Kneading theory for tree maps

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Abstract. Using the techniques developed in [ASR], we generalize to tree maps the Milnor and Thurston results about zeta function, semiconjugacy and topological entropy of interval maps.

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1. Introduction
Motivated by some applications concerning the study of two-dimensional real and one-dimensional complex dynamics, continuous maps from a tree into itself have been studied in recent years by several authors (see de Carvalho and Hall [CH]). It is well-known that important problems such as: how to obtain estimates for the topological entropy of a tree map (see Alseda & Al. [ABLM] and [AGLMM]), and how to characterize the set of all periods of a tree map (see Baldwin and Llibre [BL]), are closely related with the study of the Artin-Mazur zeta function of a tree map.

Including the results which appeared in [ASR1] and in the Ph. D. Thesis of the first author, the aim of the present paper is to extend to tree maps the kneading theory developed by Milnor and Thurston in [MTh].

The main difficulty in generalizing the kneading theory consists into obtain a good substitution for the kneading matrix introduced by Milnor and Thurston for interval maps. To circumvent this difficulty we introduce the pair of linear endomorphisms $(\#_0(f), \#_1(f))$.

Section 2 is devoted to the construction of this pair which is a natural extension of the pair $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ introduced in [ASR] for interval maps. Also here this pair of linear endomorphisms presents two fundamental properties which play an important role. As an immediate consequence of the functoriality of $\epsilon_{\#0}$ and $\epsilon_{\#1}$ we obtain

$$\epsilon_{\#0}(f^n) = \epsilon_{\#0}(f^n) \quad \text{and} \quad \epsilon_{\#1}(f^n) = \epsilon_{\#1}(f^n)$$

which show that these endomorphisms have a good behavior in the presence of iteration. The nature of the second property is exclusively algebraic. We will prove
that the pair of linear endomorphisms \((\epsilon_{\#0}(f), \epsilon_{\#1}(f))\) has finite rank, and this will enable us to define the trace and the determinant of this pair (see appendix). This determinant is the characteristic polynomial of a finite matrix \(M(t)\) with entries in the ring \(\mathbb{Q}[t]\), and will play an analogous role to the kneading determinant. Observe also that, for the special case of Markov maps, this determinant coincides with the characteristic polynomial of the Markov transition matrix of the map.

Section 3 is devoted to the computation of the Artin-Mazur \([\text{AM}]\) zeta function \(\zeta(t)\) of a tree map. Another approach to computing \(\zeta(t)\) can be found in Baillif \([\text{Ba}]\), see also Hofbauer and Keller \([\text{HK}]\), Mori \([\text{Mo}]\), Baladi-Ruelle \([\text{BR}]\) and Flatto-Lagarias \([\text{FL}]\). The main result of this section shows that the determinant of \((\epsilon_{\#0}(f), \epsilon_{\#1}(f))\) plays a relevant role in the computation of \(\zeta(t)\), and extends for tree maps the Milnor and Thurston relation

\[
\exp \sum_{n \geq 1} \left(2N_{f^n} - 1 \right) \frac{t^n}{n} = D_f(t)^{-1}
\]

where \(D_f(t)\) denotes the kneading determinant of the interval map \(f : I \to I\), and \(N_{f^n}\) denotes the number of fixed points of \(f^n\) of negative type (a fixed point of \(f^n\), lying in the interior of \(I\), is of negative type if \(f^n\) is decreasing in a neighborhood of that point).

In Section 4 we study the existence of piecewise linear models for tree maps. Recently, Baillif and de Carvalho \([\text{BC}]\) generalized for tree maps the theorems of Parry \([\text{Pa}]\) and Milnor-Thurston, which state that every piecewise monotone interval map, with positive topological entropy \(\log(s)\), is semiconjugated to a piecewise linear interval map with slope \(\pm s\) everywhere. Using total variations, we give in this section an alternative way to construct these semiconjugacies. The main theorem in this section generalizes the Parry and Milnor-Thurston theorems, and furthermore shows an unexpected role played by total variations on the construction of other piecewise linear models.

Finally in Section 5 we extend for tree maps the main relation between the topological entropy \(h_t(f)\) and the growth number of fixed points of negative type (see Theorem 4.11, in Milnor-Tresser \([\text{MTr}]\)) which states that

\[
h_t(f) = \log \left( \max \left\{ 1, \limsup_n N_{f^n}^{-1} \right\} \right).
\]

2. The functors \(\epsilon_{\#0}\) and \(\epsilon_{\#1}\)

A simply connected topological space \(T\) is a tree if there exist subspaces \(I_1, \ldots, I_m\) of \(T\) such that \(T = I_1 \cup I_2 \cup \cdots \cup I_m\), and each \(I_i\) is homeomorphic to \([0, 1]\). As usual, if \(T\) is a tree and \(x \in T\) we define the valence of \(x\) to be the number of connected components of \(T \setminus \{x\}\). The valence of \(x\) will be denoted by \(\text{val}(x)\) and a point \(x \in T\) is a vertex of \(T\) if \(\text{val}(x) \neq 2\). The set of all vertices of \(T\) is denoted by \(V_T\) and the closures of the connected components of \(T \setminus V_T\) are called the edges of \(T\). A connected (and non-empty) set \(I\) contained in some edge of \(T\) is called an interval of \(T\). Given \(x, y \in T\) we denote by \(\langle x, y \rangle\) the convex hull of \(\{x, y\}\), in other
words \((x,y)\) is the smallest connected set containing \(\{x,y\}\). Notice that, in general, \((x,y)\) is not an interval of \(T\).

Let \(g : T \rightarrow T'\) be a continuous map between trees and let \(I\) be an interval of \(T\) such that \(\text{int}(I) \neq \emptyset\) and \(g(I)\) is an interval of \(T'\). Under this conditions, we will say that \(g\) is monotone on \(I\) if either \(g\) is constant on \(I\) or \(g\) is injective on \(I\). We will say that \(g\) is piecewise monotone if it is possible to cover \(T\) by finitely many intervals of monotonicity of \(g\). If, in addition to this, the map \(g\) has no intervals of constancy, then it will be called piecewise strictly monotone.

By a pointed tree we mean a pair \((T,K)\) where \(T\) is a tree and \(K\) is a finite subset of \(T\) containing all vertices of \(T\). If \((T,K)\) is a pointed tree, then the elements of \(K\) are the vertices of \((T,K)\) and the closures of the connected components of \(T\setminus K\) are called the edges of \((T,K)\). A connected and non-empty set \(I \subseteq T\) is called an interval of \((T,K)\) if there exists an edge \(T_i\) of \((T,K)\) such that \(I \subseteq T_i\). Given pointed trees \((T,K)\) and \((T',K')\), a map \(g : T \rightarrow T'\) will be called vertex preserving if \(g(K) \subseteq K'\).

To define the functors \(\epsilon_{\#0}\) and \(\epsilon_{\#1}\) we have to consider an additional structure on the pointed tree \((T,K)\). By an oriented tree we mean a pointed tree \((T,K)\), where each edge \(T_i\) is equipped with a linear ordering \(\geq_i\), satisfying the following: there exists a homeomorphism \(\varphi_i\), from \(T_i\) into \([0,1] \subset \mathbb{R}\), that verifies:

\[
y \geq_i x \text{ if and only if } \varphi_i(y) \geq \varphi_i(x) \text{, for all } x, y \in T_i.
\]

Let \(x\) and \(y\) be points of \(T\), lying in a same edge \(T_i\) of the oriented tree \((T,K)\). If \(y \geq_i x\) we define the closed interval

\[
[x,y] \overset{def}{=} \{z \in T_i : y \geq_i z \geq_i x\}
\]

Observe that every closed interval of \((T,K)\) has this form, and more generally, any interval of \((T,K)\) has one of the following forms: \([x,y]\) \overset{def}{=} \([x,y] \setminus \{x\}\), \([x,y]\) \overset{def}{=} \([x,y] \setminus \{y\}\) and \([x,y]\) \overset{def}{=} \([x,y] \setminus \{x,y\}\).

Let \(g : T \rightarrow T'\) be a piecewise monotone map. If \((T,K)\) and \((T',K')\) are oriented trees and \(I = [x,y]\) is a closed interval of \((T,K)\) on which \(g\) is monotone, we know that there exists an edge \(T'_j\) of \((T',K')\) such that \(g([x,y]) \subseteq T'_j\), and therefore we can define the sign

\[
\epsilon_{g,1}([x,y]) = \begin{cases} 0 & \text{if } g(x) = g(y) \\ -1 & \text{if } g(x) >_j g(y) \\ 1 & \text{if } g(y) >_j g(x) \end{cases} \tag{2}
\]

Let \(T\) be a tree and let \(S_0(T,\mathbb{R})\) be the \(\mathbb{R}\)-vector space whose basis is the set of formal symbols \(x\), with \(x \in T\). Denote by \(S_1(T,\mathbb{R})\) the subspace of \(S_0(T,\mathbb{R})\) which is generated by

\[
\{y-x \in S_0(T,\mathbb{R}) : x,y \in T\}.
\]

† More generally, for an arbitrary set \(X\), we will denote by \(S_0(X,\mathbb{R})\) the \(\mathbb{R}\)-vector space whose basis is the set of formal symbols \(x\), with \(x \in X\). The subspace of \(S_0(X,\mathbb{R})\) which is generated by \(\{y-x \in S_0(X,\mathbb{R}) : x,y \in X\}\) will be denoted by \(S_1(X,\mathbb{R})\).

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If \((T, K)\) is an oriented tree and \(I = [x, y]\) is a closed interval of \((T, K)\), we will use the symbol \(I\) to denote the vector \(y - x \in S_1(T, \mathbb{R})\). More generally, if \(X \neq \emptyset\) is a closed and connected subset of \(T\) and \(T_1, \ldots, T_k\) are the edges of \((T, K)\) which intersects \(X\), we define the vector
\[
\overrightarrow{X} = I_1 + \ldots + I_k \in S_1(T, \mathbb{R})
\] (3)
with \(I_i = X \cap T_i\). Observe that, with this notation, we can say that \(S_1(T, \mathbb{R})\) is the subspace of \(S_0(T, \mathbb{R})\) which is generated by
\[
\mathcal{I} = \{ \overrightarrow{I} : I \text{ is a closed interval of } (T, K) \}.
\] (4)

Now we can define the functor \(\epsilon_{\#1}\). Any tree map \(g : T \to T'\) induces the linear maps
\[
g_{\#0} : S_0(T, \mathbb{R}) \to S_0(T', \mathbb{R}) \quad \text{and} \quad g_{\#1} : S_1(T, \mathbb{R}) \to S_1(T', \mathbb{R}),
\]
defined by: \(g_{\#0}(x) = g(x)\), for all \(x \in T\), and \(g_{\#1} = g_{\#0}|_{S_1(T, \mathbb{R})}\). In other words \(g_{\#1}\) is the unique linear map from \(S_1(T, \mathbb{R})\) into \(S_1(T', \mathbb{R})\) that verifies
\[
g_{\#1}(y - x) = g(y) - g(x), \text{ for all } x, y \in T.\] (5)

Assume now that \((T, K)\) and \((T', K')\) are oriented trees. If \(g : T \to T'\) is a piecewise monotone map, we know that there exists a finite family \(\{I_k\}\) of intervals of \((T, K)\) on which \(g\) is monotone and such that \(T = \cup I_k\). In algebraic terms, this means that the subset of \(\mathcal{I}\)
\[
\mathcal{I}_g = \{ \overrightarrow{I} \in \mathcal{I} : g \text{ is monotone on } I \}
\] (6)
generates \(S_1(T, \mathbb{R})\), and from (5) we conclude that \(g_{\#1}\) is the unique linear map from \(S_1(T, \mathbb{R})\) into \(S_1(T', \mathbb{R})\) that verifies
\[
g_{\#1}(\overrightarrow{I}) = \epsilon_{g_{\#1}}(I) \cdot \overrightarrow{g(I)}, \text{ for all } \overrightarrow{I} \in \mathcal{I}_g.
\] (7)

Using \(\epsilon_{g_{\#1}}\) and \(g_{\#1}\) we define \(\epsilon_{\#1}(g)\)

**Definition 1.** Let \((T, K)\) and \((T', K')\) be oriented trees and let \(g : T \to T'\) be a piecewise monotone map. Define \(\epsilon_{\#1}(g)\) to be the unique linear map from \(S_1(T, \mathbb{R})\) into \(S_1(T', \mathbb{R})\) that verifies
\[
\epsilon_{\#1}(g)(\overrightarrow{I}) = \epsilon_{g_{\#1}}(I) \cdot g_{\#1}(\overrightarrow{I}), \text{ for all } \overrightarrow{I} \in \mathcal{I}_g.
\]
Observe that, from (7), \(\epsilon_{\#1}(g)\) can be regarded as the unique linear map from \(S_1(T, \mathbb{R})\) into \(S_1(T', \mathbb{R})\) that verifies
\[
\epsilon_{\#1}(g)(\overrightarrow{I}) = \overrightarrow{g(I)}, \text{ for all } \overrightarrow{I} \in \mathcal{I}_g,
\] (8)
and as an immediate consequence of this we have the following:
Proposition 2. Let $\mathcal{L}$ be the category whose objects are the oriented trees, and whose morphisms are piecewise monotone maps between oriented trees. Then the association

$$(T, K) \rightarrow S_0(T, \mathbb{R})$$

$g \rightarrow \epsilon_{\#1}(g)$$

is a covariant functor from $\mathcal{L}$ to the category of vector spaces over $\mathbb{R}$.

