lambda(t)} \Big)^{q/(1-q)} dt \Big)^{1-q}. \end{aligned}$$

The sufficiency of (26) follows.

In the proof of necessity we need the following simple observation.

Lemma 4.3 *Let $\{v_1, \dots, v_N\}$ be a finite collection in the space $C(\Omega) \cap L^1_p(\Omega)$, $p \in [1, \infty)$. Then for $x \in \Omega$ the function*

$$x \mapsto v(x) = \max\{v_1(x), \dots, v_N(x)\}$$

belongs to the same space and

$$\|\nabla v\|_{L_1(\Omega)} \leq \sum_{i=1}^{N} \|\nabla v_i\|_{L_1(\Omega)}.$$

Proof. An induction argument reduces consideration to the case $N = 2$. Here

$$v = (v_1 + v_2 + |v_1 - v_2|)/2.$$

Furthermore

$$\nabla v = \frac{1}{2}(\nabla v_1 + \nabla v_2 + \operatorname{sgn}(v_1 + v_2)(\nabla v_1 - \nabla v_2))$$

almost anywhere in Ω. Therefore,

$$|\nabla v(x)| \leq \max\{|\nabla v_1(x)|, |\nabla v_2(x)|\}$$

for almost all $x \in \Omega$. The last inequality gives

$$|\nabla v(x)| \leq |\nabla v_1(x)| + |\nabla v_2(x)|,$$

thus concluding the proof.

Continuation of Proof of Theorem 4.2 (Necessity) First we remark that the claim implies $m_n(\Omega) < \infty$ and that $\lambda(t) > 0$ for all $t \in (0, m_n(\Omega)]$. Let j be any integer satisfying $2^j \leq m_n(\Omega)$. Then there exists a subset g_j of Ω such that

$$m_n(g_j) \geq 2^j, \text{ and } \mathcal{H}_{n-1}(\Omega \cap g_j) \leq 2\lambda(2^j).$$

By the definition of λ and the coarea formula there is a function $u_j \in C^\infty(\Omega)$ subject to $u_j \geq 1$ on g_j, $u_j = 0$ on B and

$$\|\nabla u_j\|_{L_1(\Omega)} \leq 4\lambda(2^j).$$

Let s be the integer for which $2^s \leq m_n(\Omega) < 2^{s+1}$. For any integer $r < s$, we put

$$f_{r,s}(x) = \max_{r \leq j \leq s} \beta_j u_j(x), \;\; x \in \Omega,$$

where

$$\beta_j = (2^j/\lambda(2^j))^{1/(1-q)}.$$

By the above lemma

$$\|\nabla f_{r,s}\|_{L_1(\Omega)} \leq c \sum_{j=r}^{s} \beta_j \|\nabla u_j\|_{L_1(\Omega)},$$

and one obtains the following upper bound for $\|\nabla f_{r,s}\|_{L_1(\Omega)}$:

$$\|\nabla f_{r,s}\|_{L_1(\Omega)} \le c \sum_{j=r}^{s} \beta_j \lambda(2^j). \tag{30}$$

We now derive a lower bound for the norm of $f_{r,s}$ in $L_q(\Omega)$. Since $f_{r,s}(x) \ge \beta_j$ for $x \in g_j$, $r \le j \le s$, and $m_n(g_j) \ge 2^j$, the inequality

$$m_n(\{x \in \Omega : |f_{r,s}(x)| > \tau\}) < 2^j$$

implies $\tau \ge \beta_j$. Hence

$$f^*_{r,s}(t) \ge \beta_j \text{ for } t \in (0, 2^j), \quad r \le j \le s,$$

where $f^*_{r,s}$ is the nonincreasing rearrangement of $f_{r,s}$. Then

$$\int_0^{m_n(\Omega)} (f^*_{r,s}(t))^q dt \ge \sum_{j=r}^{s} \int_{2^{j-1}}^{2^j} (f^*_{r,s})^q dt \ge \sum_{j=r}^{s} \beta_j^q 2^{j-1},$$

which implies

$$\|f_{r,s}\|^q_{L_q(\Omega,\mu)} \ge \sum_{j=r}^{s} \beta_j^q 2^{j-1}. \tag{31}$$

Next, we note that if inequality (14) holds for all $u \in C^\infty(\Omega) \cap L_1^1(\Omega)$, then it holds for all $u \in C(\Omega) \cap L_1^1(\Omega)$. In particular,

$$\|f_{r,s}\|_{L_q(\Omega)} \le C \|\nabla f_{r,s}\|_{L_1(\Omega)}.$$

Now (30) and (31) in combination with the last inequality give

$$C \ge c \frac{\left(\sum_{j=r}^{s} \beta_j^q 2^j\right)^{1/q}}{\sum_{j=r}^{s} \beta_j (2^j)} = c \Big(\sum_{j=r}^{s} \frac{2^{j/(1-q)}}{(\lambda(2^j))^{q/(1-q)}} \Big)^{(1-q)/q}.$$

By letting $r \to -\infty$ and by the monotonicity of λ, we obtain

$$C \ge c \Big(\sum_{j=-\infty}^{s} \Big(\frac{2^j}{\lambda(t)}\Big)^{\frac{q}{1-q}} 2^j \Big)^{\frac{1-q}{q}} \ge c \Big(\int_0^{m_n(\Omega)} \Big(\frac{t}{\lambda(t)}\Big)^{\frac{q}{1-q}} dt \Big)^{\frac{1-q}{q}}.$$

This completes the proof of Theorem 4.2. □

Example 4.4 Consider the Nikodym domain depicted in Figure 6. Let $\varepsilon_m = \delta(2^{-m-1})$ where δ is a Lipschitz function on $[0, 1]$ such that $c_1\delta(t) \le \delta(2t) \le c_2\delta(t)$. Then $c_3\delta(s) \le \lambda(s) \le c_4\delta(s)$ (see [M4], Sect. 3.4). Therefore, inequality (14) holds if and only if

$$\int_0^1 \Big(\frac{s^{1/q}}{\delta(s)}\Big)^{q/(1-q)} \frac{ds}{s} < \infty.$$

5 Imbeddings into fractional Sobolev spaces

Definition 5.1 We introduce the seminorm

$$\langle u\rangle_{q,\mu} = \Big(\int_\Omega\int_\Omega |u(x)-u(y)|^q \mu(dx,dy)\Big)^{1/q}, \tag{32}$$

where μ is a measure on $\Omega\times\Omega$, $\mu(\mathcal{E},\mathcal{F}) = \mu(\mathcal{F},\mathcal{E})$, and Ω is any open set.

Here we shall deal with the inequality

$$\langle u\rangle_{q,\mu} \le C\int_\Omega |\nabla u|dx, \tag{33}$$

where $u\in C^\infty(\Omega)\cap L_1^1(\Omega)$.

We show that inequality (33) is equivalent to an isoperimetric inequality of a new type.

Theorem 5.2 *Inequality* (33) *holds with* $q\ge 1$ *if and only if for any* $g\subset\Omega$ *such that* $\Omega\cap\partial g$ *is smooth, the isoperimetric inequality*

$$\mu(g,\Omega\setminus g)^{1/q} \le 2^{-1/q} C\mathcal{H}_{n-1}(\Omega\cap\partial g) \tag{34}$$

holds.

Proof. (Sufficiency) Denote by u_+ and u_- the positive and negative parts of u, so that $u = u_+ - u_-$. Since,

$$\langle u\rangle_{q,\mu} \le \langle u_+\rangle_{q,\mu} + \langle u_-\rangle_{q,\mu}$$

and

$$\int_\Omega |\nabla u_+|dx + \int_\Omega |\nabla u_-|dx = \int_\Omega |\nabla u|dx,$$

it suffices to prove (33) for nonnegative Lipschitz functions u. Clearly

$$\langle u\rangle_{q,\mu}^q = \int_\Omega\int_\Omega |u(x)-u(y)|^q\mu(dx,dy) = \int_\Omega\int_\Omega \Big|\int_{u(y)}^{u(x)} dt\Big|^q \mu(dx,dy).$$

By Minkowski's inequality

$$\begin{aligned}\langle u\rangle_{q,\mu} &\le 2^{1/q}\int_0^\infty \Big(\int_\Omega\int_\Omega \chi(u(x)>t>u(y))\mu(dx,dy)\Big)^{1/q} dt\\ &= 2^{1/q}\int_0^\infty \mu(N_t,\Omega\setminus N_t)^{1/q} dt.\end{aligned}$$

By using (34) we obtain

$$\langle u\rangle_{q,\mu} \le C\int_0^\infty \mathcal{H}_{n-1}(E_t)dt = C\int_\Omega |\nabla u|dx.$$

(Necessity) Let $\{w_m\}$ be the sequence from Lemma 2.4. Then

$$\langle w_m\rangle \le 2^{1/2}C\int_\Omega |\nabla w_m|dx \to 2^{1/2}C\mathcal{H}_{n-1}(\Omega\cap\partial g) \ \text{ as } m\to\infty$$

and

$$\lim_{m\to\infty}\int_\Omega\int_\Omega |w_m(x)-w_m(y)|^q\mu(dx,dy)$$
$$= 2^{1/q}\int_g\int_{\Omega\setminus g}\mu(dx,dy) = 2^{1/q}\mu(g,\Omega\setminus g)^{1/q}.$$

The result follows.

Corollary 5.3 *(One-dimensional case) Let*

$$\Omega = (\alpha,\beta),\ \ -\infty \le \alpha < \beta \le \infty.$$

The inequality

$$\Big(\int_\Omega\int_\Omega |u(x)-u(y)|^q\mu(dx,dy)\Big)^{1/q} \le C\int_\Omega |u'(x)|dx$$

with $q\ge 1$ holds for all $u\in C^\infty(\Omega)$ if and only if

$$\mu(I,\Omega\setminus I)^{1/q} \le 2^{-1/q}C \tag{35}$$

for all intervals I, $\overline{I}\subset\Omega$, and

$$\mu(I,\Omega\setminus I)^{1/q} \le 2^{1-1/q}C \tag{36}$$

for all intervals $I\subset\Omega$ such that $\overline{I}$ contains one of the ends of Ω.

