#### Multiple-Conclusion Logics PART 3: "Is that all that there is?"

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Uni-Log 2005

Montreux, CH

#### Introductory (and Motivational) Course



Recall, again, the axioms of T-logics: (C1)  $\Gamma, \beta \Vdash \beta, \Delta$  overlap (C2)  $(\exists \Theta \subseteq S)(\forall \langle \Sigma, \Pi \rangle \in \operatorname{QPart}(\Theta)) \Sigma, \Gamma \Vdash \Delta, \Pi \Rightarrow \Gamma \Vdash \Delta$  full cut (C3)  $\Gamma \Vdash \Delta \Rightarrow \Sigma, \Gamma \Vdash \Delta, \Pi$  dilution

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How can such logics be characterized abstractly?

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Examples:

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- each of Łukasiewicz's  $L_n$ , for  $n \in \mathbb{N}$ , is genuinely *n*-valued
- Lukasiewicz's  $L_{\aleph_0}$  is genuinely  $2^{\aleph_0}$ -valued [Shoesmith & Smiley 1971]

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- Łukasiewicz's  $L_{\aleph_0}$  is genuinely  $2^{\aleph_0}$ -valued [Shoesmith & Smiley 1971]
- Intuitionistic Logic is not genuinely finitely-valued [Gödel 1932]
- most usual Normal Modal Logics are not genuinely finitely-valued [Dugundji 1940]

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But then again, which logics *have* adequate matrix semantics?

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Consider now the following axiom:

(C4)  $[\Gamma_k]_{k\in K} \Vdash [\Delta_k]_{k\in K} \Rightarrow \Gamma_k \Vdash \Delta_k$ , for any  $k \in K$ , whenever  $\{\Gamma_k, \Delta_k\}_{k\in K}$  is a disconnected family of theories cancellation

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#### Or, in a single-conclusion version:

(C4<sup>s</sup>)  $[\Gamma_k]_{k \in K}, \Gamma \Vdash \beta \Rightarrow \Gamma \Vdash \beta$ , s-cancellation whenever  $\Gamma \cup \{\beta\}, [\Gamma_k]_{k \in K}$  are pairwise disconnected, and no  $\Gamma_k$  is  $\mathcal{L}$ -trivializing

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**Theorem**: Suppose |At| = |S|. Then: [Shoesmith & Smiley 1971] A single-conclusion logic  $\mathcal{L}$  has an adequate matrix semantics iff  $\mathcal{L}$  is a substitutional T-logic that respects cancellation [(C4)].

[check also Wójcicki 1969–1970]

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- Which property would play the role of cancellation, *in general*, for multiple-conclusion logics?

# Logics with matrix semantics

**Theorem**: Suppose |At| = |S|. Then: [Shoesmith & Smiley 1971] A single-conclusion logic  $\mathcal{L}$  has an adequate matrix semantics iff  $\mathcal{L}$  is a substitutional T-logic that respects cancellation [(C4)].

Other results, and questions that remain (as far as I know!!):

- The above theorem is also valid for compact multiple-conclusion logics [Shoesmith & Smiley 1978]
- Which property would play the role of cancellation, *in general*, for multiple-conclusion logics?
- How can genuinely finite-valued logics be abstractly characterized? (they are all *compact* and *decidable*, but L<sub>×0</sub> also is...)

Recall that logics with matrix semantics are based on fixed sets  $\mathcal{V}$ ,  $\mathcal{D}$ ,  $\mathcal{U}$ , and a family of mappings Sem = { $\S_k : S \to \mathcal{V}$ } s.t.:

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Some facts and open questions:

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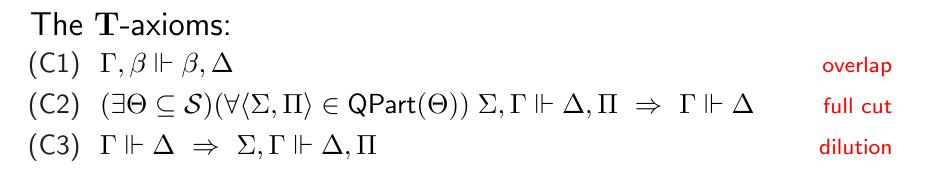
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And now some variations on the T-axioms: (C1)  $\Gamma, \beta \Vdash \beta, \Delta$  overlap (C2)  $(\exists \Theta \subseteq S)(\forall \langle \Sigma, \Pi \rangle \in \mathsf{QPart}(\Theta)) \Sigma, \Gamma \Vdash \Delta, \Pi \Rightarrow \Gamma \Vdash \Delta$  full cut (C3)  $\Gamma \Vdash \Delta \Rightarrow \Sigma, \Gamma \Vdash \Delta, \Pi$  dilution