Now we define the linear map $\epsilon_{\#0}(g)$. In order to obtain a functor we define the linear map $\epsilon_{\#0}(g)$ on $S_0(T\backslash K, \mathbb{R})$. Let $(T, K)$ and $(T', K')$ be oriented trees and let $g : T \rightarrow T'$ be a piecewise monotone map. Define the map $\epsilon_{\#0} : T \rightarrow \{-1, 0, 1\}$ as follows: for any $x \in T\backslash K$, lying in the interior of some interval $I$ on which $g$ is monotone, define $\epsilon_{g,0}(x) = \epsilon_{g,1}(I)$; for the remaining points of $T$, define $\epsilon_{g,0}(x) = 0$.

It is clear that, in general, the linear map $g_{\#0} : S_0(T, \mathbb{R}) \rightarrow S_0(T', \mathbb{R})$ does not map $S_0(T\backslash K, \mathbb{R})$ into $S_0(T'\backslash K', \mathbb{R})$. But, if $g(x) \in K'$, then $\epsilon_{g,0}(x) = 0$, and therefore

$$\epsilon_{g,0}(x) : g_{\#0} \in S_0(T'\backslash K', \mathbb{R}), \text{ for all } x \in T\backslash K.$$ 

Definition 3. Let $(T, K)$ and $(T', K')$ be oriented trees and let $g : T \rightarrow T'$ be a piecewise monotone map. Define $\epsilon_{\#0}(g)$ to be the unique linear map from $S_0(T\backslash K, \mathbb{R})$ into $S_0(T'\backslash K', \mathbb{R})$ that verifies

$$\epsilon_{\#0}(g)(x) = \epsilon_{g,0}(x) : g_{\#0}(x), \text{ for all } x \in T\backslash K.$$ 

Let $(T, K)$, $(T', K')$ and $(T'', K'')$ be oriented trees, and let $g : T \rightarrow T'$ and $h : T' \rightarrow T''$ be piecewise monotone maps. In general we have

$$\epsilon_{\#0}(h) \circ \epsilon_{\#0}(g) \neq \epsilon_{\#0}(h \circ g),$$

however, if $g^{-1}(K') \subseteq K$ or $h(K') \subseteq K''$, then

$$\epsilon_{\#0}(h) \circ \epsilon_{\#0}(g) = \epsilon_{\#0}(h \circ g).$$

Thus we can write:

Proposition 4. Let $\mathcal{L}'$ be the category whose objects are oriented trees $(T, K)$, and whose morphisms are vertex preserving piecewise monotone tree maps. Then the association

$$(T, K) \rightarrow S_0(T\backslash K, \mathbb{R})$$

$g \rightarrow \epsilon_{\#0}(g)$$

is a covariant functor from $\mathcal{L}'$ to the category of vector spaces over $\mathbb{R}$.

3. The determinant of $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$

Our goal is to study the endomorphisms of an oriented tree $(T, K)$. From now on, an endomorphism of $\mathcal{L}$ (see Proposition 2) will be called a $K$-piecewise monotone map. An endomorphism of $\mathcal{L}'$ (see Proposition 4) will be called a $K$-preserving piecewise monotone map. In other words, a $K$-piecewise monotone map is nothing

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else than a piecewise monotone map \( f : T \to T \), regarded as an endomorphism of \((T, K)\), and a \( K \)-preserving piecewise monotone map is a \( K \)-piecewise monotone map that verifies \( f(K) \subseteq K \). So, for each \( K \)-piecewise monotone map \( f \), we obtain two linear endomorphisms \( \epsilon_{\#0}(f) \) and \( \epsilon_{\#1}(f) \), defined in different subspaces of \( S_0(T, \mathbb{R}) \): the first one is defined on \( S_0(T \setminus K, \mathbb{R}) \), and the second one is defined on \( S_1(T, \mathbb{R}) \).

Notice that, by Proposition 2, if \( f \) is a \( K \)-piecewise monotone map, then we have

\[
\epsilon_{\#1}(f^n) = \epsilon_{\#1}(f)^n,
\]

for all \( n \geq 0 \). But, as we have mentioned before, the equation \( \epsilon_{\#0}(f^n) = \epsilon_{\#0}(f)^n \) may not hold for a map which be only \( K \)-piecewise monotone. Nevertheless, as a consequence of Proposition 4, if \( f \) is a \( K \)-preserving piecewise monotone map, then we obtain

\[
\epsilon_{\#0}(f^n) = \epsilon_{\#0}(f)^n,
\]

for all \( n \geq 0 \). Indeed, this is the reason why we consider \( K \)-preserving maps.

These relations will play an essential role in the sequel, as well as the following theorem which states that the pair of linear endomorphisms \( (\epsilon_{\#0}(f), \epsilon_{\#1}(f)) \) has finite rank (see Definition 42). This fact enable us to define the trace and the determinant of this pair.

**Theorem 5.** Let \((T, K)\) be an oriented tree and let \( f \) be a \( K \)-piecewise monotone map. Then the pair of linear endomorphisms \((\epsilon_{\#0}(f), \epsilon_{\#1}(f))\) has finite rank.

Since the linear endomorphisms \( \epsilon_{\#0}(f) \) and \( \epsilon_{\#1}(f) \) are defined in different subspaces of \( S_0(T, \mathbb{R}) \), to prove Theorem 5 we introduce the endomorphisms \( \theta_0 \) and \( \theta_1 \) that are extensions of \( \epsilon_{\#0}(f) \) and \( \epsilon_{\#1}(f) \) respectively to \( S_0(T, \mathbb{R}) = S_0(T \setminus K, \mathbb{R}) + S_1(T, \mathbb{R}) \).

**Definition 6.** Let \((T, K)\) be an oriented tree, and let \( f \) be a \( K \)-piecewise monotone map. Define \( \theta_0 \) to be the unique linear endomorphism of \( S_0(T, \mathbb{R}) \) that verifies: the diagram

\[
\begin{array}{ccc}
S_0(T \setminus K, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R}) \\
\epsilon_{\#0}(f) & \downarrow & \theta_0 \\
S_0(T \setminus K, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R})
\end{array}
\]

is commutative, and \( \theta_0(x) = 0 \), for all \( x \in K \).

**Definition 7.** Let \((T, K)\) be an oriented tree, and let \( f \) be a \( K \)-piecewise monotone map. Define \( \theta_1 \) to be the unique linear endomorphism of \( S_0(T, \mathbb{R}) \) that verifies: the diagram

\[
\begin{array}{ccc}
S_1(T, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R}) \\
\epsilon_{\#1}(f) & \downarrow & \theta_1 \\
S_1(T, \mathbb{R}) & \hookrightarrow & S_0(T, \mathbb{R})
\end{array}
\]

is commutative, and \( \theta_1(\sum_{x \in K} x) = 0 \).

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We have then

\[ \theta_0(y) = \begin{cases} \epsilon \#_0(f)(y) & \text{if } y \in T \setminus K \\ 0 & \text{if } y \in K \end{cases} \in S_0(T \setminus K, \mathbb{R}), \]

and

\[ \theta_1(y) = \frac{1}{\#_K} \sum_{x \in K} \epsilon \#_1(f)(y - x) \in S_1(T, \mathbb{R}) \]

for all \( y \in T \), and consequently \( \text{Im}(\theta_0) \subseteq S_0(T \setminus K, \mathbb{R}) \) and \( \text{Im}(\theta_1) \subseteq S_1(T, \mathbb{R}) \). Thus, from Proposition 43, to prove Theorem 5 it suffices to show that \( \theta_1 - \theta_0 \) has finite rank. Let \( T_1, \ldots, T_m \) be the edges of the oriented tree \( (T, K) \). For each \( x \in T \) define the linear form \( \omega_x \) on \( S_0(T, \mathbb{R}) \) as follows: set \( X_0 = \{x\} \) and denote by \( X_1, \ldots, X_{\text{val}(x)} \) the connected components of \( T \setminus \{x\} \). For each \( j = 0, 1, \ldots, \text{val}(x) \), define

\[ \nu(X_j) = \#(X_j \cap K) \text{ and } \epsilon(X_j) = \lim_{y \to x} \epsilon_{f, 0}(y). \]

Define \( \omega_x \) to be the unique linear form on \( S_0(T, \mathbb{R}) \) that verifies

\[ \omega_x(y) = -\epsilon(X_1) + (m + 1)^{-1} \sum_{j=0}^{\text{val}(x)} \epsilon(X_j) \nu(X_j), \tag{12} \]

for all \( y \in X_j \) and \( l = 0, 1, \ldots, \text{val}(x) \). Observe that the linear form \( \omega_x \) measures the discontinuity of the map \( \epsilon_{f, 0} \) at \( x \). Indeed, if we define

\[ C_f \overset{\text{def}}{=} \{ x \in T : \epsilon_{f, 0} \text{ is discontinuous at } x \}, \tag{13} \]

then we obtain

\[ C_f = \{ x \in T : \omega_x \neq 0 \}. \tag{14} \]

From the definition of \( \epsilon_{f, 0} \), we also have that \( C_f \) is a finite set. Thus, as a consequence of the following proposition we see that \( \theta_1 - \theta_0 \) has finite rank. The proof of the next proposition is a simple verification.