Proof. Necessity follows directly from (34) by setting $g=I$. Let us prove the sufficiency of (35). Represent an arbitrary open set $g\subset\Omega$ as the sum of non-overlapping intervals I_k. Then by (35) and (36)

$$\begin{aligned}\mu(g,\Omega\setminus g)^{1/q} &= \big(\sum_k \mu(I_k,\Omega\setminus g)\big)^{1/q} \le \big(\sum_k \mu(I_k,\Omega\setminus I_k)\big)^{1/q}\\ &\le \sum_k \mu(I_k,\Omega\setminus I_k)^{1/q} \le 2^{-1/q}C\sum_k \mathcal{H}_0(\Omega\cap\partial I_k)\end{aligned}$$

which is the same as (34). The result follows from Theorem 5.2.

Example 5.4 We deal with functions in $\mathbb{R}^n$ and prove the inequality:

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^q}{|x-y|^{n+\alpha q}}dxdy\Big)^{1/q}\le C\int_{\mathbb{R}^n}|\nabla u|dx, \tag{37}$$

where $u\in C_0^\infty(\mathbb{R}^n)$, $n>1$, $0<\alpha<1$ and $q=n/(n-1+\alpha)$.

Let us introduce the set function

$$g\to\mathcal{I}(g):=\int_g\int_{\mathbb{R}^n\backslash g}\frac{dxdy}{|x-y|^{n+\alpha q}}.$$

By Theorem 5.2 we only need to prove the isoperimetric inequality

$$(\mathcal{I}(g))^{\frac{n-1}{n-\alpha q}}\le c(\alpha,n)\mathcal{H}_{n-1}(\partial g) \tag{38}$$

for $q=n/(n-1+\alpha)$. Let Δ be the Laplace operator in $\mathbb{R}^n$. If $u=r^\lambda$, we may write

$$\Delta u=\frac{1}{r^{n-1}}(r^{n-1}u_r)_r=\lambda(\lambda+n-2)r^{\lambda-2}.$$

Setting $\lambda=2-n-\alpha q$, we arrive at

$$\Delta_y|x-y|^{2-n-\alpha q}=(n-2+\alpha q)|x-y|^{-n-\alpha q}.$$

Using (2) and Example 1.3, we obtain

$$\begin{aligned}\mathcal{I}(g)&=\frac{1}{\alpha q(n-2+\alpha q)}\int_g\int_{\mathbb{R}^n\backslash g}\Delta_y|x-y|^{2-n-\alpha q}dydx\\&=\frac{1}{\alpha q(n-2+\alpha q)}\int_g\int_{\partial g}\frac{\partial}{\partial\nu_y}|x-y|^{2-n-\alpha q}dydx\\&\le\frac{1}{\alpha q}\int_{\partial g}\int_g|x-y|^{n-1+\alpha q}dxds_y\\&\le\frac{nv_n^{1-\frac{1-\alpha q}{n}}}{\alpha q(1-\alpha q)}(m_ng)^{\frac{1-\alpha q}{n}}\mathcal{H}_{n-1}(\partial g)\le\frac{(nv_n)^{1-\frac{1-\alpha q}{n-1}}}{\alpha q(1-\alpha q)}\mathcal{H}_{n-1}(\partial g)^{1+\frac{1-\alpha q}{n-1}}.\end{aligned}$$

Since

$$1-\alpha q=\frac{(n-1)(1-\alpha)}{n-1+\alpha},$$

inequality (38) follows.

Remark 5.5 Inequality (37) can be interpreted as the imbedding

$$\mathring{L}_1^1(\mathbb{R}^n)\subset\mathring{W}_q^\alpha(\mathbb{R}^n)$$

where $\mathring{L}_1^1(\mathbb{R}^n)$ is the completion of the space $C_0^\infty(\mathbb{R}^n)$ in the norm $\|\nabla u\|_{L_1(\mathbb{R}^n)}$ and $\mathring{W}_q^\alpha(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ in the fractional Sobolev norm

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^q}{|x-y|^{n+\alpha q}}dxdy\Big)^{1/q}.$$

6 Capacity minimizing functions and their applications to Sobolev type inequalities

Definition 6.1 Wiener's capacity of a compact set $F \subset \Omega$ with respect to Ω is defined by

$$\mathrm{cap} F = \inf_{u \geq 1 \text{ on } F} \int_\Omega |\nabla u(x)|^2 dx, \tag{39}$$

where $u \in C_0^\infty(\Omega)$. Its obvious generalization is the p-capacity

$$\mathrm{cap}_p F = \inf_{u \geq 1 \text{ on } F} \int_\Omega |\nabla u(x)|^p dx, \tag{40}$$

where $u \in C_0^\infty(\Omega)$, $p \geq 1$.

For basic properties of the p-capacity see [M4], Ch. 2.

The following arguments are very convincing but not fully rigorous because of the presence of critical points. The complete proof can be found in [M4], Ch. 2.

Let $u \in C_0^\infty(\Omega)$. We write $|u|$ in the form of a composition $\lambda(v)$, where $v(x)$ is the volume of the set bounded by the level surface of $|u|$ passing through the point x. By the coarea formula (4),

$$\|\nabla u\|_{L_p(\Omega)} = \Big\{ \int_0^{m_n(\Omega)} |\lambda'(v)|^p \int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x) dv \Big\}^{1/p}. \tag{41}$$

We note that by Hölder's inequality we have the following estimate for the area $s(v)$ of the surface $\{x : v(x) = v\}$

$$\begin{aligned} [s(v)]^p &= \Big(\int_{v(x)=v} \Big(\frac{dv}{d\nu}\Big)^{(p-1)/p} \Big(\frac{d\nu}{dv}\Big)^{(p-1)/p} ds \Big)^p \\ &\leq \int_{v(x)=v} \Big(\frac{dv}{d\nu}\Big)^{p-1} ds \Big(\int_{v(x)=v} \frac{d\nu ds}{dv} \Big)^{p-1}, \end{aligned}$$

where $d\nu$ is an element of the trajectory orthogonal to a level surface. Because of the obvious identity

$$\int_{v(x)=v} d\nu ds = dv$$

we find

$$\int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x) \geq [s(v)]^p, \tag{42}$$

where $s(v)$ is the area of the level surface $v(x) = v$.

Minimizing the integral

$$\int_\Omega |\nabla(\lambda(v(x))|^p dx$$

over all smooth functions on the segment $[0, m_n\Omega]$ such that $\lambda(0) = 0$ subject to the inequality $\lambda(v) \geq 1$ for $v \leq m_n(F)$, we obtain another expression for the p-capacity

$$\mathrm{cap}_p(F) = \inf\Big\{\int_{m_n(F)}^{m_n(\Omega)} \Big[\int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x)\Big]^{1/(1-p)} dv\Big\}^{1-p} \qquad (43)$$

Here the infimum is taken over all functions u in the definition (40). This useful identity is known as the Dirichlet principle with prescribed level surfaces, Pólya-Szegö [1951].

Estimating the integral over the level surface $v(x) = v$ with the aid of (42), we derive the following lower estimate from (43) for p-capacity

$$\mathrm{cap}_p(F) \geq \inf\Big\{\int_{m_n(F)}^{m_n(\Omega)} \frac{dv}{[s(v)]^{p/(p-1)}}\Big\}^{1-p} \qquad (44)$$

From (44) and (2) we obtain

$$\mathrm{cap}_p(F) \geq n v_n^{\frac{p}{n}} \Big|\frac{p-n}{p-1}\Big|^{p-1} |[m_n(\Omega)]^{\frac{p-n}{n(p-1)}} - [m_n(F)]^{\frac{p-n}{n(p-1)}}|^{1-p}, \qquad (45)$$

if $p \neq n$, and

$$\mathrm{cap}_p(F) \geq n^n v_n \Big[\log \frac{m_n(\Omega)}{m_n(F)}\Big]^{1-n}, \qquad (46)$$

if $p = n$. In particular, if $n > p$ then

$$\mathrm{cap}_p(F) \geq n v_n^{p/n} \Big(\frac{n-p}{p-1}\Big)^{p-1} [m_n(F)]^{(n-p)/n}. \qquad (47)$$

The application of p-capacity to imbedding theorems is based on following inequality which plays the same role for $p \geq 1$ as the coarea formula for $p = 1$.

Theorem 6.2 (see [M3], [M4]) *The inequality*

$$\int_0^\infty \mathrm{cap}_p N_t \, d(t^p) \leq \frac{p^p}{(p-1)^{p-1}} \int_\Omega |\nabla u|^p dx \qquad (48)$$

holds, where $u \in C_0^\infty(\Omega)$, $p \geq 1$ *and the constant is the best possible.*

For $p > 1$ this inequality is obtained in the following manner. In view of (41)

$$\|\nabla u\|^p_{L_p(\Omega)} = \int_0^{\varphi(m_n\Omega)} \Big|\frac{d}{d\varphi}\lambda(v(\varphi))\Big|^p d\varphi, \tag{49}$$

where φ is a new independent variable defined by the formula

$$\varphi(v) = \int_v^{m_n(\Omega)} \Big[\int_{v(x)=v} |\nabla v(x)|^{p-1} ds(x)\Big]^{1/(1-p)} dv.$$

In view of Hardy's inequality the right hand side of (49) majorizes

$$\Big(\frac{p-1}{p}\Big)^p \int_0^\infty \Big[\frac{\lambda(v(\varphi))}{\varphi}\Big]^p d\varphi = \frac{(p-1)^{p-1}}{p^p} \int_0^\infty \varphi^{1-p} d[\lambda(v(\varphi))]^p.$$

It remains to apply identity (43) which implies

$$\mathrm{cap}_p(N_{\lambda(v(\varphi))}) \le \varphi^{1-p}. \tag{50}$$

Remark 6.3 Inequality (48) with a rougher constant can be obtained quite simply by the following truncation argument.