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SC-interpretations. Similar to tarskian, but: [G. Malinowski 1990s]

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$$\mathcal{D} \mapsto T \qquad \mathcal{R} \mapsto F \qquad \mathcal{U} \setminus \mathcal{R} \mapsto I$$

where I is some sort of 'intermediary logical value'.

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I.J-overcompleteness

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Obviously: 
$$(C0.0.0) \Rightarrow (C0.I.J)$$
  
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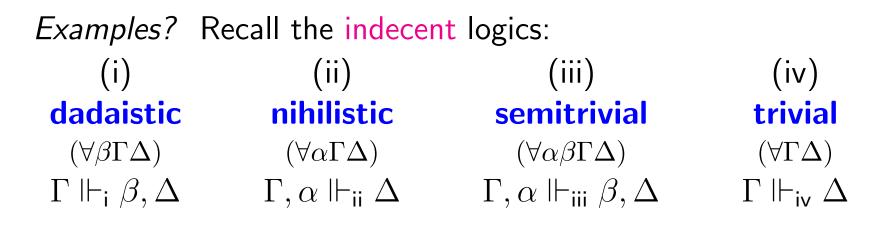
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Examples?

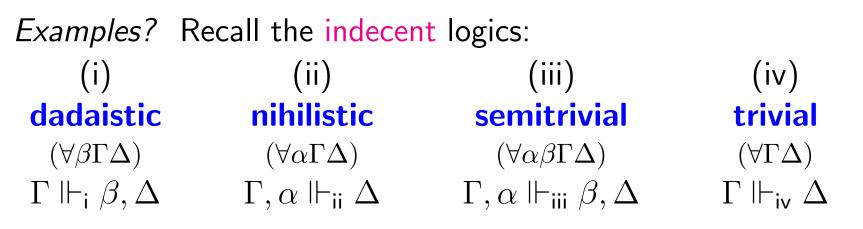
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Observe that (i)–(iv) consist in <u>compact substitutional T-logics</u>.

(C0.I.J)  $(\Gamma, [\alpha_i]_{i \leq I} \Vdash [\beta_j]_{j \leq J}, \Delta)$  I.J-overcompleteness

Obviously: (C0.0.0) 
$$\Rightarrow$$
 (C0.I.J)  
(C0.I.J)  $\Leftrightarrow$  (C0.I+K.J+L), for I, J > 0

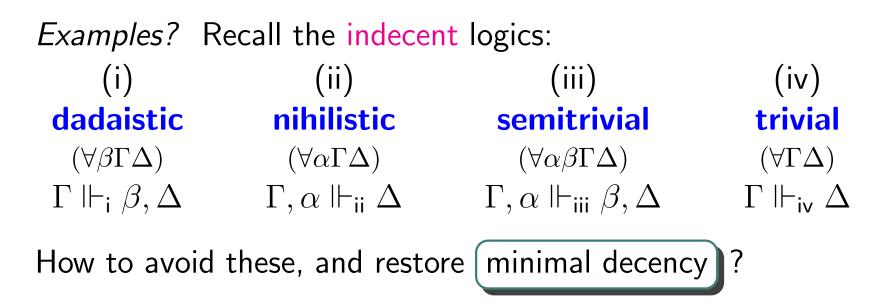
Examples?Recall the indecent logics:(i)(ii)(iii)(iii)(iii)(iv)dadaisticnihilisticsemitrivial $(\forall \beta \Gamma \Delta)$  $(\forall \alpha \Gamma \Delta)$  $(\forall \alpha \beta \Gamma \Delta)$  $(\forall \beta \Gamma \Delta)$  $(\forall \alpha \Gamma \Delta)$  $(\forall \alpha \beta \Gamma \Delta)$  $\Gamma \Vdash_i \beta, \Delta$  $\Gamma, \alpha \Vdash_{ii} \Delta$  $\Gamma, \alpha \Vdash_{iii} \beta, \Delta$ 

Observe that (i)–(iv) consist in <u>compact substitutional T-logics</u>. Note also that (iii) is the only one that does **not** have a <u>matrix semantics</u>.