**Proposition 8.** Let \( (T, K) \) be an oriented tree, and let \( f \) be a \( K \)-piecewise monotone map. Then we have

\[ \theta_0(y) = \epsilon_{f, 0}(y) \cdot f \#_0(y), \]

for all \( y \in T \), and

\[ \theta_1(y) = \sum_{x \in C_f} \omega_x \otimes f \#_0(x). \]

As a consequence of Theorem 5 we know that, for each \( K \)-piecewise monotone map \( f \), the trace of \( (\epsilon \#_0(f), \epsilon \#_1(f)) \) and the corresponding determinant (Definition 45)

\[ D(\epsilon \#_0(f), \epsilon \#_1(f))(t) \overset{\text{def}}{=} \exp \sum_{n \geq 1} -\frac{t^n}{n} \text{Tr}(\epsilon \#_0(f)^n, \epsilon \#_1(f)^n) \]

are defined. From Propositions 8 and 46, we can use the same pair \( (\theta_0, \theta_1) \) to compute the formal power series \( D(\epsilon \#_0(f), \epsilon \#_1(f))(t) \): considering the matrix with
Example 10. Let \((T,K)\) be an oriented tree and let \(f\) be a \(K\)-piecewise monotone map. Then we have

\[
D(\epsilon_{\neq 0}(f), \epsilon_{\neq 1}(f))(t) = \text{Det} \left( \text{id} - t M(t) \right).
\]

Proposition 9. Let \((T,K)\) be an oriented tree and let \(f\) be a \(K\)-piecewise monotone map. Then we have

\[
D(\epsilon_{\neq 0}(f), \epsilon_{\neq 1}(f))(t) = \text{Det} \left( \text{id} - t M(t) \right).
\]

Example 10. The matrix \(M(t)\) contains all the symbolic information on the orbits of the points of \(C_f\). If, in particular, we know the symbolic orbits of these points, then the matrix \(M(t)\) can be computed explicitly. As an example, let \((T,K)\) be the oriented tree defined by: \(K = \{1,0,i,-i\}\) and \(T = \{x \in \mathbb{C} : |x| \leq 1, \arg(x) = -\pi/2, 0, \pi/2\} = [0,1] \cup [0,i] \cup [-i,0]\). Let \(f : T \to T\) be the \(K\)-preserving piecewise monotone map defined by

\[
f(x) = \begin{cases} 
  i \left(1 + 2\sqrt{2}\right) x \left(1 - x\right) & \text{if } x \in [0,1] \\
  -x & \text{if } x \in [0,i] \\
  ix & \text{if } x \in [-i,0]
\end{cases}.
\]

We have \(C_f = \{x_1, x_2, x_3, x_4, x_5\}\) with \(x_1 = 0\), \(x_2 = \frac{1}{2}\), \(x_3 = 1\), \(x_4 = i\), and \(x_5 = -i\). From construction of \(f\), we also have \(f(x_1) = x_1\), \(f(x_3) = x_1\), \(f(x_4) = x_5\) and \(f(x_5) = x_1\). Note that the orbit of \(x_2\) is not finite, nevertheless it can be shown that:

\[
f^{3k-2}(x_2) \in \]x_1, x_4[, \ f^{3k-1}(x_2) \in \]x_5, x_1[, \ f^{9k-6}(x_2) \in \]x_2, x_3[, \ f^{9k}(x_2) \in \]x_2, x_3[ \ and \ f^{9k-3}(x_2) \in \]x_1, x_2[.
\]

for all \(k \geq 1\). With this, we obtain

\[
M(t) = \begin{bmatrix}
-1/4 & \frac{(3-3i-5t^2)(1-t^3-t^6)}{4(1-t^6)} & -1/4 & 3/4 & 5/4 \\
-1/2 & \frac{(-1+3t^3)(1-t^3-t^6)+4t^5}{2(1-t^6)} & -1/2 & 1/2 & 3/2 \\
1/4 & \frac{(1-t^3)(1-t^3-t^6)}{4(1-t^6)} & 1/4 & 1/4 & -3/4 \\
1/4 & \frac{(1-t^3)(1-t^3-t^6)}{4(1-t^6)} & 1/4 & 1/4 & 1/4 \\
1/4 & \frac{(1-t^3)(1-t^3-t^6)}{4(1-t^6)} & 1/4 & -3/4 & 1/4
\end{bmatrix},
\]

and from Proposition 9

\[
D(\epsilon_{\neq 0}(f), \epsilon_{\neq 1}(f))(t) = \frac{1-t^3-t^6}{1+t^3+t^6}.
\]
3.1. Markov maps

Now, we are going to see that, for a very special class of piecewise monotone maps, there is an alternative way for computing $D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t)$. Let $(T, K)$ be an oriented tree and let $f$ be a $K$-preserving piecewise monotone map. Denoting the edges of $(T, K)$ by $T_1, \ldots, T_m$, we say that $f$ is Markov if either $f$ is constant on $T_i$ or $f$ is injective on $T_i$, for all $i = 1, \ldots, m$.

It is easy to see that, for these maps, the determinant $D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t)$ is always a polynomial. In fact, if $f$ is Markov, then we have $f(C_f) \subseteq K$, and therefore the entries of the matrix $M(t)$, defined in (15), are rational numbers. More precisely, this matrix has the form

$$M = \begin{bmatrix}
\omega_{x_1}(f_{#0}(x_1)) & \ldots & \omega_{x_1}(f_{#0}(x_1)) \\
\ldots & \ddots & \ldots \\
\omega_{x_i}(f_{#0}(x_1)) & \ldots & \omega_{x_i}(f_{#0}(x_1))
\end{bmatrix},$$

with $\{x_1, \ldots, x_l\} = C_f$, and from Proposition 9 we obtain

$$D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t) = \text{Det} (\text{id} - tM) \quad \text{and} \quad Tr(\epsilon_{#0}(f)^n, \epsilon_{#1}(f)^n) = Tr(M^n),$$

for all $n \geq 1$. Observe also that, if $f$ is Markov, then $S_1(K, \mathbb{R})$ is an $\epsilon_{#1}(f)$ invariant subspace of $S_1(T, \mathbb{R})$, so, denoting by $P$ the matrix that represents $\epsilon_{#1}(f)|_{S_1(K, \mathbb{R})}$ with respect to the basis $T_1, T_2, \ldots, T_m$ (see (3)), we obtain

$$Tr(P^n) = Tr(\epsilon_{#1}(f)^n|_{S_1(K, \mathbb{R})})$$

for all $n \geq 0$. Notice that the matrix $P$ coincides with the usually called Markov transition matrix of $f$. In fact, from definition of $\epsilon_{#1}(f)$, we see that $P$ is a $m \times m$ matrix with coefficients in $\{0, 1\}$ and such that

$$P(i, j) = \begin{cases} 
1 & \text{if } T_i \subseteq f(T_j) \\
0 & \text{otherwise}
\end{cases}.$$ 

Using (18) and (17) we prove now that, for Markov maps, $D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t)$ coincides with the characteristic polynomial of $P$.

**Theorem 11.** Let $(T, K)$ be an oriented tree and let $f : T \rightarrow T$ be a $K$-preserving piecewise monotone map. If $f$ is Markov, then we have

$$Tr(P^n) = Tr(\epsilon_{#0}(f)^n, \epsilon_{#1}(f)^n),$$

for all $n \geq 1$, and

$$D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t) = \text{Det}(\text{id} - tP).$$

**Proof 12.** From (17) and (18) it suffices to show that

$$Tr(\epsilon_{#1}(f)^n|_{S_1(K, \mathbb{R})}) = Tr(M^n), \text{ for all } n \geq 1,$$

where $M$ is the matrix defined in (16). From Definition 7 we obtain the commutative diagram

$$\begin{array}{ccc}
0 & \rightarrow & S_1(K, \mathbb{R}) \\
\downarrow \epsilon_{#1}(f) \big|_{S_1(K, \mathbb{R})} & & \downarrow \theta_1 \big|_{S_0(K, \mathbb{R})} \\
0 & \rightarrow & S_1(K, \mathbb{R})
\end{array}$$

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with exact rows, and therefore
\[
\text{Tr}(\epsilon_{\#1}(f)^n | S_1(K, \mathbb{R})) = \text{Tr} \left( \theta^n_1 \big| S_0(K, \mathbb{R}) \right), \quad \text{for all } n \geq 1.
\]

On the other hand, from Proposition 8, we conclude also that \( \theta_0 | S_0(K, \mathbb{R}) = 0 \) and
\[
\theta_1 | S_0(K, \mathbb{R}) = \sum_{x \in C_f} (\omega_x | S_0(K, \mathbb{R})) \otimes f_{\#0}(x),
\]
and consequently
\[
\text{Tr} \left( \theta^n_1 \big| S_0(K, \mathbb{R}) \right) = \text{Tr} \left( M^n \right),
\]
for all \( n \geq 1 \), as desired. \( \Box \)

Theorem 11 shows in particular that, for Markov maps, \( D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t) \) defines a holomorphic function on \( \mathbb{C} \). For piecewise monotone maps in general this is not true (see Example 10), however, from (12), we see easily that
\[
|\omega_x (\theta^n_0 (y))| = |\epsilon_{f,0}(y) \cdots \epsilon_{f,0}(f^{n-1}(y)) \omega_x (f_{\#0}(y))| \leq 2,
\]
for all \( x, y \in T \), and this shows that the entries of the matrix \( M(t) \) of (15) are holomorphic on the unit disk \( \mathbb{D} = \{ t \in \mathbb{C} : |t| < 1 \} \). Thus, as a consequence of Proposition 9, we obtain:

**Proposition 13.** Let \((T, K)\) be an oriented tree and let \( f \) be a \( K \)-piecewise monotone map. Then the function \( D_{(\epsilon_{\#0}(f), \epsilon_{\#1}(f))}(t) \) is holomorphic on \( \mathbb{D} \).

4. The Artin-Mazur zeta function

Recall that if \( f : X \to X \) is a function defined in an arbitrary set \( X \), and each iterate \( f^n \) has only finitely many fixed points, then the Artin-Mazur zeta function of \( f \) is defined to be the following formal power series
\[
\zeta(t) = \exp \sum_{n \geq 1} N_{f^n} t^n / n,
\]
where \( N_{f^n} \) denotes the total number of fixed points of the iterate \( f^n \). The aim of this section is to show that the determinant of the pair \( (\epsilon_{\#0}(f), \epsilon_{\#1}(f)) \) plays a fundamental role in the computation of \( \zeta(t) \). We begin to introduce some notation. Let \( g : T \to T \) be a piecewise strictly monotone map and denote by \( \mathbb{T} \) the set whose elements are the pairs \((x, C)\) such that \( x \in T \) and \( C \) is a connected component of \( T \setminus \{x\} \). Consider the map
\[
G : T \to \mathbb{T},
\]
induced by \( g \), and defined by: \( G(x, C) = (x', C') \) if and only if \( g(x) = x' \) and there exists a neighborhood \( V \) of \( x \) such that \( g(V \cap C) \subset C' \). Notice that a pair \((x, C) \in \mathbb{T} \) is a fixed point of \( G \) if and only if \( x \) is a fixed point of \( g \) and there exists a neighborhood \( V \) of \( x \) such that \( g(V \cap C) \subset C \).
Definition 14. Let $T$ be a tree and let $g : T \to T$ be a piecewise strictly monotone map with isolated fixed points. If $(x, C)$ is a fixed point of $G$ and there exists a neighborhood $V$ of $x$ such that $g(y)$ lies between $x$ and $y$, for all $y \in V \cap C$, then $(x, C)$ will be called a formally stable pair of $g$. A fixed point of $G$ which is not formally stable will be called a formally unstable pair of $g$. The number of all formally stable pairs of $g$ is denoted by $S_g$.

So, if $f : T \to T$ is a strictly piecewise monotone map, and each iterate $f^n$ has only finitely many fixed points, we can define the formal power series

$$ S(t) \overset{def}{=} \exp \sum_{n \geq 1} \frac{S_{f^n}}{n} t^n. $$

Consider now a pointed tree $(T, K)$ with $m$ edges and assume that $f$ is a $K$-piecewise monotone map. Under this conditions we define the transition matrices $Q$ and $R$ as follows. Denoting by $x_1, \ldots, x_{m+1}$ the elements of $K$, $Q$ is a $(m+1) \times (m+1)$ matrix defined by

$$ Q(i, j) = \begin{cases} 1 & \text{if } f(x_j) = x_i \\ 0 & \text{otherwise} \end{cases}. \quad (21) $$

Denoting by $F : T \to T$ the map induced by $f$ and by $X_1, \ldots, X_{2m}$ the elements of $\{(x, C) \in T : x \in K\}$, $R$ is a $2m \times 2m$ matrix defined by

$$ R(i, j) = \begin{cases} 1 & \text{if } F(X_j) = X_i \\ 0 & \text{otherwise} \end{cases}. \quad (22) $$

Finally we are in position to state the main theorem of this section. An analogous result can be provided for $K$-piecewise strictly monotone maps (see [AI99, AI99]).