Let as before $N_t = \{x : |u(x)| \ge t\}$. Clearly,

$$\begin{aligned}
\int_\Omega |\nabla u|^p dx &= \sum_{k=-\infty}^{\infty} \int_{N_{2^k}\setminus N_{2^{k+1}}} |\nabla u|^p dx = \sum 2^{kp} \int_{N_{2^k}\setminus N_{2^{k+1}}} \Big|\nabla \frac{u-2^k}{2^k}\Big|^p dx \\
&\ge \sum_{k=-\infty}^{\infty} 2^{kp}\mathrm{cap}_p N_{2^{k+1}} \ge C(p) \sum_{k=\infty}^{\infty} \mathrm{cap}_p N_{2^k}(2^{(k+1)p} - 2^{kp}) \\
&\ge C(p) \sum_k \int_{2^k}^{2^{k+1}} \mathrm{cap}_p N_t d(t^p) = C(p) \int_0^\infty \mathrm{cap}_p(N_t) d(t^p),
\end{aligned}$$

concluding the proof.

Remark 6.4 Inequalities (44)-(47) may be called *isocapacitary inequalities.* In the next theorem we deal with the isocapacitary inequality

$$\mu(g)^{p/q} \le C\mathrm{cap}_p \overline{g} \tag{51}$$

where μ is an arbitrary measure, g is an arbitrary open set with smooth boundary such that $\overline{g} \subset \Omega$, and $q \ge p \ge 1$. By (4), this inequality coincides with the isoperimetric inequality (52) for $p = 1$.

The following criterion shows the importance of (51).

Theorem 6.5 *(i) Assume that there exists a constant C such that (51) holds with $q \geq p \geq 1$ Then the inequality*

$$\Big(\int_\Omega |u|^q d\mu \Big)^{1/q} \leq D \|\nabla u\|_{L_p(\Omega)} \tag{52}$$

holds for all $u \in C_0^\infty(\Omega)$ with

$$D \leq p(p-1)^{(1-p)/p} C^{1/p}.$$

(ii) Conversely, if (52) holds for all $u \in C_0^\infty(\Omega)$ with $q > 0$ and $p \geq 1$, then the isocapacitary inequality (51) holds with

$$D \geq C^{1/p}.$$

Proof. The assertion (ii) follows directly from the definition of p-capacity.

Let us prove (i). We have

$$\int_\Omega |u|^q d\mu = \int_0^\infty \mu(N_t) d(t^q) = q \int_0^\infty \mu(N_t)^{p/q} t^{p-1} \mu(N_t)^{(q-p)/q} t^{q-p} dt.$$

Since $\mu(N_t)$ is a nonincreasing function,

$$\mu(N_t)^{p/q} t^p \leq p \int_0^t \mu(N_\tau)^{p/q} \tau^{p-1} d\tau.$$

It follows that

$$\begin{aligned} \int_\Omega |u|^q d\mu &\leq q \int_0^\infty \mu(N_t)^{p/q} t^{p-1} \Big(p \int_0^t \mu(N_\tau)^{p/q} \tau^{p-1} d\tau \Big)^{\frac{q-p}{p}} dt \\ &= \Big(p \int_0^\infty \mu(N_\tau)^{p/q} \tau^{p-1} d\tau \Big)^{q/p}. \end{aligned}$$

Now by (51) and (48)

$$\begin{aligned} \Big(p \int_0^\infty \mu(N_\tau)^{p/q} \tau^{p-1} d\tau \Big)^{q/p} &\leq C^{q/p} \Big(p \int_0^\infty \mathrm{cap}_p(N_\tau) \tau^{p-1} d\tau \Big)^{q/p} \\ &\leq C^{q/p} \Big(\frac{p^p}{(p-1)^{p-1}} \int_\Omega |\nabla u|^p dx \Big)^{q/p}. \end{aligned}$$

The proof is complete.

Definition 6.6 We introduce the capacity minimizing function

$$\nu_p(s) := \inf_{\{g:\mu(g)\geq s\}} \text{cap}_p g$$

which coincides with the area minimizing function

$$\lambda(s) = \inf_{\{g:\mu(g)\geq s\}} \mathcal{H}_{n-1}(\partial g)$$

for $p = 1$.

Remark 6.7 Clearly, the isocapacitary inequality (51) is equivalent to

$$\frac{s^{p/q}}{\nu_p(s)} \leq C.$$

Let $\mu(\Omega) < \infty$. One can show that for $q \geq p \geq 1$ the ball $\{u \in C_0^\infty(\Omega) : \|\nabla u\|_{L_p(\Omega)} \leq 1\}$ is precompact in $L_q(\Omega, \mu)$ if and only if

$$\frac{s^{p/q}}{\nu_p(s)} \to 0 \text{ as } s \to 0$$

(see [MP], Sect. 8.6).

Making obvious changes in the proof of Theorem 4.2 one can show that inequality (52) with $q < p$ holds if and only if

$$\int_0^{\mu(\Omega)} \Big(\frac{s^{p/q}}{\nu_p(s)}\Big)^{q/(p-q)} \frac{ds}{s} < \infty.$$

If $\mu(\Omega) < \infty$, the same condition is necessary and sufficient for the precompactness of the ball $\{u \in C_0^\infty(\Omega) : \|\nabla u\|_{L_p(\Omega)} \leq 1\}$ in $L_q(\Omega, \mu)$, $q < p$ (see [MP], Sect. 8.5, 8.6).

Remark 6.8 The following important statement formulated in terms of the p-capacity minimizing function shows that the Sobolev type inequality (52) is a consequence of a certain weighted integral inequality for functions of one variable. This result leads to the best constants in inequalities of type (52) and, as most of the previous results, can be directly extended to functions on Riemannian manifolds.

Theorem 6.9 *Let $p \geq 1$ and $q > 0$. Assume that the capacity minimizing function $\nu_p(s)$ has the inverse ν_p^{-1}. If for all absolutely continuous functions h on $(0, \infty)$ such that $h(0) = 0$:*

$$\Big(\int_0^\infty |h(\tau)|^q |d\nu_p^{-1}(1/\tau)|\Big)^{1/q} \leq D\Big(\int_0^\infty |h'(\tau)|^p d\tau\Big)^{1/p}, \tag{53}$$

then (52) holds for all $u \in C_0^\infty(\Omega)$.

Proof. Let

$$\psi(t) = \int_t^\infty \Big[\int_{|u(x)|=\tau} |\nabla u(x)|^{p-1} ds(x) \Big]^{1/(1-p)} d\tau.$$

and let $t(\psi)$ denote the inverse function. Then

$$\int_\Omega |u|^q d\mu = \int_0^\infty \mu(N_{t(\psi)}) d(t(\psi)^q)$$

and

$$\int_\Omega |\nabla u|^p dx = \int_0^\infty |t'(\psi)|^p d\psi$$

(see Sect 2.2 and 2.3. of [M4] for more details). Clearly,

$$\mu(N_{t(\psi)}) \le \nu_p^{-1}(\mathrm{cap}_p(N_{t(\psi))}))$$

and it remains to note that

$$\mathrm{cap}_p(N_{t(\varphi)}) \le \frac{1}{\psi(t)^{p-1}}$$

(see (50) and Lemma 2.2.2/1 in [M4]).

Remark 6.10 It is obvious that similar results with the same proofs remain valid for functions with unrestricted boundary values. This influences only the definition of the corresponding p-capacity. In particular, if the set of admissible functions in the definition of $\mathrm{cap}_p(F)$ consists of functions vanishing on a ball B, $\overline{B} \subset \Omega$, and such that $u \ge 1$ on F, then everything said about Sobolev inequalities before in this section holds for functions with unrestricted boundary values which are equal to zero on B.

A few words on the so-called *logarithmic Sobolev inequalities*. Let μ be a measure in Ω, $\mu(\Omega) = 1$, $p \ge 1$ and let ν_p be the capacity minimizing function generated by μ (see Definition 6.6). The inequality

$$\exp\left(- \int_\Omega \log^+ \frac{1}{|u|} \, d\mu \right) \le 4 \|\nabla u\|_{L_p(\Omega)} \exp\left(-\frac{1}{p} \int_0^1 \log \nu_p(s) \, ds \right) \qquad (54)$$

for all $u \in \overset{\circ}{L}{}^1_p(\Omega)$ was proved in 1968 by Maz'ya and Havin [MH]. It shows, in particular, that

$$\int_0^1 \nu_p(s) \, ds = +\infty$$

implies

$$\int_0^1 \log^+ \frac{1}{|u|}\, d\mu = +\infty$$

for all $u \in L_p^1(\Omega)$. This fact allows for certain applications of (54) to complex function theory [MH].

Inequality (54) is completely different from the logarithmic Sobolev inequality obtained in 1978 by Weissler [We]:

$$\exp\left(\frac{4}{n}\int_{\mathbb{R}^n} |u|^2 \log|u|\, dx\right) \le \frac{2}{\pi e n}\int_{\mathbb{R}^n} |\nabla u|^2\, dx\ ,$$

where $\|u\|_{L_2(\mathbb{R}^n)} = 1$, which is equivalent (see Beckner and Pearson [BP]) to the well-known Gross inequality of 1975 [Gr]

$$\int_{\mathbb{R}^n} u^2 \log\left(u^2\Big/\int_{\mathbb{R}^n} u^2\, d\mu\right) d\mu \le C\int_{\mathbb{R}^n} |\nabla u|^2\, d\mu\ , \tag{55}$$

where

$$d\mu = (2\pi)^{-n/2}\exp(-|x|^2/2)\, dx\ .$$

Various extensions, proofs and applications of (55) were the subject of many studies.