#### Thou shalt not trivialize!

(C0.I.J)  $(\Gamma, [\alpha_i]_{i \leq I} \Vdash [\beta_j]_{j \leq J}, \Delta)$  I.J-overcompleteness

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Obviously: 
$$(C0.0.0) \Rightarrow (C0.I.J)$$
  
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Examples?Recall the indecent logics:(i)(ii)(iii)(iv)dadaisticnihilisticsemitrivialtrivial $(\forall \beta \Gamma \Delta)$  $(\forall \alpha \Gamma \Delta)$  $(\forall \alpha \beta \Gamma \Delta)$  $(\forall \Gamma \Delta)$  $\Gamma \Vdash_i \beta, \Delta$  $\Gamma, \alpha \Vdash_{ii} \Delta$  $\Gamma, \alpha \Vdash_{iii} \beta, \Delta$  $\Gamma \Vdash_{iv} \Delta$ How to avoid these, and restoreminimal decency?Through:(PNO)  $\neg$ (C0.1.1)Principle of Non-Overcompleteness

Here is a typical subclassical rule of negation:

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 $(\Gamma \Vdash \sim \beta, \beta, \Delta)$  casus judicans

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And here is its **dual**:

*pseudo-scotus*  $(\Gamma, \alpha, \sim \alpha \Vdash \Delta)$ 

Here is a typical subclassical rule of negation:

*ex contradictione sequitur quodlibet* 

 $(\Gamma, \alpha, \sim \alpha \Vdash \beta, \Delta)$ 

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*ex contradictione sequitur quodlibet* 

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 $\begin{array}{ll} \text{And} & \text{Mere} &$ 

No

Notice that *ex contradictione* and *pseudo-scotus* are distinct rules!

Here is a typical subclassical rule of negation:

*ex contradictione sequitur quodlibet* 

$$(\Gamma, \alpha, \sim \alpha \Vdash \beta, \Delta)$$

And here is its **dual**:

$$(\Gamma, \alpha \Vdash \sim \beta, \beta, \Delta)$$

*quodlibet sequitur ad casos* 

Here is a typical subclassical rule of negation:

 $(\Gamma,\beta \Vdash \sim \beta,\Delta) / (\Gamma \Vdash \sim \beta,\Delta)$ 

consequentia mirabilis

And Merelis/its/dual/

 $(\Gamma, \alpha \Vdash \sim \beta, \beta, \Delta)$ 

*quodlibet sequitur ad casos* 

Here is a typical subclassical rule of negation:

 $(\Gamma, \beta \Vdash \sim \beta, \Delta) / (\Gamma \Vdash \sim \beta, \Delta)$  consequentia

mirabilis

#### And here is its **dual**:



*causa mirabilis*  $(\Gamma, \sim \alpha \Vdash \alpha, \Delta) / (\Gamma, \sim \alpha \Vdash \Delta)$ 

Here is a typical subclassical rule of negation:

 $\begin{array}{c} (\Gamma, \beta \Vdash \alpha, \Delta \text{ and } & left \\ \Gamma', \sim \beta \vDash \alpha, \Delta') \ / & redundancy, or \\ (\Gamma', \Gamma \vDash \alpha, \Delta, \Delta') & proof by cases \end{array}$   $\begin{array}{c} \text{And Metric Metric Metric Metric Action } \\ \text{causa} \\ \text{mirabilis} & (\Gamma, \sim \alpha \vDash \alpha, \Delta) \ / \ (\Gamma, \sim \alpha \vDash \Delta) \end{array}$ 