Theorem 15. Let $(T, K)$ be an oriented tree and let $f$ be a $K$-preserving piecewise strictly monotone map. If each iterate $f^n$ has only finitely many fixed points, then

$$ \zeta(t) = \frac{D_R(t) \ S(t)}{D_{(\epsilon_{\#0(f), \#1(f))}}(t) \ D_Q(t)}, $$

where $D_Q(t)$ (resp. $D_R(t)$) is the characteristic polynomial of $Q$ (resp. $R$).

Remark 16. In the particular of the oriented tree $(\{a, b\}, \{a, b\})$ it is easy to see that, for a given $\{a, b\}$-preserving piecewise strictly monotone, the matrices $P$ and $Q$ and coincide, and from Theorem 15 we obtain

$$ \zeta(t) = \frac{S(t)}{D_{(\epsilon_{\#0(f), \#1(f))}}(t)}. $$

Observe also that Theorem of Julia and Fatou (see [Ju] and [Fa]), valid for polynomials of degree $n \geq 2$, and Theorem of Singer (see [Si]), valid for piecewise monotone interval maps with negative Schwarzian derivative, enable to conclude that, for these maps, $S(t)^{-1}$ is a cyclotomic polynomial, i.e.,

$$ S(t)^{-1} = (1 - t^{n_1})(1 - t^{n_2}) \ldots (1 - t^{n_k}). $$

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For an arbitrary piecewise strictly monotone tree map \( f : T \rightarrow T \) the computation of \( S(t) \) may be very complicated. Nevertheless, defining a formally stable \( p_o \)-periodic orbit of \( f \) as a \( p_o \)-periodic orbit of \( F : T \rightarrow T \) containing a stable pair of \( f^{p_o} \), we have

\[
S(t) = \prod_{o \in \mathbb{O}} (1 - t^{p_o}),
\]

where \( \mathbb{O} \) denotes the set (eventually not finite) of all formally stable periodic orbits of \( f \). As an example, let \( f \) be the map described in Example 10. It can be shown that there exists one and only one formally stable periodic orbit of \( f \). Since the period of this orbit is 9, and, on the other hand \( D_9(t) = 1 - t \) and \( D_9(t) = 1 - t^3 \), we obtain

\[
\zeta(t) = \frac{(1 + t^3 + t^6)(1 - t^3)}{(1 - t^3 - t^6)(1 - t)(1 - t^6)}.
\]

4.1. Traces and fixed points (Proof of Theorem 14) Let \((T, K)\) be oriented tree and let \( g \) be a \( K \)-piecewise monotone map. We know, from Theorem 8, that the trace of the pair \((\epsilon_{g,0}(g), \epsilon_{g,1}(g))\) is defined. Now we will show that there exists an important relationship between this trace and the number of fixed points of \( g \). First we introduce some more notation. Denote by \( \text{Fix}_g \) the set of all fixed points of \( g \). If \( x \in \text{Fix}_g \), then we say that \( x \) is a fixed point of \( g \) of positive (resp. negative) type if \( \epsilon_{g,0}(x) > 0 \) (resp. \( \epsilon_{g,0}(x) < 0 \)). If \( x \in \text{Fix}_g \) and \( \epsilon_{g,0}(x) = 0 \), then we say that \( x \) is a fixed point of \( g \) of critical type. We will denote by \( \text{Fix}_g^+ \) (resp. \( \text{Fix}_g^- \)) the set of all fixed points of \( g \) of positive (resp. negative) type, and by \( \text{Fix}_g^0 \) the set of all fixed points of \( g \) of critical type. In other words, we have

\[
\text{Fix}_g^+ = \{ x \in \text{Fix}_g : \epsilon_{g,0}(x) > 0 \}, \quad \text{Fix}_g^- = \{ x \in \text{Fix}_g : \epsilon_{g,0}(x) < 0 \}
\]

and

\[
\text{Fix}_g^0 = \{ x \in \text{Fix}_g : \epsilon_{g,0}(x) = 0 \}.
\]

Notice that \( \text{Fix}_g^+ \), \( \text{Fix}_g^- \), and \( \text{Fix}_g^0 \) are independent of the order of the pointed tree \((T, K)\), nevertheless these sets depend on \( K \). The following figure shows the graph of a piecewise monotone map \( g : [-1, 1] \rightarrow [-1, 1] \). Regarding \( g \) as a \([-1, 1]\)-piecewise monotone map, we obtain \( \text{Fix}_g = \{ 0 \} \) and \( \text{Fix}_g^0 = \emptyset \). However for the same map, regarded as a \([-1, 0, 1]\)-piecewise monotone map, we obtain \( \text{Fix}_g^- = \emptyset \) and \( \text{Fix}_g^0 = \{ 0 \} \).

It is clear that the cardinals

\[
N_g^+ \overset{\text{def}}{=} \# \text{Fix}_g \quad \text{and} \quad N_g^- \overset{\text{def}}{=} \# \text{Fix}_g^+,
\]

may be not finite, however, since \( g \) is piecewise monotone, we see at once that

\[
N_g^- \overset{\text{def}}{=} \# \text{Fix}_g^- \quad \text{and} \quad N_g^0 \overset{\text{def}}{=} \# \text{Fix}_g^0
\]

are both natural numbers.

Observe that, if \((T, K)\) is an oriented tree and \( g \) is a \( K \)-piecewise monotone map, then the pair of linear endomorphisms \((\epsilon_{g,0}(g), \epsilon_{g,1}(g))\) depends on the orientation of \((T, K)\). However, as an immediate consequence of the next result, we see that the trace and the determinant of this pair is independent of the orientation of \((T, K)\).
Theorem 17. Let \((T, K)\) be an oriented tree and let \(g\) be a piecewise \(K\)-monotone map. Then we have

\[
\text{Tr} (\epsilon_\#_0(g), \epsilon_\#_1(g)) = 2N_g^- - 1 + N_g^0.
\]

Let \(T_1, \ldots, T_m\) be the edges of the oriented tree \((T, K)\) and consider the oriented trees \((T_1, K_1), \ldots, (T_m, K_m)\), with \(K_j = K \cap T_j\). For each \(j = 1, \ldots, m\), let \(i_j : T_j \to T\) be the injection of \(T_j\) in \(T\), and \(p_j : T \to T_j\) the projection of \(T\) into \(T_j\) (\(p_j\) is the unique continuous map that verifies \(p_j(x) = x\), for all \(x \in T_j\), and \(p_j\) is constant on each connected component of \(T \setminus T_j\)). Clearly \(i_j\) and \(p_j\) are piecewise monotone maps, so if \(g : T \to T\) is a piecewise monotone map, then the interval map

\[
g_j = p_j \circ g \circ i_j : T_j \to T_j
\]

(23)
can be regarded as a \(K_j\)-piecewise monotone map, and by Lemma 2.23 of [ASR] †, we know that

\[
\text{Tr} (\epsilon_\#_0(g_j), \epsilon_\#_1(g_j)) = 2N_{g_j}^- - 1 + N_{g_j}^0.
\]

Thus, Theorem 17 is an immediate consequence of the two following lemmas.

† Actually this lemma is proved for piecewise strictly monotone interval maps, but with small changes it can be shown for the wider class of piecewise monotone interval maps. This remark is important because even for a piecewise strictly monotone tree map \(g\) the maps \(g_j\) may not be strictly monotone.
LEMMA 18. Let \((T, K)\) be an oriented tree with \(m\) edges, and let \(g\) be a \(K\)-piecewise monotone map. Then we have

\[
Tr(\epsilon_{\#0}(g), \epsilon_{\#1}(g)) = \sum_{j=1}^{m} Tr(\epsilon_{\#0}(g_j), \epsilon_{\#1}(g_j)).
\]

PROOF 19. Setting \(\varphi = \epsilon_{\#0}(g), \psi = \epsilon_{\#1}(g), U = S_0(T, \mathbb{R}), V = S_0(T\setminus K, \mathbb{R}), W = S_1(T, \mathbb{R}), V_j = S_0(T_j\setminus K_j, \mathbb{R}),\) and \(W_j = S_1(T_j, \mathbb{R}),\) it is easy to check that we are under the conditions of Proposition 44, and therefore

\[
Tr(\epsilon_{\#0}(g), \epsilon_{\#1}(g)) = \sum_{j=1}^{m} Tr(p_{V_j} \circ \epsilon_{\#0}(g) \circ i_{V_j}, p_{W_j} \circ \epsilon_{\#1}(g) \circ i_{W_j}).
\]

So it suffices to show that

\[
p_{V_j} \circ \epsilon_{\#0}(g) \circ i_{V_j} = \epsilon_{\#0}(g_j) \text{ and } p_{W_j} \circ \epsilon_{\#1}(g) \circ i_{W_j} = \epsilon_{\#1}(g_j).
\]

From definition of \(\epsilon_{\#0},\) we verify easily that \(p_{V_j} = \epsilon_{\#0}(p_j)\) and \(i_{V_j} = \epsilon_{\#0}(i_j),\) where \(i_j : T_j \to T\) and \(p_j : T \to T_j\) are the piecewise monotone maps of (23). Since \(i_j^{-1}(V) = V_j \text{ and } p_j(V) = V_j,\) it follows from (9) that

\[
p_{V_j} \circ \epsilon_{\#0}(g) \circ i_{V_j} = \epsilon_{\#0}(p_j) \circ \epsilon_{\#0}(g) \circ \epsilon_{\#0}(i_j) = \epsilon_{\#0}(p_j \circ g \circ i_j) = \epsilon_{\#0}(g_j).
\]

By definition of \(\epsilon_{\#1},\) we have \(p_{W_j} = \epsilon_{\#1}(p_j)\) and \(i_{W_j} = \epsilon_{\#1}(i_j),\) and from Proposition 2

\[
p_{W_j} \circ \epsilon_{\#1}(g) \circ i_{W_j} = \epsilon_{\#1}(p_j) \circ \epsilon_{\#1}(g) \circ \epsilon_{\#1}(i_j) = \epsilon_{\#1}(p_j \circ g \circ i_j) = \epsilon_{\#1}(g_j),
\]

as desired.\\

LEMMA 20. Let \((T, K)\) be an oriented tree with \(m\) edges, and let \(g\) be a \(K\)-piecewise monotone map. Then we have

\[
2N_g^- - 1 + N_g^0 = \sum_{j=1}^{m} \left(2N_{g_j}^- - 1 + N_{g_j}^0\right).
\]

PROOF 21. Let \(x \in K\) be one of the \(m+1\) vertices of the oriented tree \((T, K),\) and let \(T_{x_{val(x)}}\) be the edges of \((T, K)\) that contain \(\{x\}\). From definition of \(g_j,\) we have

\[
\sum_{j=1}^{val(x)} \#(Fix^0_{g_{x_j}} \cap \{x\}) = \#(Fix^0_g \cap \{x\}) - 1 + val(x),
\]

for all \(x \in K.\) Thus we can write

\[
\sum_{j=1}^{m} \#(Fix^0_{g_j} \cap K) = \sum_{x \in K} \sum_{j=1}^{val(x)} \#(Fix^0_{g_{x_j}} \cap \{x\})
\]

\[
= \sum_{x \in K} \left(\#(Fix^0_g \cap \{x\}) - 1 + val(x)\right)
\]

\[
= \#(Fix^0_g \cap K) - (m+1) + \sum_{x \in K} val(x)
\]

\[
= \#(Fix^0_g \cap K) - (m+1) + 2m
\]

\[
= \#(Fix^0_g \cap K) + m - 1.
\]
and therefore
\[
\sum_{j=1}^{m} \left(2N_{g_j}^- - 1 + N_{g_j}^0 \right) = 2N_g^- - m + \# (\text{Fix}_g^0 \cap (T \setminus K)) + \sum_{j=1}^{m} \# (\text{Fix}_{g_j}^0 \cap K) \\
= 2N_g^- - 1 + \# (\text{Fix}_g^0 \cap (T \setminus K)) + \# (\text{Fix}_g^0 \cap K) \\
= 2N_g^- - 1 + N_g^0,
\]
as desired. \(\square\)

Observe that Theorem 17 holds for any \(K\)-piecewise monotone map, even for maps with an infinite number of fixed points. Assuming that \(g\) is a piecewise strictly monotone map with isolated fixed points, it is possible to establish a precise relationship between the trace of the pair \((\epsilon_{#0}(g), \epsilon_{#1}(g))\) and the total number of fixed points of \(g\).