Remark 6.11 The inequality

$$\int_\Omega u^2\, d\mu \le c\int_\Omega |\nabla u|^2 dx \tag{56}$$

holds if and only if

$$\sup_F \frac{\mu(F)}{\mathrm{cap}(F)} < \infty\ ,$$

where F is an arbitrary compact set in Ω. For the case $\Omega = \mathbb{R}^n$, $n > 2$, other criteria for the validity of (56) are known. The following one is due to Kerman and Sawyer [KeS] :

For every open ball B in $\mathbb{R}^n$,

$$\int_B\int_B \frac{d\mu(x)d\mu(y)}{|x-y|^{n-2}} \le c\ \mu(B)\ .$$

Another two criteria for (56) were obtained by Maz'ya and Verbitsky [MV1]:

(i) The pointwise inequality

$$I_1(I_1\mu)^2(x) \le c\ I_1(\mu)(x) < \infty \quad \text{a.e.}$$

holds, where I_1 stands for the Riesz potential of order 1, i.e. $I_1\mu = |x|^{1-n} \star \mu$.

(ii) For every compact set $F \subset \mathbb{R}^n$,

$$\int_F (I_1\mu)^2 \, dx \le c \, \mathrm{cap}(F) \, .$$

One more condition necessary and sufficient for (56) was found by Verbitsky [Ve]:

For every dyadic cube P in $\mathbb{R}^n$,

$$\sum_{Q \subset P} \left[\frac{\mu(Q)}{|Q|^{1-1/n}} \right]^2 |Q| \le c \, \mu(P) \, ,$$

where the sum is taken over all dyadic cubes Q contained in P, and c does not depend on P.

We now state the main result of the paper [MV2] by the author and Verbitsky, characterizing arbitrary complex-valued distributions V subject to the inequality

$$\Big| \int_{\mathbb{R}^n} |u|^2 V \, dx \Big| \le c \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \quad \text{for all } u \in C_0^\infty \, . \tag{57}$$

This characterization reduces the case of distributional potentials V to that of nonnegative absolutely continuous weights.

Theorem 6.12 *Let $V \in (C_0^\infty)'$, $n > 2$. Then the inequality (57) holds, if and only if there is a vector-field $\mathbf{\Gamma} \in L_2(\mathbb{R}^n, \mathrm{loc})$ such that $V = \mathrm{div}\, \mathbf{\Gamma}$, and*

$$\int_{\mathbb{R}^n} |u(x)|^2 |\mathbf{\Gamma}(x)|^2 \, dx \le C \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx \, ,$$

for all $u \in C_0^\infty$. The vector field $\mathbf{\Gamma}$ can be chosen in the form $\mathbf{\Gamma} = \nabla \Delta^{-1} V$.

7 Isoperimetric function and a Brezis-Gallouët-Wainger type inequality

Theorem 7.1 *Let Ω be a domain with $m_n(\Omega) < \infty$ and let*

$$\int_0^{m_n(\Omega)/2} \frac{d\mu}{[\lambda(\mu)]^{p'}} < \infty \, ,$$

where $p+p' = pp'$ *and* $\lambda(\mu)$ *denotes* $\lambda_M(\mu)$ *for* $M = m_n(\Omega)/2$*. Furthermore, let for some* $r \in (1,p)$

$$\int_0^{m_n(\Omega)/2} \frac{d\mu}{[\lambda(\mu)]^{r'}} = \infty \ .$$

Then for any $\varepsilon \in (0, m_n(\Omega)/2)$ *and* $u \in L_p^1(\Omega)$

$$\begin{aligned} \mathrm{osc}_\Omega u \ &\leq \ c(p,r)\Big\{\Big(\int_0^\varepsilon \frac{d\mu}{[\lambda(\mu)]^{p'}}\Big)^{1/p'} \|\nabla u\|_{L_p(\Omega)} \\ &+\Big(\int_\varepsilon^{m_n(\Omega)/2} \frac{d\mu}{[\lambda(\mu)]^{r'}}\Big)^{1/r'} \|\nabla u\|_{L_r(\Omega)}\Big\} \ . \qquad (58) \end{aligned}$$

Proof. Let the numbers T and t be chosen so that

$$m_n\{x : u(x) > T\} \leq m_n(\Omega)/2 \leq m_n\{x : u(x) \geq T\},$$

$$m_n\{x : u(x) > t\} \leq \varepsilon \leq m_n\{x : u(x) \geq t\} \ .$$

Furthermore, let $S := \operatorname{ess\,sup}_\Omega u$. Then

$$(S-t)^p c_p(K_{t,s}) \leq \int |\nabla u|^p \, dx$$

and

$$(t-T)^r c_r(K_{T,t}) \leq \int |\nabla u|^r \, dx \ .$$

Quite similar to (44) we obtain

$$c_p(K_{t,S}) \geq \Big(\int_0^\varepsilon \frac{d\mu}{[\lambda(\mu)]^{p'}}\Big)^{1-p}$$

and

$$c_r(K_{T,t}) \geq \Big(\int_\varepsilon^{m_n(\Omega)/2} \frac{d\mu}{[\lambda(\mu)]^{r'}}\Big)^{1-r} \ .$$

This leads to the estimate

$$\begin{aligned} \operatorname*{ess\,sup}_\Omega u - T \ &\leq \ \Big(\int_0^\varepsilon \frac{d\mu}{[\lambda(\mu)]^{p'}}\Big)^{1/p'} \|\nabla u\|_{L_p(u\geq T)} \\ &+\Big(\int_\varepsilon^{m_n(\Omega)/2} \frac{d\mu}{[\lambda(\mu)]^{r'}}\Big)^{1/r'} \|\nabla u\|_{L_r(u\geq T)} \ . \end{aligned}$$

An analogous estimate for $T - \operatorname{ess\,inf}_\Omega u$ is proved in the same way. Adding both estimates we arrive at (58). □

We specify this theorem for domains satisfying (16) with $q = r'$.

Corollary 7.2 *Let $p > r > 1$, $m_n(\Omega) < \infty$ and let $\Omega \in \mathcal{J}_{1/r'}$, i.e.*

$$\lambda(\mu) \geq C\ \mu^{1/r'}\ .$$

Then

$$\mathrm{osc}_\Omega u \leq c_0\Big(\varepsilon^{(p-r)/pr}\|\nabla u\|_{L_p(\Omega)} + \Big(\log\frac{m_n(\Omega)}{2\varepsilon}\Big)^{1/r'}\|\nabla u\|_{L_r(\Omega)}\Big) \tag{59}$$

where $\varepsilon \in (0, m_n(\Omega)/2)$ and c_0 depends only on C, p and r.

Remark 7.3 Minimizing the right-hand side of (59) in ε we see that

$$\mathrm{osc}_\Omega u \leq c_1(1 + |\log(c_2\|\nabla u\|_{L_p(\Omega)})|)^{1/r'} \tag{60}$$

provided

$$1 = \|\nabla u\|_{L_r(\Omega)} \leq c_2\ \|\nabla u\|_{L_p(\Omega)}\ .$$

For the cusp (17) with $\varphi(x_n) = x_n^\beta$, $\beta > 1$, we have by (18) that

$$\lambda(\mu) \sim \mu^{\beta(n-1)/(\beta(n-1)+1)}$$

and we may take $r = \beta(n-1)+1$.

This example can be used to show that the exponent $1/r'$ of the power of logarithm in (59) is sharp by setting $u(x) = \log\log(a x_n^{-1})$ into (59).

8 Conductor inequalities for a Dirichlet-type integral on a topological space

Let $\mathcal{X}$ denote a locally compact Hausdorff space and let $C(\mathcal{X})$ stand for the space of continuous real valued functions given on $\mathcal{X}$. By $C_0(\mathcal{X})$ we denote the set of functions $f \in C(\mathcal{X})$ with compact supports in $\mathcal{X}$.

We introduce an operator $\mathcal{F}_p$ defined on a subset $\mathrm{dom}(\mathcal{F}_p)$ of $C(\mathcal{X})$ and taking values in the cone of nonnegative locally finite Borel measures on $\mathcal{X}$. We suppose that $1 \in \mathrm{dom}(\mathcal{F}_p)$ and $\mathcal{F}_p$ is positively homogeneous of order $p \geq 1$, i.e. for every real α, $f \in \mathrm{dom}(\mathcal{F}_p)$ implies $\alpha f \in \mathrm{dom}(\mathcal{F}_p)$ and

$$\mathcal{F}_p[\alpha f] = |\alpha|^p \mathcal{F}_p[f]. \tag{61}$$

It is also assumed that $\mathcal{F}_p$ is contractive, that is $\lambda(f) \in \mathrm{dom}(\mathcal{F}_p)$ and

$$\mathcal{F}_p[\lambda(f)] \leq \mathcal{F}_p[f], \tag{62}$$

for all $f \in \operatorname{dom}(\mathcal{F}_p)$, where λ is an arbitrary real valued Lipschitz function on the line $\mathbb{R}$ such that $|\lambda'| \le 1$ and $\lambda(0) = 0$. We suppose that the following locality condition holds:

$$f(x) = c \in \mathbb{R} \;\; on \;\; a \;\; compact \;\; set \;\; \mathcal{C} \;\; \Longrightarrow \;\; \int_{\mathcal{C}} \mathcal{F}_p[f] = \int_{\mathcal{C}} \mathcal{F}_p[c]. \tag{63}$$

In this section we deal with the functional

$$f \to \int_{\mathcal{X}} \mathcal{F}_p[f] \tag{64}$$

which is a far reaching generalization of the Dirichlet-type integral

$$\int_{\Omega} (\Phi(x, \nabla f))^p dx + \int_{\Omega} |f|^p d\nu, \tag{65}$$

where ν is a measure and the function

$$\Omega \times \mathbb{R}^n \ni (x, z) \to \Phi(x, z) \in \mathbb{R} \tag{66}$$

is continuous, positively homogeneous of degree 1 with respect to z. One can take the space of locally Lipschitz functions on Ω as $\operatorname{dom}(\mathcal{F}_p)$.