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 $(\Gamma, \beta \Vdash \alpha, \Delta \text{ and } left$   $\Gamma', \sim \beta \Vdash \alpha, \Delta') / redundancy, or$   $(\Gamma', \Gamma \Vdash \alpha, \Delta, \Delta') \text{ proof by cases}$ And here is its dual:  $right \qquad (\Gamma, \beta \vDash \alpha, \Delta \text{ and} \\ \Gamma', \beta \vDash \alpha, \Delta \text{ and} \\ \Gamma', \beta \vDash \alpha, \Delta') / (\Gamma', \Gamma, \beta \vDash \Delta, \Delta')$ 

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**CRR** : 'Consistency-Related Rules'

 $\begin{array}{ll} \textit{dextro-levo} & (\Gamma \Vdash \alpha, \Delta) / (\Gamma, \sim \alpha \Vdash \Delta) \\ \textit{symmetry} & (\Gamma \Vdash \sim \alpha, \Delta) / (\Gamma, \alpha \Vdash \Delta) \\ & (``\alpha \text{ and } \sim \alpha \text{ are not both true''}) \end{array}$ 

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'Determinedness-Related Rules' : DRR  $(\Gamma, \beta \Vdash \Delta) / (\Gamma \Vdash \sim \beta, \Delta)$  levo-dextro  $(\Gamma, \sim \beta \Vdash \Delta) / (\Gamma \Vdash \beta, \Delta)$  symmetry

(" $\alpha$  and  $\sim \alpha$  are not both false")

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'Determinedness-Related Rules' : DRR  $(\Gamma, \beta \Vdash \Delta) / (\Gamma \Vdash \sim \beta, \Delta)$  levo-dextro  $(\Gamma, \sim \beta \Vdash \Delta) / (\Gamma \Vdash \beta, \Delta)$  symmetry

#### Interlude on Paranormality:

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#### Interlude on Paranormality:

Some **CRR** must be failed by paraconsistent logics.

Some **DRR** must be failed by paracomplete logics.

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reductio ad absurdum

 $(\Gamma, \beta \Vdash \alpha, \Delta \text{ and } \Gamma', \beta \Vdash \sim \alpha, \Delta') / (\Gamma', \Gamma \Vdash \sim \beta, \Delta, \Delta')$ 

 $(\Gamma, \sim\!\!\beta \Vdash \alpha, \Delta \text{ and } \Gamma', \sim\!\!\beta \Vdash \sim\!\!\alpha, \Delta') \not \ (\Gamma', \Gamma \Vdash \beta, \Delta, \Delta')$ 

Some more general rules are:

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 $(\Gamma, \beta \Vdash \alpha, \Delta \text{ and } \Gamma', \beta \Vdash \sim \alpha, \Delta') / (\Gamma', \Gamma \Vdash \sim \beta, \Delta, \Delta')$  $(\Gamma, \sim \beta \Vdash \alpha, \Delta \text{ and } \Gamma', \sim \beta \Vdash \sim \alpha, \Delta') / (\Gamma', \Gamma \Vdash \beta, \Delta, \Delta')$ 

And its so far unsuspected **dual**:

reductio ex evidentia

 $\begin{array}{c} (\Gamma, \boldsymbol{\beta} \Vdash \boldsymbol{\alpha}, \Delta \text{ and } \Gamma', \boldsymbol{\sim} \boldsymbol{\beta} \Vdash \boldsymbol{\alpha}, \Delta') \neq (\Gamma', \Gamma, \boldsymbol{\sim} \boldsymbol{\alpha} \Vdash \Delta, \Delta') \\ (\Gamma, \boldsymbol{\beta} \Vdash \boldsymbol{\sim} \boldsymbol{\alpha}, \Delta \text{ and } \Gamma', \boldsymbol{\sim} \boldsymbol{\beta} \Vdash \boldsymbol{\sim} \boldsymbol{\alpha}, \Delta') \neq (\Gamma', \Gamma, \boldsymbol{\alpha} \Vdash \Delta, \Delta') \end{array}$ 

Some more general rules are:

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 $(\Gamma, \beta \Vdash \alpha, \Delta \text{ and } \Gamma', \beta \Vdash \sim \alpha, \Delta') / (\Gamma', \Gamma \Vdash \sim \beta, \Delta, \Delta')$  $(\Gamma, \sim \beta \Vdash \alpha, \Delta \text{ and } \Gamma', \sim \beta \Vdash \sim \alpha, \Delta') / (\Gamma', \Gamma \Vdash \beta, \Delta, \Delta')$ 