Once again it is convenient to consider first the case of the oriented tree \(([a, b], \{a, b\})\). Let \(g\) be a \([a, b]\)-piecewise monotone map with isolated fixed points. In order to relate the numbers \(N_g\) and \(2N_g^- - 1\), Milnor and Thurston proved in [MTh] that the difference \(N_g - (2N_g^- - 1)\) is equal to the number of formally left stable fixed points of \(g\) plus the number of formally right stable fixed points of \(g\).

In other words
\[
N_g - (2N_g^- - 1 + N_g^0) = \sum_{x \in I} l_g(x) + \sum_{x \in I} r_g(x) - \sum_{x \in I} n_g^0(x),
\]
where the numbers \(l_g(x), r_g(x)\) and \(n_g^0(x)\) are defined by: \(l_g(x) = 1\) if \(x\) is a formally left stable fixed point of \(g\), and \(l_g(x) = 0\) for the remaining points of \(I\); \(r_g(x) = 1\) if \(x\) is a formally right stable fixed point of \(g\), and \(r_g(x) = 0\) for the remaining points of \(I\); \(n_g^0(x) = 1\) if \(x\) is a fixed point of \(g\) of zero type, and \(n_g^0(x) = 0\) for the remaining points of \(I\). Since \(l_g(a) = 1\) if and only if \(a\) is a fixed point of \(g\), and \(r_g(b) = 1\) if and only if \(b\) is a fixed point of \(g\), we arrive at
\[
\# (\text{Fix}_g \cap [a, b]) - (2N_g^- - 1 + N_g^0) = \sum_{x \in [a, b]} (l_g(x) + r_g(x) - n_g^0(x)) + (24) \\
r_g(a) - n_g^0(a) + l_g(b) - n_g^0(b).
\]

Using (24) we prove now the following general result:

**Theorem 22.** Let \((T, K)\) be an oriented tree, let \(g\) be a \(K\)-piecewise strictly monotone map, with isolated fixed points, and \(G : T \to T\) the map induced by \(g\). Define
\[
A_g = \# (\text{Fix}_g \cap K) \quad \text{and} \quad B_g = \# \{(x, C) \in T : x \in K \text{ and } (x, C) \in \text{Fix}_G\}.
\]
Then we have
\[
N_g - (2N_g^- - 1 + N_g^0) = A_g - B_g + S_g.
\]

\(\dagger\) Recall that, following Milnor and Thurston, an isolated fixed point \(x_0\) of \(g\), not of negative type, is formally left stable if \(g(x) > x\) throughout some interval \([x_0 - \epsilon, x_0]\), or also if \(x_0\) is the leftmost point of the entire interval \(I\). Similarly, an isolated fixed point \(x_0\) of \(g\), not of negative type, is formally right stable if \(g(x) < x\) throughout some interval \([x_0, x_0 + \epsilon]\), or also if \(x_0\) is the rightmost point of the entire interval \(I\).
Finally the proof follows from the following equalities:

\[ N_g - (2N_g^0 - 1 + N_g^0) = A_g + \sum_{j=1}^{m} \left( \# \left( \text{Fix}_{g_j} \cap [a_j, b_j] \right) - (2N_g^0 - 1 + N_g^0) \right) \]

\[ = A_g + \sum_{j=1}^{m} \sum_{x \in [a_j, b_j]} (l_{g_j}(x) + r_{g_j}(x) - n_{g_j}^0(x)) + \sum_{j=1}^{m} (r_{g_j}(a_j) - n_{g_j}^0(a_j)) + \sum_{j=1}^{m} (l_{g_j}(b_j) - n_{g_j}^0(b_j)). \]

Finally the proof follows from the following equalities:

\[ r_{g_j}(a_j) - n_{g_j}^0(a_j) \begin{cases} 
-1 & \text{if } (a_j, C_j) \text{ is an unstable pair of } g, \text{ for all } j = 1, \ldots, m; \\
0 & \text{otherwise}
\end{cases} \]

\[ l_{g_j}(b_j) - n_{g_j}^0(b_j) \begin{cases} 
-1 & \text{if } (b_j, D_j) \text{ is an unstable pair of } g, \text{ for all } j = 1, \ldots, m; \\
0 & \text{otherwise}
\end{cases} \]

and

\[ l_{g_j}(x) + r_{g_j}(x) - n_{g_j}^0(x) = \# \{ (x, C) \in T : (x, C) \text{ is a stable pair of } g \}, \]

for all \( x \in [a_j, b_j] \) and \( j = 1, \ldots, m. \)

We are now in position to state a fundamental relationship between the determinant

\[ D_{(\varepsilon_{\#0(f)}, \varepsilon_{\#1(f)})}(t) \overset{\text{def}}{=} \exp \sum_{n \geq 1} - \text{Tr} (\varepsilon_{\#0(f)^n}, \varepsilon_{\#1(f)^n}) t^n/n, \]

and a modified zeta function of \( f \) defined by

\[ \hat{\zeta}(t) \overset{\text{def}}{=} \exp \sum_{n \geq 1} \left( 2N_{f^n} - 1 + N_{f^n}^0 \right) t^n/n. \]

Since, by (10) and Theorem 17, we have

\[ \text{Tr} (\varepsilon_{\#0(f)^n}, \varepsilon_{\#1(f)^n}) = 2N_{f^n} - 1 + N_{f^n}^0, \]

for all \( n \geq 1 \), we can write (compare with (1)):

**Theorem 24.** Let \((T, K)\) be an oriented tree and let \( f \) be a \( K \)-preserving piecewise monotone map. Then we have

\[ \hat{\zeta}(t) = D_{(\varepsilon_{\#0(f)}, \varepsilon_{\#1(f)})}(t)^{-1}. \]

Finally we can prove Theorem 15. From Theorem 24 it suffices to show that

\[ \zeta(t) \hat{\zeta}(t)^{-1} = \frac{D_R(t)}{D_Q(t)} S(t). \]
In other words we have to prove that
\[ N_{f^n} - \left(2N_{f^n} - 1 + N_0^f\right) = \text{Tr}(Q^n) - \text{Tr}(R^n) + S_{f^n}, \]
for all \( n \geq 1 \), and this follows from Theorem 22, because
\[ \text{Tr}(Q^n) = A_{f^n} \quad \text{and} \quad \text{Tr}(R^n) = B_{f^n}, \quad \text{for all} \quad n \geq 1. \]

Turning our attention to the specific case of Markov maps, the following corollary is an immediate consequence of Theorem 15 and Theorem 24.

**Corollary 25.** Let \((T,K)\) be an oriented tree and let \( f : T \to T \) be a \( K\)-preserving monotone map. Assume that \( f \) is Markov. Then we have
\[ \text{Tr}(P^n) = 2N_{f^n} - 1 + N_0^f, \]
for all \( n \geq 1 \), and this follows from Theorem 22, because
\[ \text{Tr}(Q^n) = A_{f^n} \quad \text{and} \quad \text{Tr}(R^n) = B_{f^n}, \quad \text{for all} \quad n \geq 1. \]

**Example 26.** As we can see through the following examples, even for interval maps, the matrices \( P, Q \) and \( R \) play an important role in the computation of \( \zeta(t) \).

Let \( f : [-1,1] \to [-1,1] \) be the tent map defined by \( f(x) = b(|x|) + 1 \), with \( b \in ]1,2[ \). Since \( b > 1 \), we have \( S(t) = 1 \), and from Remark 16 we can regard \( f \) as a \([-1,1]\)-preserving piecewise monotone map to obtain
\[ \zeta(t) = \frac{1}{D_Q(t)D_R(t)D_P(t)} \]
where \( Q \) and \( R \) are the transition matrices of \( f \) defined in (21) and (22).

If in particular the orbit of 0 is finite, Corollary 25 gives an alternative way for computing \( \zeta(t) \). For \( b = \frac{\sqrt{5} + 1}{2} \), the set \( K = \{ f^n(x) = x = -1,0,1 \} \) is finite because \( f^3(0) = 0 \). So, we can consider the oriented tree \((-1,1),K\).

Regarding \( f \) as a \( K\)-piecewise monotone map, we see that \( f \) is Markov with \( D_P(t) = 1 - 2t + t^3 \) and \( D_Q(t) = D_R(t) \), and from Corollary 25
\[ \zeta(t) = \frac{1}{D_P(t)} = \frac{1}{1 - 2t + t^3}. \]

For \( b = \sqrt{2} \), the set \( K \) is also finite (we have \( f^3(0) = f^2(0) \)), but in this case the orbit of 0 is not periodic and therefore \( D_Q(t) \neq D_R(t) \). More precisely we have:
\[ D_P(t) = 1 - t - 2t^2 + 2t^3, \quad D_Q(t) = (1 - t)^2 \quad \text{and} \quad D_R(t) = (1 - t^2) (1 - t), \] and from Corollary 25

\[
\zeta(t) = \frac{D_R(t)}{D_P(t) D_Q(t)} = \frac{(1 - t^2) (1 - t)}{(1 - t - 2t^2 + 2t^3)(1 - t)^2}.
\]

To end this section, we give another consequence of the Theorem 24 which will play a relevant role in the study of the topological entropy. Define

\[
\rho = \max \left\{ |t|^{-1} : D_{(\epsilon, 1) \epsilon (f)}(t) = 0 \right\} \cup \{1\}.
\]

**Corollary 27.** Let \((T, K)\) be an oriented tree and let \(f\) be \(K\)-preserving piecewise monotone map. Then we have

\[
\rho = \max \left\{ \lim_{n \to \infty} \sqrt[n]{N_{f^n}}, 1 \right\}.
\]

5. **Piecewise linear models**

In this section we will study the existence of piecewise linear models for a given piecewise monotone tree map. Baillif and de Carvalho [BC] showed how to construct a piecewise linear model for a given piecewise strictly monotone tree map with positive topological entropy. Our goal here is to show how to obtain, not only one, but all piecewise linear models of a given piecewise monotone tree map.

For that purpose, letting \(T\) be a tree, a continuous semimetric \(d : T \times T \to \mathbb{R}^+\) is called linear if

\[
d(x, z) = d(x, y) + d(y, z),
\]

for all \(x, y \in T\) and \(z \in \langle x, y \rangle\), where \(\langle x, y \rangle\) denotes the convex hull of \(\{x, y\}\). If \(d\) is a linear semimetric on \(T\), a piecewise monotone map \(f : T \to T\) is called \(d\)-piecewise linear with slope \(s(d) \in \mathbb{R}^+\) if

\[
d(f(x), f(y)) = s(d) \ d(x, y),
\]

for all \(x\) and \(y\) lying in a same interval on which \(f\) is injective or constant.

Notice that, for a given piecewise monotone map \(f : I = [a, b] \to I\), a continuous and increasing (not necessarily strictly increasing) map \(\lambda : I \to [0, 1]\) is a semiconjugacy from \(f\) to a piecewise linear map \(F : [0, 1] \to [0, 1]\), with slope \(\pm s\) everywhere, if and only if \(f\) is a \(d\)-piecewise linear map with slope \(s\), where \(d\) is the continuous linear semimetric on \(I\), defined by

\[
d : \ I \times I \to \mathbb{R}^+, \quad (x, y) \to |\lambda(y) - \lambda(x)|.
\]

So, with this terminology, the theorems of Parry and Milnor-Thurston can be restated as follows: if \(f : I \to I\) is a piecewise strictly monotone interval map with positive topological entropy \(\log(s)\), then there exists a continuous and linear semimetric \(d\) on \(I\) such that \(f\) is a \(d\)-piecewise linear map with slope \(s\). Our goal in this section is to extend this result for tree maps.
Let $T$ be a tree and $d : T \times T \to \mathbb{R}^+$ a linear semimetric on $T$. If $g : T \to T$ is a piecewise monotone map, we define the total variation of $g$, with respect to $d$, by:

$$
Var_d (g) = \sup_P \left\{ \sum_{i=1}^{k} d (g(x_i), g(y_i)) \right\},
$$

where $P = \{[x_1, y_1], ..., [x_k, y_k]\}$ denotes an arbitrary partition of $T$. So for each piecewise monotone map $f : T \to T$, and each compact and connected set $X \subseteq T$ we can define the formal power series

$$
V (d; X; t) \overset{def}{=} \sum_{n \geq 0} Var_d (f^n | X) t^{n+1}.
$$

These formal power series are holomorphic on $\{ t \in \mathbb{C} : |t| < s(d)^{-1} \}$, with

$$
s(d) = \lim_{n \to \infty} \sqrt[n]{Var_d (f^n)},
$$

and the next result shows that they are meromorphic on the unit disk $\mathbb{D}$ (compare with [Pr]).