Let g and G denote open sets in $\mathcal{X}$ such that the closure $\bar{g}$ is a compact subset of G. We introduce the p-conductivity of the conductor $G\backslash\bar{g}$ (in other terms, the relative p-capacity of the set $\bar{g}$ with respect to G) as

$$\operatorname{cap}_p(\bar{g}, G) = \inf\Big\{ \int_{\mathcal{X}} \mathcal{F}_p[f] \colon \; f \in \operatorname{dom}(\mathcal{F}_p), \; 0 \le f \le 1 \;\text{ on }\; G$$

$$\text{and }\; f = 1 \;\text{ on a neighborhood of } \bar{g} \;\Big\}. \tag{67}$$

I state a general conductor inequality in the integral form for the functional (64).

Theorem 8.1 *Let M denote an increasing convex (not necessarily strictly convex) function given on $[0, \infty)$, $M(0) = 0$. Then the conductor inequality*

$$M^{-1}\Big(\int_0^\infty M(t^p \operatorname{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t))\frac{dt}{t}\Big) \le c(a, p) \int_{\mathcal{X}} \mathcal{F}_p[f], \tag{68}$$

holds for all $f \in \operatorname{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ and for an arbitrary $a > 1$. Here $\mathcal{L}_t = \{x \in \mathcal{X} : |f(x)| > t\}$, M is a positive convex function on $(0, \infty)$, $M(+0) = 0$, and M^{-1} stands for the inverse of M. By cap_p we mean the p-capacity generated by the operator $\mathcal{F}_p$.

Suppose that (63) is replaced by the following more restrictive locality condition:

$$f(x) = \text{const} \quad \textit{on a compact set } \mathcal{C} \quad \Longrightarrow \int_{\mathcal{C}} \mathcal{F}_p[f] = 0, \tag{69}$$

which holds, for example, if the measure ν in (65) is zero. We choose $M(\xi) = \xi^{q/p}$ for $\xi \geq 0$.

Corollary 8.2 Let $q \geq p$ and let $\mathcal{F}_p$ satisfy the locality condition (63). Then for all $f \in \text{dom}(\mathcal{F}_p) \cap C_0(\mathcal{X})$ and for an arbitrary $a > 1$

$$\Big(\int_0^\infty (\text{cap}_p(\overline{\mathcal{L}_{at}}, \mathcal{L}_t))^{q/p} d(t^q)\Big)^{1/q} \leq C\Big(\int_{\mathcal{X}} \mathcal{F}_p[f]\Big)^{1/p}. \tag{70}$$

If additionally $\mathcal{F}_p$ is subject to (69), then one can choose

$$C = \frac{(q \log a)^{1/q}}{a - 1}.$$

9 Sharp capacitary inequalities and their applications

Let Ω denote an open set in $\mathbb{R}^n$ and let the function

$$\Omega \times \mathbb{R}^n \ni (x, z) \to \Phi(x, z) \in \mathbb{R}$$

be a continuous function, positively homogeneous of degree 1 with respect to z. Clearly, the measure

$$\mathcal{F}_p[f] := |\Phi(x, \nabla\, f(x))|^p dx$$

satisfies (61), (62), and (69). Hence, (70) implies the inequality

$$\Big(\int_0^\infty (\text{cap}_p(\overline{\mathcal{L}_t}, \mathcal{X}))^{q/p} d(t^q)\Big)^{1/q} \leq C\Big(\int_{\mathcal{X}} |\Phi(x, \nabla\, f(x))|^p dx\Big)^{1/p}, \tag{71}$$

where cap_p is the p-capacity corresponding to the integral in the right-hand side, $C = \text{const} > 0$ and f is an arbitrary function in $C_0^\infty(\Omega)$. The next assertion gives the sharp value of C for $q > p$. In the case $q = p$ the sharp value of C is given by

$$C = \frac{p}{(p-1)^{(p-1)/p}}$$

and is obtained by the same argument as Theorem 6.2.

Theorem 9.1 *Inequality* (71) *with* $q > p$ *holds withsub]Capacitary inequality with the best constant*

$$C = \Big(\frac{\Gamma(\frac{pq}{q-p})}{\Gamma(\frac{q}{q-p})\Gamma(p\frac{q-1}{q-p})}\Big)^{1/p-1/q}. \tag{72}$$

This value of C *is sharp if* $\Phi(x, y) = |y|$ *and if either* Ω *is a ball or* $\Omega = \mathbb{R}^n$.

Proof. Let

$$\psi(t) = \int_t^\infty \Big(\int_{|f(x)|=\tau} |\Phi(x, N(x))|^p |\nabla\, f(x)|^{p-1} ds(x)\Big)^{1/(1-p)} d\tau$$

with ds standing for the surface element and $N(x)$ denoting the normal vector at x directed inward $\mathcal{L}_\tau$. Further, let $t(\psi)$ denote the inverse function of $\psi(t)$. Then

$$\int_\Omega |\Phi(x, \nabla\, f(x))|^p dx = \int_0^\infty |t'(\psi)|^p d\psi \tag{73}$$

(compare with 49). By Bliss' inequality [Bl]

$$\Big(\int_0^\infty t(\psi)^q \frac{d\psi}{\psi^{1+q(p-1)/p}}\Big)^{1/q} \le \Big(\frac{p}{q(p-1)}\Big)^{1/q} C \Big(\int_0^\infty |t'(\psi)|^p d\psi\Big)^{1/p}, \tag{74}$$

with C as in (72), and by (73) this is equivalent to

$$\Big(\int_0^\infty \frac{d(t(\psi)^q)}{\psi^{q(p-1)/p}}\Big)^{1/q} \le C\Big(\int_\Omega |\Phi(x, \nabla\, f(x))|^p dx\Big)^{1/p}.$$

In order to obtain (71) with C given by (72) it remains to note that

$$\mathrm{cap}_p(\overline{\mathcal{L}_t}) \le \frac{1}{\psi(t)^{p-1}} \tag{75}$$

(compare with (50)) The constant (72) is the best possible since (75) becomes equality for radial functions. $\square$

We introduce the weighted perimeter minimizing function σ on $(0, \infty)$ by

$$\mathcal{C}(m) := \inf \int_{\partial g} |\Phi(x, N(x))| ds(x), \tag{76}$$

where the infimum is extended over all bounded open sets g with smooth boundaries subject to

$$m_n(g) \ge m.$$

Similarly to (44), the following isocapacitary inequality holds

$$\mathrm{cap}_p(\overline{g}, G) \ge \Big(\int_{m_n(g)}^{m_n(G)} \frac{dm}{\mathcal{C}(m)^{p'}}\Big)^{1-p}. \tag{77}$$

Therefore, (71) leads to

Corollary 9.2 *For, all $f \in C_0^\infty(\Omega)$,*

$$\Big(\int_0^\infty \Big(\int_{m_n(\mathcal{L}_t)}^{m_n(\Omega)} \frac{dm}{\mathcal{C}(m)^{p'}}\Big)^{-q/p'} d(t^q)\Big)^{1/q}$$
$$\leq C\Big(\int_\Omega |\Phi(x, \nabla f(x))|^p dx\Big)^{1/p} \tag{78}$$

with $q > p$ and C defined by (72). *For $p = 1$ the last inequality should be replaced by*

$$\Big(\int_0^\infty \mathcal{C}(m_n(\mathcal{L}_t))^q d(t^q)\Big)^{1/q} \leq \int_\Omega |\Phi(x, \nabla f(x))| dx \tag{79}$$

with $q \geq 1$.

By the way, this corollary, combined with the isoperimetric inequality

$$s(\partial g) \geq n^{1/n'} \omega_n^{1/n} m_n(g)^{1/n'},$$

immediately gives the following well-known sharp result.

Corollary 9.3 *Let $n > p \geq 1$ and $q = pn(n-p)^{-1}$. Then every $f \in C_0^\infty(\mathbb{R}^n)$ satisfies the Sobolev inequality*

$$\|f\|_{L_{\frac{pn}{n-p}}(\mathbb{R}^n)} \leq c\, \|\nabla f\|_{L_p(\mathbb{R}^n)}$$

with the best constant

$$c = \pi^{-1/2} n^{-1/2} \Big(\frac{p-1}{n-p}\Big)^{1/p'} \Big(\frac{\Gamma(n)\Gamma(1+n/2)}{\Gamma(n/p)\Gamma(1+n-n/p)}\Big)^{1/n}.$$

The next assertion resulting from (73) and (75) shows that a quite general capacitary inequality is a consequence of a certain inequality for functions of one variable.

Theorem 9.4 *Let α and β be positive nondecreasing functions on $(0, \infty)$ such that*

$$\sup \int_0^\infty \beta(\psi^{1-p})\, d(\alpha(t(\psi))) < \infty, \tag{80}$$

with the supremum taken over all absolutely continuous functions $[0, \infty) \ni \psi \to t(\psi) \geq 0$ subject to $t(0) = 0$ and

$$\int_0^\infty |t'(\psi)|^p d\psi \leq 1. \tag{81}$$

Then

$$\sup \int_0^\infty \beta(\mathrm{cap}_p(\overline{\mathcal{L}_t}, \Omega))\, d\alpha(t) < \infty \tag{82}$$

with the supremum extended over all f subject to

$$\int_\Omega |\Phi(x, \nabla f(x))|^p dx \le 1. \tag{83}$$

The least upper bounds (80) *and* (82) *coincide.*

In fact, the above Theorem 9.1 is a particular case of Theorem 9.4 corresponding to the choice

$$\alpha(t) = t^q \qquad \text{and} \quad \beta(\xi) = \xi^{q/p}.$$

The next result is another consequence of Theorem 9.4.