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Some have claimed that *reductio* (*ad absurdum*) rules are enough so as to characterize classical negation... [See, e.g., Béziau 1994]

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reductio ad absurdum  $\Rightarrow$  casus judicans  $(\Vdash \sim \beta, \beta)$ 

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Indeed, in **multiple-conclusion**: ad absurdum ( ex evidentia

In particular:

 $\begin{array}{ll} \textit{reductio ad absurdum} \ \Rightarrow \ \textit{casus judicans} & (\Vdash \sim \beta, \beta) \\ & ex \ \textit{contradictione} & (\alpha, \sim \alpha \Vdash \beta) \end{array}$ 

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Indeed, in multiple-conclusion:  $ad \ absurdum \quad \nleftrightarrow \quad ex \ evidentia$ In particular:  $reductio \ ad \ absurdum \quad \Rightarrow \quad casus \ judicans \quad (\Vdash \sim \beta, \beta)$  $ex \ contradictione \quad (\alpha, \sim \alpha \Vdash \beta)$ 

reductio ad absurdum



pseudo-scotus

 $(\alpha, \sim \alpha \Vdash)$ 

There are several other usual rules for negation, such as:

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Double negation introduction

 $(\Gamma, \gamma \Vdash \sim \sim \gamma, \Delta)$ 

Double negation elimination

 $(\Gamma, \mathop{\sim} \mathop{\sim} \gamma \Vdash \gamma, \Delta)$ 

There are several other usual rules for negation, such as:

Double negation manipulation

 $\begin{array}{c} (\Gamma, \gamma \Vdash \delta, \Delta) \ / \ (\Gamma, \sim \sim \gamma \Vdash \sim \sim \delta, \Delta) \\ (\Gamma, \sim \sim \gamma \Vdash \delta, \Delta) \ / \ (\Gamma, \gamma \Vdash \sim \sim \delta, \Delta) \\ (\Gamma, \gamma \Vdash \sim \sim \delta, \Delta) \ / \ (\Gamma, \sim \sim \gamma \Vdash \delta, \Delta) \\ (\Gamma, \sim \sim \gamma \Vdash \sim \sim \delta, \Delta) \ / \ (\Gamma, \gamma \Vdash \delta, \Delta) \end{array}$ 

There are several other usual rules for negation, such as:

(Contextual) Contraposition  $(\Gamma, \gamma \Vdash \delta, \Delta) / (\Gamma, \sim \delta \Vdash \sim \gamma, \Delta)$   $(\Gamma, \sim \gamma \Vdash \delta, \Delta) / (\Gamma, \sim \delta \Vdash \gamma, \Delta)$   $(\Gamma, \gamma \Vdash \sim \delta, \Delta) / (\Gamma, \delta \Vdash \sim \gamma, \Delta)$   $(\Gamma, \sim \gamma \Vdash \sim \delta, \Delta) / (\Gamma, \delta \Vdash \gamma, \Delta)$ 

There are several other usual rules for negation, such as:

Contextual Replacement (for negation)  $(\Gamma, \gamma \dashv \models \delta, \Delta) / (\Gamma, \sim \gamma \dashv \models \sim \delta, \Delta)$   $(\Gamma, \sim \gamma \dashv \models \delta, \Delta) / (\Gamma, \gamma \dashv \models \sim \delta, \Delta)$   $(\Gamma, \gamma \dashv \models \sim \delta, \Delta) / (\Gamma, \sim \gamma \dashv \models \delta, \Delta)$  $(\Gamma, \sim \gamma \dashv \models \sim \delta, \Delta) / (\Gamma, \gamma \dashv \models \delta, \Delta)$ 

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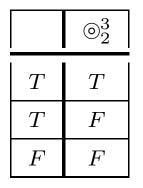
Is there a set of <u>indisputable rules</u> for negation??