**Theorem 28.** Let $(T, K)$ be an oriented tree, let $f$ be a $K$-piecewise monotone tree map, and let $d$ be a linear semimetric on $T$. Then, for any compact and connected set $X \subseteq T$, the function $V (d; X; t)$ has a meromorphic extension to the unit disk whose poles are contained in the set of zeros of $D(\epsilon_{\#_0 (f)}, \epsilon_{\#_1 (f)}) (t)$. Furthermore, if $s(d) > 1$, then $V (d; T; t)$ has a pole at $s(d)^{-1}$.

Notice that, if $s(d) > 1$, Theorem 28 shows that the function

$$
\frac{V (d; X; t)}{V (d; T; t)}
$$

is meromorphic on $\mathbb{D}$, and (since $V (d; X; t) \leq V (d; T; t)$, for all $t \in [0, s(d)^{-1}]$) has a removable singularity at $s(d)^{-1}$. Thus we can construct a new linear semimetric $\Lambda (d) : T \times T \to [0, 1]$, defined by

$$
\Lambda (d)(x, y) = \lim_{t \to s(d)^{-1}} \frac{V (d; (x, y); t)}{V (d; T; t)}, \text{ for all } (x, y) \in T \times T.
$$

where $\langle x, y \rangle$ denotes the convex hull of $\{x, y\}$. We are now in position to prove the main theorem of this section.

**Theorem 29.** Let $f : T \to T$ be a piecewise monotone tree map and let $d$ be a linear semimetric on $T$ such that $s(d) > 1$. Then $\Lambda (d)$ is a continuous and linear semimetric on $T$ such that $f$ is $\Lambda (d)$-piecewise linear with slope $s(d)$.

**Proof 30.** Assume that $d$ is a linear semimetric on $T$ such that $s(d) > 1$. We have

$$
\begin{align*}
V (d; \langle x, y \rangle; t) &= \sum_{n \geq 0} Var_d \left( f^n | \langle x, y \rangle \right) t^{n+1} \\
&= d (x, y) t + \sum_{n \geq 0} Var_d \left( f^{n-1} | \langle f(x), f(y) \rangle \right) t^{n} \\
&= d (x, y) t + tV (d; \langle f(x), f(y) \rangle; t),
\end{align*}
$$

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for all $x$ and $y$ lying in a same interval on which $f$ is injective or constant, but from Theorem 28 we know that $V(d; T; t)$ has a pole at $s(d)^{-1}$ and therefore

$$\Lambda(d)(x, y) = \lim_{t \to s(d)^{-1}} \frac{V(d; (x, y); t)}{V(d; T; t)} = s(d)^{-1} \Lambda(d)(f(x), f(y)).$$

This shows that $f$ is a $\Lambda(d)$-piecewise linear map, with slope $s(d)$, since $f$ is piecewise monotone map and $s(d) > 1$, it follows that $\Lambda(d)$ is continuous on $T \times T$. □

**Remark 31.** Recall that Misiurewicz and Szlenk proved in [MSz] that the topological entropy of a piecewise strictly monotone interval map $f$ coincides with the logarithm of

$$\max \left\{ \lim_n \sqrt[n]{\text{Var}(f^n)}, 1 \right\}.$$

Assume now that $f : T \to T$ is a piecewise strictly monotone tree map with positive topological entropy $\log(s)$. If $d$ is an arbitrary linear metric on $T$, the same arguments can be used to show that

$$s = \lim_n \sqrt[n]{\text{Var}_d(f^n)},$$

and from Theorem 29 we can conclude that $f$ is $\Lambda(d)$-piecewise linear map with slope $s$.

But Theorem 29 also shows that, for the same map $f$, other semiconjugacies may exist. As an example, if $f$ is Markov, we can use the eigenvalues of the transition matrix $P$ (19) to construct other semiconjugacies. Let $T_1 = [a_1, b_1], \ldots, T_m = [a_m, b_m]$ be the edges of the oriented tree $(T, K)$, and suppose that $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_m)^T$ (with $\alpha_1 \geq 0, \ldots, \alpha_m \geq 0$) is an eigenvector of $P^T$, associated to an eigenvalue $r > 1$. If $d_{\overrightarrow{\alpha}} : T \times T \to \mathbb{R}^+$ is a linear semimetric such that $d(a_i, b_i) = \alpha_i$, for $i = 1, \ldots, m$, then we have

$$\text{Var}_{d_{\overrightarrow{\alpha}}}(f^n) = (\alpha_1 \ldots \alpha_m) P^n (1 \ldots 1)^T = r^n (\alpha_1 + \ldots + \alpha_m)$$

and $s(d_{\overrightarrow{\alpha}}) = r > 1$, and from Theorem 29 we conclude that $f$ is a $\Lambda(d_{\overrightarrow{\alpha}})$-piecewise linear map with slope $r$ (compare with [ALM]).

Finally, notice that any piecewise linear model of a tree map with $s(d) > 1$ can be obtained (less a multiplicative constant) through $\Lambda$. Indeed if $f : T \to T$ is a $d$-piecewise linear map with slope $s(d) > 1$, then we check easily $\Lambda(d) = \text{Var}_d(id_T)^{-1}.d$.

### 5.1. Total variations (Proof of Theorem 28)

Let $d$ be a linear semimetric on $T$ and let $f : T \to T$ be piecewise monotone map. We have seen in the previous section the importance of the pair $(\epsilon_{\#0}(f), \epsilon_{\#1}(f))$ in the computation of the number of fixed points of $f^n$. To prove Theorem 28 we will show that the same pair of linear endomorphisms plays an equally important role in the computation of the total variation of $f^n$ with respect to $d$. We begin to introduce the linear form $\xi_d$. Since $\mathcal{I}$ (see (4)) generates $\mathcal{S}_1(T, \mathbb{R})$, we can define this linear form as follows:
Definition 32. Let \((T, K)\) be an oriented tree and let \(d\) be a linear semimetric on \(T\). Define \(\xi_d\) to be the unique linear form on \(S_1(T, \mathbb{R})\) that verifies

\[
\xi_d(y - x) = d(x, y), \quad \text{for all } y - x \in \mathcal{I}.
\]

Let \((T, K)\) be an oriented tree. If \(g : T \to T\) is a piecewise monotone map and \(d\) is a linear semimetric \(d\) on \(T\), we can construct another linear semimetric \(d_g\) as follows:

\[
d_g(x, y) \stackrel{\text{def}}{=} Var_d(g |_{(x, y)}),
\]

for all \((x, y) \in T \times T\), and in this way we can consider the linear form \(\xi_{d_g}\). Observe that, if \([x, y]\) is a monotonicity interval of \(g\), then we have

\[
\xi_{d_g}(y - x) = d_g(x, y) = Var_d(g |_{(x, y)}) = d(g(x), g(y)) = \xi_d(\epsilon_{\#1}(g)(y - x)),
\]

and, since \(\mathcal{I}_g\) generates \(S_1(T, \mathbb{R})\) (see (6)), we obtain

\[
\xi_{d_g} = \xi_d \circ \epsilon_{\#1}(g).
\]

So, for a given piecewise monotone map \(f : T \to T\), we conclude from (10) that

\[
\xi_d \circ \epsilon_{\#1}(f)^n(\overline{X}) = \xi_d \circ \epsilon_{\#1}(f^n)(\overline{X}) = \xi_{d,f^n}(\overline{X}) = Var_d(f^n |_{X}),
\]

for all \(n \geq 0\), where \(X\) denotes an arbitrary compact and connected subset of \(T\) and \(\overline{X}\) denotes the corresponding vector of \(S_1(T, \mathbb{R})\) (see (3)). These equalities allows us to establish an important relationship between the determinant of the pair \((\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \epsilon_d \otimes \overline{X})\) and the formal power series \(V(d; X; t)\).

Indeed, since \(\epsilon_{\#1}(f)\) and \(\epsilon_{\#1}(f) + \epsilon_d \otimes \overline{X}\) are defined both in \(S_1(T, \mathbb{R})\), and \(\epsilon_{\#1}(f) + \epsilon_d \otimes \overline{X} - \epsilon_{\#1}(f) = \epsilon_d \otimes \overline{X}\), it follows immediately that the pair \((\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \epsilon_d \otimes \overline{X})\) has finite rank, and from Proposition 46

\[
D(\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \epsilon_d \otimes \overline{X})(t) = 1 - \sum_{n \geq 0} \xi_d \circ \epsilon_{\#1}(f)^n(\overline{X}) t^{n+1} = 1 - \sum_{n \geq 0} Var_d(f^n |_{X}) t^{n+1} = 1 - V(d; X; t).
\]

Observe also that we can use the determinant of the pair \((\epsilon_{\#0}(f), \epsilon_{\#1}(f))\) to decompose \(D(\epsilon_{\#1}(f), \epsilon_{\#1}(f) + \epsilon_d \otimes \overline{X})(t)\). In fact, if \(\overline{\xi}_d\) denotes an extension of \(\xi_d\) to \(S_0(T, \mathbb{R})\), then we have a commutative diagram with exact arrows

\[
\begin{array}{ccccccccc}
0 & \to & S_1(T, \mathbb{R}) & \to & S_0(T, \mathbb{R}) & \to & \frac{S_0(T, \mathbb{R})}{S_1(T, \mathbb{R})} & \to & 0 \\
\downarrow \varphi & & \downarrow \psi & & & & \downarrow 0 & & \\
0 & \to & S_1(T, \mathbb{R}) & \to & S_0(T, \mathbb{R}) & \to & \frac{S_0(T, \mathbb{R})}{S_1(T, \mathbb{R})} & \to & 0
\end{array}
\]
where \( \varphi = (\epsilon_{#1}(f) + \xi_d \otimes \overline{X})^n - \epsilon_{#1}(f)^n \), and \( \psi = (\theta_1 + \overline{\xi_d} \otimes \overline{X})^n - \theta_1^n \). Consequently
\[
Tr \left( (\epsilon_{#1}(f) + \xi_d \otimes \overline{X})^n - \epsilon_{#1}(f)^n \right) = Tr \left( (\theta_1 + \overline{\xi_d} \otimes \overline{X})^n - \theta_1^n \right)
\]
that is
\[
Tr \left( \epsilon_{#1}(f)^n, \left( \epsilon_{#1}(f) + \xi_d \otimes \overline{X} \right)^n \right) = Tr \left( \epsilon_{#0}(f)^n, \left( \epsilon_{#1}(f) + \xi_d \otimes \overline{X} \right)^n \right) - Tr \left( \epsilon_{#0}(f)^n, \epsilon_{#1}(f)^n \right),
\]
and thus
\[
D_{(\epsilon_{#1}(f), \epsilon_{#1}(f) + \xi_d \otimes \overline{X})}(t) = D_{(\epsilon_{#0}(f), \epsilon_{#1}(f) + \xi_d \otimes \overline{X})}(t)D_{(\epsilon_{#0}(f), \epsilon_{#1}(f))}(t)^{-1}.
\]
So, by (26), we obtain (compare with [Pr]):