Theorem 9.5 *For every $c > 0$*

$$\sup \int_0^\infty \exp\Big(\frac{-c}{\mathrm{cap}_p(\overline{\mathcal{L}_t}, \Omega)^{1/(p-1)}}\Big) d(\exp(c\, t^{p'})) < \infty, \tag{84}$$

where the supremum is taken over all $f \in C_0^\infty(\Omega)$ subject to (83) *and $p' = p/(p-1)$, $p > 1$.*

Proof. It follows from a theorem by Jodeit [Jo], that

$$\sup \int_0^\infty \exp(t(\psi)^{p'} - \psi) d\psi < \infty, \tag{85}$$

with the supremum taken over all absolutely continuous functions $[0, \infty) \ni \psi \to t(\psi) \ge 0$ subject to $t(0) = 0$ and (81). Hence, for every $c > 0$,

$$\sup \int_0^\infty \exp(c\, t(\psi)^{p'} - c\, \psi) d\psi < \infty.$$

It remains to refer to Theorem 9.4 with

$$\alpha(t) = \exp(c\, t^{p'}) \quad \text{and} \quad \beta(\xi) = \exp(-c\, \xi^{1/(1-p)}).$$

A direct consequence of Theorem 9.5 and the isocapacitary inequality (46) is the following Moser's result.

Corollary 9.6 (Moser [Mos]) *Let $m_n(\Omega) < \infty$ and let*

$$\{f\} := \{f \in C_0^\infty(\Omega) : \|\nabla f\|_{L_n(\Omega)} \le 1\}.$$

Then

$$\sup_{\{f\}} \int_\Omega \exp(n\,\omega_n^{1/(n-1)}\,|f(x)|^{n'})dx < \infty.$$

Proof. The isocapacitary inequality (46) can be written as

$$m_n(\overline{g}) \le m_n(G)\exp\Big(-n\,\omega_n^{1/(n-1)}\,\mathrm{cap}_n(\overline{g}, G)^{1/(1-n)}\Big).$$

Hence, putting $c = n\,\omega_n^{1/(n-1)}$ and $p = n$ in (84), we obtain

$$\int_0^\infty m_n(\mathcal{L}_t)d\exp(n\omega_n^{1/(n-1)}t^{n'}) < \infty.$$

The result follows. □

One needs no changes in proofs to see that the main results of this section, Theorems 9.1 - 9.5 hold if Ω is an open subset of a Riemannian manifold $\mathfrak{R}_n$, and ∇f is the Riemannian gradient. As an application, we obtain another Moser's inequality [Mos].

Corollary 9.7 *Let Ω be a proper subdomain of the unit sphere S^2 and let $\{f\}$ be defined as in Corollary* 9.6*. Then*

$$\sup_{\{f\}} \int_\Omega \exp(4\pi f^2(\omega))\,ds_\omega < \infty\ .$$

Proof. By Theorem 9.5 with $c = 4\pi$ we have the capacitary integral inequality

$$\sup_{\{f\}} \int_0^\infty \exp\Big(\frac{-4\pi}{\mathrm{cap}_2(\overline{\mathcal{L}_t}, \Omega)}\Big)\,d(\exp(4\pi t^2)) < \infty\ . \tag{86}$$

The classical isoperimetric inequality on S^2

$$s(\partial g)^2 \ge m_2(g)(4\pi - m_2(g))$$

(see Rado [Ra]), combined with (77) implies the isocapacitary inequality

$$\mathrm{cap}_2(\bar{g}, G) \ge 4\pi\left(\log\frac{m_2(G)(4\pi - m_2(g))}{m_2(g)(4\pi - m_2(G))}\right)^{-1}$$

Setting here $G = \Omega$, $g = \mathcal{L}_t$, and using (86), we complete the proof. □

Remark 9.8 We can go even further extending the above results to the measure valued operator $\mathcal{F}_p[f]$ in Section 8 subject to the condition

$$\mathcal{F}_p[\lambda(f)] = |\lambda'(f)|^p \mathcal{F}_p[f] \tag{87}$$

with the same λ as in (62). In fact, (87) implies

$$\int_{\mathcal{X}} \mathcal{F}_p[f] = \int_0^\infty |t'(\psi)|^p d\psi, \tag{88}$$

where $t(\psi)$ is the inverse of the function

$$\psi(t) = \int_t^\infty \Big| \frac{d}{d\tau} \mathcal{F}_p[f](\mathcal{L}_\tau) \Big|^{1/(1-p)} d\tau.$$

Identity (88) is the core of the proof of the results in the present section.

10 Properties of Sobolev spaces generated by quadratic forms with variable coefficients

In the preceding sections I showed that rather general inequalities, containing the integral $\int_\Omega [\Phi(x, \nabla u)]^p \, dx$, are equivalent to isocapacitary inequalities which relate (p, Φ)-capacity and measures. Although such criteria are of primary interest, we should note that their verification in particular cases is often difficult. Even for rather simple quadratic forms

$$[\Phi(x, \xi)]^2 = \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j$$

the estimates for the corresponding capacities by measures are unknown.

Thus, the general necessary and sufficient conditions obtained in the present chapter can not diminish the value of straightforward methods of investigation of integral inequalities without using capacity. In the present section this will be illustrated using as an example the quadratic form

$$[\Phi(x, \xi)]^2 = (|x_n| + |x'|^2) \xi_n^2 + |\xi'|^2 \,,$$

where $x' = (x_1, \ldots, x_{n-1})$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$.

The inequality

$$\int_{\mathbb{R}^{n-1}} [u(x', 0)]^2 \, dx' \le c \int_{\mathbb{R}^n} [\Phi(x, \nabla u)]^2 \, dx \tag{89}$$

holds for all $u \in C_0^\infty(\mathbb{R}^n)$ if and only if the isocapacitary inequality

$$m_{n-1}(\{x \in g, x_n = 0\}) \le c\,(2, \Phi)\text{-cap}(g)$$

holds for any admissible set g. A straightforward proof of the preceding isocapacitary inequality is unknown to the author. Nevertheless, the estimate (89) is true and will be proved in the sequel.

Theorem 10.1 *Let*

$$[\Phi(x, \nabla u)]^2 = (|x_n| + |x'|^2)(\partial u/\partial x_n)^2 + \sum_{i=1}^{n-1} (\partial u/\partial x_i)^2 \ .$$

Then (89) *is valid for all* $u \in C_0^\infty(\mathbb{R}^n)$.

Proof. Let the integral in the right-hand side of (89) be denoted by $Q(u)$. For any $\delta \in (0, 1/2)$ we have

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |u(x', 0)|^2\, dx' &\le 2 \int_{\mathbb{R}^n} \frac{(|x_n| + |x'|^2)^{1/2}}{|x_n|^{(1-\delta)/2} |x'|^\delta} \left| u \frac{\partial u}{\partial x_n} \right| dx \\ &\le 2\,[Q(u)]^{1/2} \left(\int_{\mathbb{R}^n} |x_n|^{\delta-1} |x'|^{-2\delta} |u|^2\, dx \right)^{1/2} . \end{aligned} \tag{90}$$

To give a bound for the last integral we use the following well-known generalization of the Hardy-Littlewood inequality:

$$\int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^{n-1}} \frac{f(y)\, dy}{|x' - y|^{n-1-\delta}} \right)^2 \frac{dx'}{|x'|^{2\delta}} \le c \int_{\mathbb{R}^{n-1}} [f(y)]^2\, dy \ . \tag{91}$$

(For the proof of this estimate see Lizorkin [Li]. Since the convolution with the kernel $|x'|^{\delta+1-n}$ corresponds to the multiplication by $|\xi'|^{-\delta}$ of the Fourier transform, (91) can be written as

$$\int_{\mathbb{R}^{n-1}} |u|^2 |x'|^{-2\delta}\, dx' \le c \int_{\mathbb{R}^{n-1}} [(-\Delta_{x'})^{\delta/2} u]^2\, dx' \ ,$$

where $(-\Delta_{x'})^{\delta/2}$ is the fractional power of the Laplace operator. Now we find that the right-hand side in (90) does not exceed

$$c\Big(Q(u) + \int_{\mathbb{R}^n} |x_n|^{\delta-1} [(-\Delta_{x'})^{\delta/2} u]^2\, dx \Big) \ . \tag{92}$$

From the almost obvious estimate

$$\int_0^\infty g^2 t^{\delta-1}\, dt \le c\Big(\int_0^\infty (g')^2 t\, dt + \int_0^\infty g^2\, dt\Big)$$

it follows that

$$|\xi'|^{2\delta} \int_{\mathbb{R}^n} |(F_{x'\to\xi'} u)(\xi', x_n)|^2 |x_n|^{\delta-1} dx_n$$

$$\le c\Big(\int_{\mathbb{R}^1} \Big|\Big(F_{x'\to\xi'} \frac{\partial u}{\partial x_n}\Big)(\xi', x_n)\Big|^2 |x_n|\, dx_n + |\xi'|^2 \int_{\mathbb{R}^1} |(F_{x'\to\xi'} u)(\xi', x_n)|^2\, dx_n\Big)$$

where $F_{x'\to\xi'}$ is the Fourier transform in $\mathbb{R}^{n-1}$. So the second integral in (92) does not exceed

$$c\int_{\mathbb{R}^n} (|x_n|(\partial u/\partial x_n)^2 + (\nabla_{x'} u)^2)\, dx\ .$$

The result follows. □

The next assertion shows that Theorem 10.1 is exact in a certain sense.

Theorem 10.2 *The space of restrictions to* $\mathbb{R}^{n-1} = \{x \in \mathbb{R}^n : x_n = 0\}$ *of functions in the set* $\{u \in C_0^\infty(\mathbb{R}^n) : Q(u) + \|u\|^2_{L_2(\mathbb{R}^n)} \le 1\}$ *is not relatively compact in* $L_2(B_1^{(n-1)})$, *where* $B_\varrho^{(n-1)} = \{x' \in \mathbb{R}^{n-1} : |x'| < \varrho\}$.

Proof. Let φ denote a function in $C_0^\infty(B_1^{(n-1)})$ such that $\varphi(y) = \varphi(-y)$, $\|\varphi\|_{L_2(\mathbb{R}^{n-1})} = 1$ and introduce the sequence $\{\varphi_m\}_{m=1}^\infty$ defined by $\varphi_m(y) = m^{(n-1)/2}\varphi(my)$. Since this sequence is normalized and weakly convergent to zero in $L_2(B_1^{(n-1)})$, it contains no subsequences converging in $L_2(B_1^{(n-1)})$. Further, let $\{v_m\}_{m=1}^\infty$ be the sequence of functions in $\mathbb{R}^n$ defined by

$$v_m(x) = F^{-1}_{\eta'\to x'} \exp\{-\langle\eta\rangle^2 |x_n|\} F_{x'\to\eta'} \varphi_m\ ,$$

where $\eta \in \mathbb{R}^{n-1}$, $\langle\eta\rangle = (|\eta|^2 + 1)^{1/2}$.

Consider the quadratic form

$$T(u) = \int_{\mathbb{R}^n} \Big[(|x_n| + |x'|^2)\Big|\frac{\partial u}{\partial x_n}\Big|^2 + |\nabla_{x'} u|^2 + |u|^2\Big]\, dx\ .$$

It is clear that

$$T(u) = (2\pi)^{1-n} \int_{\mathbb{R}^n} \Big(|x_n|\Big|\frac{\partial F u}{\partial t}\Big|^2 + \Big|\frac{\partial}{\partial t}\nabla_\eta F u\Big|^2 + \langle\eta\rangle^2 |Fu|^2\Big)\, d\eta\, dx_n\ .$$

Differentiating the function $T(v_m)$, we obtain from the last equality that $T(v_m)$ does not exceed

$$c\int_{\mathbb{R}^n} [(1+\langle\eta\rangle^2|x_n|+\langle\eta\rangle^4|x_n|^3)\langle\eta\rangle^2|F\varphi_m|^2+\langle\eta\rangle^4|\nabla F\varphi_m|^2] \exp(-2\langle\eta\rangle^2|x_n|) d\eta dx_n\ .$$

Thus we obtain

$$\begin{aligned} T(v_m) &\leq c\int_{\mathbb{R}^{n-1}} (\langle\eta\rangle^2|\nabla F\varphi_m|^2 + |F\varphi_m|^2)d\eta \\ &= c_1\Big(\sum_{i=1}^{n-1} \|x_i\varphi_m\|^2_{W_2^1(\mathbb{R}^{n-1})} + \|\varphi_m\|^2_{L_2(\mathbb{R}^{n-1})}\Big) \leq \text{const} . \end{aligned}$$

Let $\psi \in C_0^\infty(B_2^{(n-1)})$, $\psi = 1$ on $B_1^{(n-1)}$. It is clear that $(v_m\psi)|_{\mathbb{R}^{n-1}} = \varphi_m$ and $T(v_m\psi) \leq$ const. The sequence $\{v_m\psi/(T(v_m\psi))^{1/2}\}_{m=1}^\infty$ is the required counter-example. The theorem is proved. $\square$

11 Sharp Hardy-Leray inequality for axisymmetric divergence-free fields

Let $\mathbf{u}$ denote a $C_0^\infty(\mathbb{R}^n)$ vector field in $\mathbb{R}^n$. The following n-dimensional generalization of the one-dimensional Hardy inequality,

$$\int_{\mathbb{R}^n} \frac{|\mathbf{u}|^2}{|x|^2}dx \leq \frac{4}{(n-2)^2}\int_{\mathbb{R}^n} |\nabla\mathbf{u}|^2 dx \tag{93}$$

appears for $n = 3$ in the pioneering Leray's paper on the Navier-Stokes equations [Le]. The constant factor on the right-hand side is sharp. Since one frequently deals with divergence-free fields in hydrodynamics, it is natural to ask whether this restriction can improve the constant in (93).

It is shown in the paper [CM] that this is the case indeed if $n > 2$ and the vector field $\mathbf{u}$ is axisymmetric by proving that the aforementioned constant can be replaced by the (smaller) optimal value

$$\frac{4}{(n-2)^2}\left(1 - \frac{8}{(n+2)^2}\right) \tag{94}$$

which, in particular, evaluates to 68/25 in three dimensions. This result is a special case of a more general one concerning a divergence-free improvement of the multi-dimensional sharp Hardy inequality

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2}|\mathbf{u}|^2 dx \leq \frac{4}{(2\gamma+n-2)^2}\int_{\mathbb{R}^n} |x|^{2\gamma}|\nabla\mathbf{u}|^2 dx . \tag{95}$$

Let $\boldsymbol{\phi}$ be a point on the $(n-2)$-dimensional unit sphere S^{n-2} with spherical coordinates $\{\theta_j\}_{j=1,\dots,n-3}$ and ϕ, where $\theta_j \in (0,\pi)$ and $\varphi \in [0,2\pi)$. A point $x \in \mathbb{R}^n$ is represented as a triple $(\rho, \theta, \boldsymbol{\phi})$, where $\rho > 0$ and $\theta \in [0, \pi]$. Correspondingly, we write $\mathbf{u} = (u_\rho, u_\theta, \mathbf{u}_{\boldsymbol{\phi}})$ with $\mathbf{u}_{\boldsymbol{\phi}} = (u_{\theta_{n-3}}, ..., u_{\theta_1}, u_{\boldsymbol{\phi}})$.

The *condition of axial symmetry* means that $\mathbf{u}$ depends only on ρ and θ.

For higher dimensions, our result is as follows.

Theorem 11.1 *Let $\gamma \neq 1 - n/2$, $n > 2$, and let $\mathbf{u}$ be an axisymmetric divergence-free vector field in $C_0^\infty(\mathbb{R}^n)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 1 - n/2$. Then*

$$\int_{\mathbb{R}^n} |x|^{2\gamma-2}|\mathbf{u}|^2 dx \leq C_{n,\gamma} \int_{\mathbb{R}^n} |x|^{2\gamma}|\nabla \mathbf{u}|^2 dx \tag{96}$$

with the best value of $C_{n,\gamma}$ given by

$$C_{n,\gamma} = \frac{4}{(2\gamma + n - 2)^2}\left(1 - \frac{2}{n + 1 + (\gamma - n/2)^2}\right) \tag{97}$$

for $\gamma \leq 1$, and by

$$C_{n,\gamma}^{-1} = \left(\frac{n}{2} + \gamma - 1\right)^2 + \min\left\{n - 1,\ 2 + \min_{x \geq 0}\left(x + \frac{4(n-1)(\gamma - 1)}{x + n - 1 + (\gamma - n/2)^2}\right)\right\}$$

for $\gamma > 1$.

The two minima in the last equality can be calculated in closed form, but their expressions for arbitrary dimensions turn out to be unwieldy, and we omit them.

However, the formula for $C_{3,\gamma}$ is simple.

Corollary 11.2 *For $n = 3$ inequality* (96) *holds with the best constant*

$$C_{3,\gamma} = \begin{cases} \dfrac{4}{(2\gamma+1)^2} \cdot \dfrac{2 + (\gamma - 3/2)^2}{4 + (\gamma - 3/2)^2}, & \text{for } \gamma \leq 1 \\[2ex] \dfrac{4}{8 + (1 + 2\gamma)^2}, & \text{for } \gamma > 1. \end{cases} \tag{98}$$

For $n = 2$, we obtain the sharp constant in (96) without axial symmetry of the vector field.

Theorem 11.3 *Let $\gamma \neq 0$, $n = 2$, and let $\mathbf{u}$ be a divergence-free vector field in $C_0^\infty(\mathbb{R}^2)$. We assume that $\mathbf{u}(\mathbf{0}) = \mathbf{0}$ for $\gamma < 0$. Then inequality* (96) *holds with the best constant*

$$C_{2,\gamma} = \begin{cases} \gamma^{-2}\dfrac{1 + (1-\gamma)^2}{3 + (1 - \gamma)^2} & \text{for } \gamma \in [-\sqrt{3} - 1, \sqrt{3} - 1] \\[2ex] (\gamma^2 + 1)^{-1} & \text{otherwise.} \end{cases} \tag{99}$$

12 An application of p-capacity to Poincaré's inequality

The p-capacity has other applications to the theory of Sobolev spaces, quite different form those dealt with before. To give an example, I shall discuss the usefulness of p-capacity in the study of the Poincaré type inequality for functions defined on the cube $\{x \in \mathbb{R}^n : |x_i| < d/2, i = 1, \dots, n\}$

$$\int_{Q_d} |u(x)|^p dx \le C \int_{Q_d} |\nabla u|^p dx. \tag{100}$$

We assume that the function u vanishes on a compact subset F of $\overline{Q}_d$.

Clearly, (100) fails if F is empty. For the one-dimensional case, one point ($F = \{0\}$) is sufficient for the Poincaré inequality to hold:

$$u(x) = u(x) - u(0) = \int_0^x u'(t)dt,$$

and thus

$$\int_0^1 |u(x)|^p dx \le \int_0^1 \Big| \int_0^x u'(t)dt \Big|^p dx \le \int_0^1 |u'|^p dx.$$

By using Sobolev's imbedding theorem one can prove that this is also sufficient (and, of course, necessary) for (100) if $p > n > 1$. However, for $1 < p \le n$ a one-point set F is not sufficient for (100). Let us check this for the more difficult case $p = n$.