**Theorem 33.** Let \((T, K)\) be an oriented tree, and let \(d\) be a linear semimetric on \(T\). If \(f\) is a \(K\)-piecewise monotone map and \(X\) is compact and connected subset of \(T\), then
\[
V(d; X; t) = 1 - \frac{D_{(\epsilon_{#0}(f), \epsilon_{#1}(f) + \xi_d \otimes \overline{X})}(t)}{D_{(\epsilon_{#0}(f), \epsilon_{#1}(f))}(t)}.
\]

Notice that, we can compute \(D_{(\epsilon_{#0}(f), \epsilon_{#1}(f) + \xi_d \otimes \overline{X})}(t)\) if we know an extension, \(\overline{\xi_d}\), of \(\xi_d\) to \(S_0(T, \mathbb{R})\). Indeed, Proposition 46 shows that
\[
D_{(\epsilon_{#0}(f), \epsilon_{#1}(f) + \xi_d \otimes \overline{X})}(t) = D_{(\theta_0, \theta_1 + \xi_d \otimes X)}(t) = \text{Det}(id - tN(t)),
\]
where \(N(t)\) is the matrix, with entries in \(\mathbb{R}[[t]]\), defined by:
\[
\begin{bmatrix}
\sum_{n \geq 0} \omega_{x_1} (\theta^n_0 (y_1)) t^n & \ldots & \sum_{n \geq 0} \omega_{x_1} (\theta^n_0 (y_1)) t^n & \ldots & \sum_{n \geq 0} \omega_{x_1} \left( \theta^n_0 \left( \overline{X} \right) \right) t^n \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\sum_{n \geq 0} \omega_{x_1} (\theta^n_0 (y_1)) t^n & \ldots & \sum_{n \geq 0} \omega_{x_1} (\theta^n_0 (y_1)) t^n & \ldots & \sum_{n \geq 0} \omega_{x_1} \left( \theta^n_0 \left( \overline{X} \right) \right) t^n \\
\sum_{n \geq 0} \overline{\xi_d} (\theta^n_0 (y_1)) t^n & \ldots & \sum_{n \geq 0} \overline{\xi_d} (\theta^n_0 (y_1)) t^n & \ldots & \sum_{n \geq 0} \overline{\xi_d} \left( \theta^n_0 \left( \overline{X} \right) \right) t^n
\end{bmatrix},
\]
with \(\{x_1, \ldots, x_l\} = C_f, \ y_1 = f_{#0}(x_1), \ldots, \ y_l = f_{#0}(x_l)\).

Finally, we are in position to prove Theorem 28. Recall that Proposition 13 shows that the function \(D_{(\epsilon_{#0}(f), \epsilon_{#1}(f))}(t)\) is holomorphic on \(\mathbb{D} = \{ t \in \mathbb{C} : |t| < 1 \}\). The same argument can be used to prove that the function \(D_{(\epsilon_{#0}(f), \epsilon_{#1}(f) + \xi_d \otimes \overline{X})}(t)\) is also holomorphic on \(\mathbb{D}\). Indeed, it is easy to see that all entries of \(N(t)\) are holomorphic on \(\mathbb{D}\). Thus, as an immediate consequence of Theorem 33, we conclude...
that \( V(d; X; t) \) has a meromorphic extension to \( \mathbb{D} \) whose poles are contained in the set of zeros of \( D_{(\epsilon_{\neq 0}(f), \epsilon_{\neq 1}(f))}(t) \).

Assume now that \( s(d) > 1 \). In this case, since the radius of convergence of \( V(d; T; t) \) is precisely \( s(d) \), the meromorphic function \( V(d; T; t) \) has a pole in \( \{ t \in \mathbb{C} : |t| = s(d)^{-1} \} \). Since the coefficients of the formal power series \( V(d; T; t) \) are positives, a standard argument (see [MSt, MST]) shows that \( s(d)^{-1} \) is a pole of \( V(d; T; t) \).

6. Topological entropy

The aim of this section is to extend for tree maps an important relationship between topological entropy and the growth rate of the sequence \( N_{f^n} \), introduced in Section 2. Denoting the topological entropy of a piecewise monotone tree map \( f \) by \( h_t(f) \), the main theorem of this section can be stated as follows:

**Theorem 34.** Let \((T, K)\) be a pointed tree and let \( f : T \to T \) be a \( K \)-preserving piecewise strictly monotone map. Then we have

\[
 h_t(f) = \log \left( \max \left\{ 1, \lim_{n \to \infty} \sqrt[n]{N_{f^n}} \right\} \right).
\]

It is known that, for Markov maps, the topological entropy of \( f \) coincides with the logarithm of the spectral radius of its transition matrix \( P \). Note that, from Theorem 11, a complex number \( t^{-1} \) is an eigenvalue of \( P \) if and only if \( D_{(\epsilon_{\neq 0}(f), \epsilon_{\neq 1}(f))}(t) = 0 \), therefore the spectral radius of \( P \) coincides with

\[
 \max \left\{ |t|^{-1} : D_{(\epsilon_{\neq 0}(f), \epsilon_{\neq 1}(f))}(t) = 0 \right\}
\]

and

\[
 h_t(f) = \log \left( \max \left\{ |t|^{-1} : D_{(\epsilon_{\neq 0}(f), \epsilon_{\neq 1}(f))}(t) = 0 \right\} \right). \tag{27}
\]

Theorem 34 is an immediate consequence of the next result and Corollary 27. This theorem also shows that equality (27) holds for any piecewise monotone tree map.

**Theorem 35.** Let \((T, K)\) be an oriented tree and let \( f \) be a \( K \)-preserving piecewise strictly monotone map. Then we have \( h_t(f) = \log(\rho) \), where \( \rho \) is the number defined in (25).

**Remark 36.** Similar results to Theorems 34 and 35 can also be provided for \( K \)-piecewise strictly monotone maps (see [AI 99]).

In order to prove Theorem 35 we have to recall a well-known relationship between topological entropy and the growth rate of lap numbers due to Misiurewicz, Szlenk and Ziemian, see [MSz] and [MZ]. Let \((T, K)\) be an oriented tree with \( m \) edges and let \( f \) be a \( K \)-piecewise strictly monotone map. Define the lap number \( \ell(f) \) to be the number of maximal intervals of monotonicity of \( f \). In other words

\[
 \ell(f) = m + \# \{ x \in T \setminus K : \epsilon_{f,0}(x) = 0 \}.
\]
So we can define the sequence \( \ell (f^n) \) and the corresponding formal power series

\[
L(t) \overset{\text{def}}{=} \sum_{n \geq 1} \ell (f^n) t^n.
\]

Note that, if \( f \) and \( g \) are piecewise strictly monotone tree maps, then

\[
\ell (f \circ g) \leq \ell (f) \ell (g),
\]

and therefore

\[
\ell (f^n + m) \leq \ell (f^n) \ell (f^m)
\]

for all \( n, m \geq 0 \). Using these inequalities, a standard argument proves that \( \sqrt[\ell(n)]{f^n} \) converges to a limit \( s \). For interval maps, Misiurewicz and Szlenk proved in \([\text{MSz}]\) that the topological entropy of \( f \) coincides with the logarithm of \( s \). For tree maps the importance of the number

\[
s \overset{\text{def}}{=} \lim_{n} \sqrt[\ell(n)]{f^n}
\]

is the same. Indeed, the same arguments can be easily adapted to show that

\[
h_t(f) = \log(s)
\]

holds for piecewise strictly monotone tree maps.

6.1. *Laps and total variations (Proof of Theorem 30)* In order to prove Theorem 35 we will establish a relationship between \( L(t) \) and the formal power series \( V(d; T; t) \) of the previous section.

**Definition 37.** For each \( c \in T \), define the linear semimetric \( d_c : T \times T \to \mathbb{R}^+ \) by

\[
d_c(x, y) = \# ((x, y) \cap \{c\}) - \# (\{x\} \cap \{c\}) + \# (\{y\} \cap \{c\}) / 2.
\]

Note that, if \( g : T \to T \) is a piecewise strictly monotone map and \( c \in T \), then we have

\[
\text{Var}_{d_c}(g) = \sum_{x \in g^{-1}(c)} \text{val}(x)/2.
\]

Consider now an oriented tree \((T, K)\) and let \( f : T \to T \) be a \( K \)-preserving piecewise strictly monotone map. For each \( c \in \mathcal{C}_f \overset{\text{def}}{=} C_f \setminus K \), define the sequence \( \gamma_n(c) \), by:

\[
\gamma_0(c) = 1; \quad \gamma_{n+1}(c) = \begin{cases} 0 & \text{if } f^{n+1}(c) \in \mathcal{C}_f, \\ \gamma_n(c) & \text{if } f^{n+1}(c) \notin \mathcal{C}_f, \end{cases}
\]

for all \( n \geq 0 \), and the corresponding formal power series:

\[
\gamma_c(t) \overset{\text{def}}{=} \sum_{n \geq 0} \gamma_n(c) t^n.
\]

Observe that, since \( f(K) \subseteq K \) and \( c \notin K \), we see from (30) that

\[
V(d_c; T; t) \overset{\text{def}}{=} \sum_{n \geq 0} \text{Var}_{d_c}(f^n) t^{n+1} = \sum_{n \geq 0} \# \{x \in T : f^n(x) = c\} t^{n+1},
\]

for all \( c \in \mathcal{C}_f \).
Proposition 38. Let \((T, K)\) be an oriented tree with \(m\) edges and let \(f\) be a \(K\)-preserving piecewise strictly monotone map. Then we have:

\[
L(t) = \frac{mt}{1-t} + \sum_{c \in C} \gamma_c(t) V(d_c; t).
\]

Proof 39. From definition of \(\ell(f^n)\) we have:

\[
\ell(f^n) = m + \# \{ x \in T \setminus K : \varepsilon_{f^n,0}(x) = 0 \}.
\]

Since \(f(K) \subseteq K\), we have

\[
\varepsilon_{f^n,0}(x) = \varepsilon_{f,0}(x) \varepsilon_{f,0}(f(x)) \cdots \varepsilon_{f,0}(f^{n-1}(x)),
\]

for all \(x \in T \setminus K\). Thus, for each \(x \in T \setminus K\), we have: \(\varepsilon_{f^n,0}(x) = 0\) if and only if \(\{x, f(x), \ldots, f^{n-1}(x)\} \cap \hat{C} \neq \emptyset\), therefore

\[
\ell(f^n) = m + \# \left( \bigcup_{c \in C} \bigcup_{i=0}^{n-1} \{ x \in T \setminus K : f^i(x) = c \} \right)
\]

\[
= m + \sum_{c \in C} \sum_{i=0}^{n-1} \gamma_{n-1-i}(c) \# \{ x \in T : f^i(x) = c \},
\]

for all \(n \geq 1\), and from (32) we obtain

\[
L(t) = \sum_{n \geq 1} \ell(f^n) t^n = \frac{mt}{1-t} + \sum_{c \in C} \gamma_c(t) V(d_c; t),
\]

as desired. \(\square\)

Observe that, since the functions \(\gamma_c(t)\) are holomorphic in the unit disk \(D\), the following result is an immediate consequence of Theorem 28 and Proposition 38.

Corollary 40. Let \((T, K)\) be an oriented tree and let \(f\) be a \(K\)-preserving piecewise strictly monotone map. Then the function \(L(t)\) has a meromorphic extension to the unit disk whose poles are contained in the set of zeros of \(D_{\varepsilon_{f,0}(f), \varepsilon_{f,1}(f))}(t)\).

Finally we are in position to prove Theorem 35. From (29) it suffices to show that \(\rho = s\). Since in each lap of \(f^n\) there is at most one fixed point of \(f^n\) of negative type, we conclude that \(\ell(f^n) \geq N_f^n\), for all \(n \geq 1\), and from Corollary 27 we obtain \(s \geq \rho\).