Consider the function

$$u_\varepsilon(x) := \eta\Big(\Big|\frac{\log|x|}{\log\varepsilon}\Big|\Big)$$

where η is a piecewise linear function on $(0, \infty)$ such that

$$\eta(t) = 1 \text{ for } t < 1, \ \ \eta(t) = 0 \text{ for } t > 2.$$

Clearly,

$$|\nabla u_\varepsilon(x)| \le \frac{\max|\eta'|}{|x||\log\varepsilon|},$$

and hence

$$\begin{aligned} \int_{Q_d} |\nabla u_\varepsilon|^n dx &\le \frac{1}{|\log\varepsilon|^n} \int_{\varepsilon > |x| > \varepsilon^2} \frac{dx}{|x|^n} \\ &= \frac{1}{|\log\varepsilon|^n} \int_{\varepsilon^2}^{\varepsilon} \frac{dr}{r} = \frac{1}{|\log\varepsilon|^{n-1}} \to 0, \end{aligned}$$

as $\varepsilon \to 0$, contradicting (100).

We will show that the positivity of p-capacity is necessary and sufficient for inequality (100) with $n > p \geq 1$. The case $n = p$ is similar, but differs slightly in details and therefore will not be discussed here. The complete treatment including the non-trivial case of derivatives of higher order can be found in Ch. 10 of [M4].

Let $C^{0,1}$ be the space of functions subject to the uniform Lipschitz condition in $\mathbb{R}^n$. Also, let $C_0^{0,1}$ be the subspace of $C^{0,1}$ containing functions with compact supports. For any subset $E \subset \mathbb{R}^n$ we denote by $C^{0,1}(E)$ the set of all Lipschitz functions on E.

The following inequality is well-known

$$\|u - \bar{u}\|^p_{L_p(Q_d)} \leq c\, d^p \int_{Q_d} |\nabla u|^p dx, \tag{101}$$

where $\{Q_d\}$ is the family of closed concentric cubes with edge length $d > 0$ and faces parallel to the coordinate planes, $u \in C^{0,1}(Q_d)$ and $\bar{u} = d^{-n} \int_{Q_d} u dx$ is the mean value of u.

Another classical inequality to be used in the sequel is Hardy's inequality:

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leq c \int_{\mathbb{R}^n} |\nabla u|^p dx, \tag{102}$$

where $p < n$ and u is an arbitrary function in $C_0^{0,1}(\mathbb{R}^n)$.

In this section, we deal with the p-capacity of compact sets in $\mathbb{R}^n$, i.e. we set $\Omega = \mathbb{R}^n$ in Definition 6.1.

Theorem 12.1 (see [M3] and [M4], Ch. 10) *If $u \in C^{0,1}(\overline{Q_d})$ vanishes on a compact set $F \subset \overline{Q_d}$, then*

$$\int_{Q_d} |u|^p dx \leq \frac{c_0 d^n}{\text{cap}_p F} \int_{Q_d} |\nabla u|^p dx, \tag{103}$$

where $n > p \geq 1$ and c_0 depends only on n and p.

Proof. (The proof is the same for any Lipschitz domain.) We normalize $|u|$ by $\int_{Q_d} |u|^p dx = d^n$ i.e. $\overline{|u|^p} = 1$. By the Hölder inequality we obtain

$$\overline{|u|} \leq (\overline{|u|^p})^{1/p} = 1.$$

Clearly,

$$1 - \overline{|u|} = d^{-n/p}(\|u\|_p - \|\overline{|u|}\|_p) \leq d^{-n/p}\|u - \bar{u}\|_p,$$

where $\|u\|_p = (\int_{Q_d} |u|^p dx)^{1/p}$.

Hence and by (101)

$$\|u - \overline{u}\|_p \le c\, d \|\nabla u\|_p$$

we obtain

$$\overline{1 - |u|} \le c d^{1-n/p} \Big(\int_{Q_d} |\nabla u|^p dx \Big)^{1/p}.$$

Denoting $\varphi = 1 - |u|$, we have $\bar{\varphi} \ge 0$ and we can rewrite the inequality above as

$$\bar{\varphi}^p \le c\, d^{p-n} \int_{Q_d} |\nabla \varphi|^p dx.$$

Then

$$\|\varphi\|_p = \|(\varphi - \bar{\varphi}) + \bar{\varphi}\|_p \le \|\varphi - \bar{\varphi}\|_p + \|\bar{\varphi}\|_p$$

and

$$\|\varphi\|_p \le c\, d \|\nabla \varphi\|_p, \tag{104}$$

Let us extend φ outside Q_d by reflection in the faces of Q_d, so that the extension $\tilde{\varphi}$ satisfies

$$\int_{Q_{3d}} |\nabla \tilde{\varphi}|^p dx = 3^n \int_{Q_d} |\nabla \varphi|^p dx, \ \int_{Q_{3d}} |\tilde{\varphi}|^p dx = 3^n \int_{Q_d} |\varphi|^p dx.$$

Denote by η a piecewise linear function, equal to 1 on Q_d and zero outside Q_{2d}, so that $|\nabla \eta| \le c d^{-1}$. Then

$$\text{cap}_p F \le \int_{Q_{2d}} |\nabla(\tilde{\varphi}\eta)|^p dx \le c\Big(\int_{Q_d} |\nabla \varphi|^p dx + d^{-p} \int_{Q_d} \varphi^p dx \Big).$$

Taking into account that $|\nabla \varphi| = |\nabla u|$ almost everywhere and using (104), we obtain

$$\text{cap}_p F \le c_0 \int_{Q_d} |\nabla u|^p dx.$$

The last inequality is equivalent to the desired estimate.

The following assertion shows that the previous theorem is precise in a certain sense.

Theorem 12.2 *Let $n > p \ge 1$ and let*

$$\int_{Q_{d/2}} |u|^p dx \le C \int_{Q_d} |\nabla u|^p dx \tag{105}$$

for all $u \in C^{0,1}(\overline{Q}_d)$ *vanishing on the compact set* $F \subset \overline{Q}_d$. *If*

$$\text{cap}_p F \leq \gamma d^{n-p}, \tag{106}$$

where γ *is a sufficiently small constant depending on* n *and* p, *then*

$$C \geq \frac{c\, d^n}{\text{cap}_p F}. \tag{107}$$

Proof. Let $\varepsilon > 0$ and let φ_ε be a function in $C_0^{0,1}(\mathbb{R}^n)$ such that $\varphi_\varepsilon = 1$ on F, $0 \leq \varphi_\varepsilon \leq 1$ and

$$\int_{\mathbb{R}^n} |\nabla \varphi_\varepsilon|^p dx \leq \text{cap}_p F + \varepsilon.$$

Put $u = 1 - \varphi_\varepsilon$ in (105). Then

$$(d/2)^{n/p} - \|\varphi_\varepsilon\|_{L_p(Q_{d/2})} \leq C^{1/p} (\text{cap}_p F + \varepsilon)^{1/p}. \tag{108}$$

Using Hardy's inequality (102), we obtain

$$\begin{aligned} \|\varphi_\varepsilon\|_{L_p(Q_{d/2})} &\leq c\, d \Big(\int_{\mathbb{R}^n} \varphi_\varepsilon^p \frac{dx}{|x|^p} \Big)^{1/p} \\ &\leq c_0\, d \|\nabla \varphi_\varepsilon\|_{L_p(\mathbb{R}^n)} \leq c_0\, d (\text{cap}_p F + \varepsilon)^{1/p}. \end{aligned}$$

This estimate and (108) imply

$$(d/2)^{n/p} \leq (C^{1/p} + c_0 d)(\text{cap}_p F)^{1/p}. \tag{109}$$

If the constant γ satisfies

$$\gamma^{1/p} \leq 2^{-1-n/p} c_0^{-1}$$

we obtain from (106) and (109) that

$$2^{-1} (d/2)^{n/p} \leq C^{1/p} (\text{cap}_p F)^{1/p}.$$

The result follows.

Results of a similar nature, more general than Theorems 12.1 and 12.2 can be found in [M3] and [M4], Ch.10. I give some statements.

Let F be a compact subset of Ω and let

$$\text{cap}_l(F, \Omega) = \inf \Big\{ \int |\nabla_l u(x)|^2 dx : u = 1 \text{ in a neighbourhood of } F,\ u \in C_0^\infty(\Omega) \Big\}$$

Theorem 12.3 (1) *If F is a closed subset of $\overline{Q_d}$, then the inequality*

$$\|u\|^2_{L_2(Q_d)} \le c(l,n)\Big(d^2\|\nabla u\|_{L_2(Q_d)} + \frac{d^n}{\mathrm{cap}_l(F,Q_{2d})}\|\nabla_l u\|^2_{L_2(Q_d)}\Big)$$

holds for all functions $u \in C^\infty(Q_d)$ that vanish on F.

(2) *If*

$$\|u\|^2_{L_2(Q_d)} \le C\sum_{j=1}^{l} d^{2(j-l)}\|\nabla_j u\|^2_{L_2(Q_d)}$$

for all $u \in C^\infty(Q_d)$ that vanish in a neighbourhood of F and if

$$\mathrm{cap}_l(F,Q_{2d} \le \gamma\, d^{n-2l}$$

with a sufficiently small $\gamma = \gamma(n,l)$, then

$$C \ge c(l,n)\frac{d^n}{\mathrm{cap}_l(F,Q_{2d})}.$$

For the proof see [Hed], [M4], Ch.10, and [AH].

Noting that for $2l > n$

$$c_1 d^{n-2l} \le \mathrm{cap}_l(F,Q_{2d} \le c_2\, d^{n-2l}$$

for all nonempty F, we have

Corollary 12.4 (1) *Let $u \in C^\infty(\overline{Q_d})$ and let u vanish at some point of $\overline{Q_d}$. Then*

$$\|u\|^2_{L_2(Q_d)} \le C\sum_{j=1}^{l} d^{2(j-l)}\|\nabla_j u\|^2_{L_2(Q_d)} \tag{110}$$

with

$$C \le c_1(l,n)\, d^{2l}.$$

(2) *If* (110) *holds for all $u \in C^\infty(Q_d)$ vanishing at a point in $\overline{Q_d}$, then*

$$C \ge c_1(l,n)\, d^{2l}.$$

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