Assume now that \(s > 1\). In this case, since the radius of convergence of \(L(t)\) is precisely \(s^{-1}\), we conclude from Corollary 40 that \(D_{\varepsilon_{f,0}(f), \varepsilon_{f,1}(f))}(t)\) has a zero in \(\{ t \in \mathbb{C} : |t| = s^{-1} \}\), and from (25) we obtain \(\rho \geq s\) as desired.
A. Traces and determinants of pairs of linear endomorphisms

Let $U$ be a vector space over $\mathbb{R}$ and let $\psi : U \to U$ be a linear map with finite rank. As usually we define the trace of $\psi$ by

$$\text{Tr}(\psi) \overset{\text{def}}{=} \text{Tr}(\psi_{|V}),$$

where $V$ denotes an arbitrary finite-dimensional subspace of $U$ such that $\text{Im}(\psi) \subseteq V$ (it is easy to see that the definition does not depend on $V$). If $\psi$ has finite rank, then there are vectors $u_1, \ldots, u_k \in U$ and linear forms $\omega_1, \ldots, \omega_k \in U^*$ such that

$$\psi = \sum_{i=1}^{k} \omega_i \otimes u_i.$$

Considering the matrix

$$M = \begin{bmatrix} \omega_1(u_1) & \ldots & \omega_1(u_k) \\ \ldots & \ldots & \ldots \\ \omega_k(u_1) & \ldots & \omega_k(u_k) \end{bmatrix}, \quad (33)$$

we have

$$\text{Tr}(\psi) = \text{Tr}(M).$$

More generally, if $\psi$ has finite rank, then, for each $n \in N_1$, $\psi^n$ has finite rank and

$$\text{Tr}(\psi^n) = \text{Tr}(M^n).$$

The following proposition is well known and gives an explicit method for computing the numbers $\text{Tr}(\psi^n)$, with $n \in N_1$.

**Proposition 41.** Let $\psi$ be an endomorphism with finite rank, and define the determinant of $\psi$ to be the following element of $\mathbb{R}[[t]]$

$$D_\psi(t) \overset{\text{def}}{=} \exp \sum_{n \geq 1} -\frac{t^n}{n} \text{Tr}(\psi^n).$$

Then we have

$$D_\psi(t) = \text{Det}(id - tM_\psi).$$

Now we consider a more general situation. By a pair of endomorphisms $(\varphi, \psi)$, we mean two subspaces $V$ and $W$ of the same $\mathbb{R}$-vector space $U$, and two linear maps $\varphi : V \to V$ and $\psi : W \to W$.

**Definition 42.** We say that the pair of endomorphisms $(\varphi, \psi)$ has finite rank if there exist linear maps $\tilde{\varphi}$, $\tilde{\psi}$, $\check{\varphi}$ and $\check{\psi}$ such that the diagrams with exact rows

$$\begin{array}{cccc}
0 & \to & V & \leftrightarrow & V + W & \to & V + W & \to & 0 \\
\downarrow \varphi & & \downarrow \tilde{\varphi} & & \downarrow \check{\varphi} & & V + W \\
0 & \to & V & \leftrightarrow & V + W & \to & V + W & \to & 0 \\
\end{array}$$

and

$$\begin{array}{cccc}
0 & \to & W & \leftrightarrow & V + W & \to & V + W & \to & 0 \\
\downarrow \psi & & \downarrow \tilde{\psi} & & \downarrow \check{\psi} & & V + W \\
0 & \to & W & \leftrightarrow & V + W & \to & V + W & \to & 0 \\
\end{array}$$

are commutative, and the linear maps $\tilde{\psi} - \tilde{\varphi}$, $\check{\psi}$ and $\check{\varphi}$ have finite rank.
For any pair of endomorphisms \((\varphi, \psi)\) with finite rank, it is possible to define its
trace as follows: considering linear maps \(\tilde{\varphi}, \tilde{\varphi}, \tilde{\psi}\) and \(\tilde{\psi}\) as in the previous definition, we define
\[
Tr(\varphi, \psi) = Tr(\tilde{\psi} - \tilde{\varphi}) - Tr(\tilde{\psi}) + Tr(\tilde{\varphi})
\]
(it is easy to see that the definition does not depend on \(\tilde{\varphi}, \tilde{\varphi}, \tilde{\psi}\) and \(\tilde{\psi}\)).

Observe that an endomorphism \(\psi : U \to U\) with finite rank can be regarded
as a pair of endomorphisms with finite rank. Considering the pair \((0, \psi)\), where
\(0 : U \to U\) denotes the zero map, we see that \(\psi\) has finite rank if and only if the
pair \((0, \psi)\) has finite rank, and \(Tr(0, \psi) = Tr(\psi)\). More generally, if the linear
maps \(\varphi\) and \(\psi\) both have finite ranks, then the pair \((\varphi, \psi)\) has also finite rank and
\[
Tr(\varphi, \psi) = Tr(\psi) - Tr(\varphi).
\]

Of course, in general the single traces in the previous formula are not defined. The
following proposition follows from the definition and will be useful for computing
\(Tr(\varphi, \psi)\) in the general case.

**Proposition 43.** A pair of linear endomorphisms \((\varphi, \psi)\) has finite rank if and
only if there exist extensions \(\tilde{\varphi} : V + W \to V + W\) and \(\tilde{\psi} : V + W \to V + W\) of \(\varphi\)
and \(\psi\), respectively, verifying: \(\text{Im}(\tilde{\varphi}) \subseteq V\), \(\text{Im}(\tilde{\psi}) \subseteq W\) and \(\tilde{\psi} - \tilde{\varphi}\) has finite rank.
Furthermore, if \((\varphi, \psi)\) has finite rank, then \(Tr(\varphi, \psi) = Tr(\tilde{\psi} - \tilde{\varphi})\).

Let \(\psi : U \to U\) be an endomorphism with finite rank and let \(U_1, \ldots, U_m\) be
subspaces of \(U\) such that
\[
U = \bigoplus_{i=1}^m U_i.
\]
Considering the projections \(p_{U_i} : U \to U_i\), and the injections \(i_{U_i} : U_i \to U\), it is clear that, for each \(i = 1, \ldots, m\), the endomorphism
\[
p_{U_i} \circ \psi \circ i_{U_i} : U_i \to U_i
\]
has finite rank and
\[
Tr(\psi) = \sum_{i=1}^m Tr(p_{U_i} \circ \psi \circ i_{U_i}). \tag{34}
\]

The next proposition can be regarded as a generalization of (34). Let \((\varphi, \psi)\) be a pair of endomorphisms with finite rank, and let \(V_1, \ldots, V_m\) and \(W_1, \ldots, W_m\) be
subspaces of \(V\) and \(W\) respectively. Assume that
\[
V = \bigoplus_{i=1}^m V_i \quad \text{and} \quad W = \bigoplus_{i=1}^m W_i. \tag{35}
\]
Considering the projections
\[
p_{V_i} : V \to V_i, \quad p_{W_i} : W \to W_i
\]
and the injections
\[
i_{V_i} : V_i \to V \quad \text{and} \quad i_{W_i} : W_i \to W,
\]

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we obtain, for each \(i = 1, \ldots, m\), the pair of endomorphisms
\[
(p_{V_i} \circ \varphi \circ i_{V_i}, p_{W_i} \circ \psi \circ i_{W_i}).
\]
The rank of this pair may be not finite, however, if
\[
(V_i + W_i) \cap \sum_{j \neq i} (V_j + W_j)
\]is finite dimensional for each \(i = 1, \ldots, m\), then we have the following:

**Proposition 44.** Let \((\varphi, \psi)\) be a pair of endomorphisms with finite rank, and let \(V_1, \ldots, V_m\) and \(W_1, \ldots, W_m\) be subspaces of \(V\) and \(W\) respectively, verifying (35) and (36). Then, for each \(i = 1, \ldots, m\), the pair \((p_{V_i} \circ \varphi \circ i_{V_i}, p_{W_i} \circ \psi \circ i_{W_i})\) has finite rank and
\[
\text{Tr} (\varphi, \psi) = \sum_{i=1}^{m} \text{Tr} (p_{V_i} \circ \varphi \circ i_{V_i}, p_{W_i} \circ \psi \circ i_{W_i}).
\]

Let \((\varphi, \psi)\) be a pair of endomorphisms having finite rank, and consider endomorphisms \(\tilde{\varphi}\) and \(\tilde{\psi}\) as in Proposition 43. Since \(\tilde{\psi} - \tilde{\varphi}\) has finite rank, we can consider vectors \(u_1, \ldots, u_k \in V + W\) and linear forms \(\omega_1, \ldots, \omega_k \in (V + W)^*\) such that
\[
\tilde{\psi} - \tilde{\varphi} = \sum_{i=1}^{k} \omega_i \otimes u_i.
\]
More generally, we have
\[
\tilde{\psi}^n - \tilde{\varphi}^n = \sum_{i=1}^{k} \sum_{j=1}^{n} \left( \omega_i \circ \tilde{\psi}^{n-j} \right) \otimes \tilde{\varphi}^{j-1} (u_i),
\]
for each \(n \in \mathbb{N}_1\). This shows that \(\tilde{\psi}^n - \tilde{\varphi}^n\) has finite rank, for each \(n \in \mathbb{N}_1\). Thus, once more from Proposition 43, we conclude that the pair \((\varphi^n, \psi^n)\) has finite rank and
\[
\text{Tr} (\varphi^n, \psi^n) = \text{Tr} (\tilde{\psi}^n - \tilde{\varphi}^n), \text{ for each } n \in \mathbb{N}_1.
\]

**Definition 45.** Let \((\varphi, \psi)\) be a pair of endomorphisms having finite rank. We define the determinant of \((\varphi, \psi)\) to be the following element of \(\mathbb{R}[[t]]\)
\[
D_{(\varphi, \psi)} (t) \overset{\text{def}}{=} \exp \sum_{n \geq 1} - \frac{t^n}{n} \text{Tr} (\varphi^n, \psi^n).
\]
Observe that, if \(\psi\) has finite rank, then
\[
D_{(0, \psi)} (t) = D_{\psi} (t).
\]
If \(\varphi\) and \(\psi\) both have finite ranks, then
\[
D_{(\varphi, \psi)} (t) = D_{\psi} (t) \ D_{\varphi} (t)^{-1}.
\]
So, in these cases, we can use Proposition 41 for computing \(D_{(\varphi, \psi)} (t)\). Obviously, in the general case, Proposition 41 does not allow us to compute \(D_{(\varphi, \psi)} (t)\) (in general \(D_{\varphi} (t)\) and \(D_{\psi} (t)\) are not defined).
In order to compute $D_{(\varphi, \psi)}(t)$, in the general case, we generalize Proposition 41. Let $\varphi$ and $\psi$ be endomorphisms as in Proposition 43. Considering vectors $u_1, \ldots, u_k \in V + W$ and linear forms $\omega_1, \ldots, \omega_k \in (V + W)^*$ as in (37), we define the matrix

$$M(t) = \begin{bmatrix} \sum_{n \geq 0} \omega_1(\varphi^n(u_1)) t^n & \cdots & \sum_{n \geq 0} \omega_k(\varphi^n(u_k)) t^n \\ \vdots & \ddots & \vdots \\ \sum_{n \geq 0} \omega_1(\varphi^n(u_1)) t^n & \cdots & \sum_{n \geq 0} \omega_k(\varphi^n(u_k)) t^n \end{bmatrix}, \quad (38)$$

with coefficients in $\mathbb{R}[[t]]$. Observe that, if we identify an endomorphism with finite rank $\psi : U \to U$ with the corresponding pair of finite rank $(0, \psi)$, then the matrix $M(t)$ of (38) coincides with the matrix $M$ defined in (33). Thus, the next proposition, which gives an explicit method for computing $D_{(\varphi, \psi)}(t)$, can be regarded as a generalization of Proposition 41.

**Proposition 46.** Let $(\varphi, \psi)$ be a pair of endomorphisms having finite rank. Then we have

$$D_{(\varphi, \psi)}(t) = \text{Det}(\text{id} - tM(t)).$$

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**References**


References