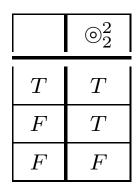
#### A semantic intuition

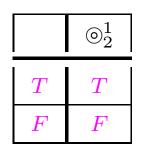
A 'binary print' of negation:

[Béziau 1996]

#### A 'binary print' of negation:

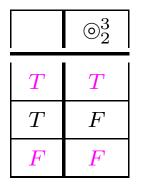


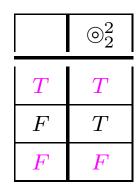


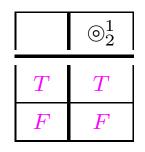


kinds of affirmation

#### A 'binary print' of negation:

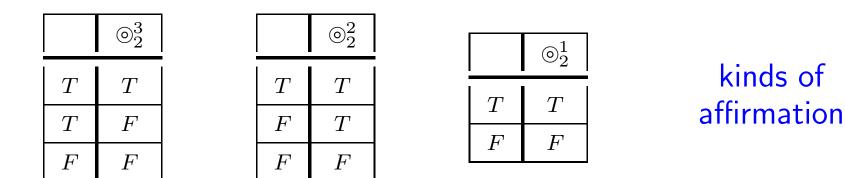


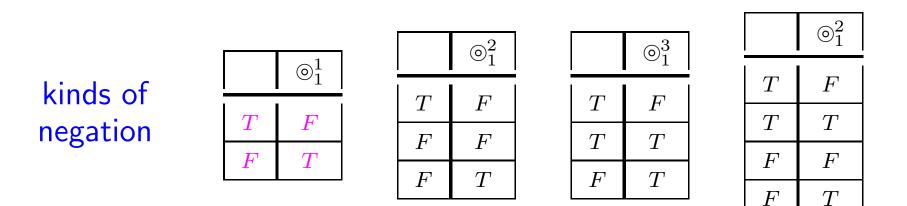




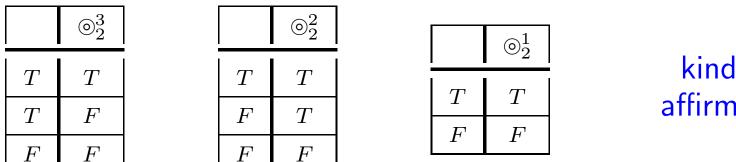
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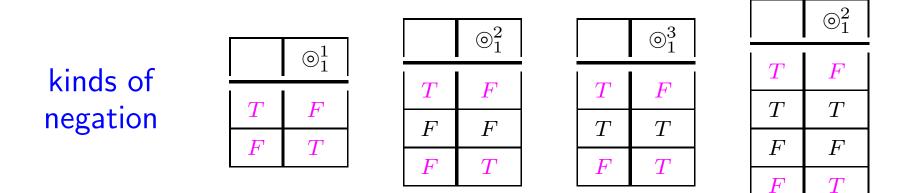




#### A 'binary print' of negation:







Let  $\sim^0 \varphi \stackrel{\text{def}}{=} \varphi$ , and  $\sim^{n+1} \varphi \stackrel{\text{def}}{=} \sim^n \sim \varphi$ .

Let 
$$\sim^0 \varphi \stackrel{\text{def}}{=} \varphi$$
, and  $\sim^{n+1} \varphi \stackrel{\text{def}}{=} \sim^n \sim \varphi$ .

On what negation **is not**:

[Marcos 2005]

Let 
$$\sim^0 \varphi \stackrel{\text{def}}{=} \varphi$$
, and  $\sim^{n+1} \varphi \stackrel{\text{def}}{=} \sim^n \sim \varphi$ .

On what negation is not:

[Marcos 2005]

 $(\mathsf{N1}.m) \quad (\Gamma, \sim^{m+1} \varphi \not\models \sim^m \varphi, \Delta) \quad m\text{-verificatio}$ 

Let 
$$\sim^0 \varphi \stackrel{\text{def}}{=} \varphi$$
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On what negation is not: [Marcos 2005] (N1.m)  $(\Gamma, \sim^{m+1}\varphi \not \vdash \sim^{m}\varphi, \Delta)$  *m-verificatio m-falsificatio*  $(\Gamma, \sim^{m}\varphi \not \vdash \sim^{m+1}\varphi, \Delta)$  (N2.m)

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$$(N1.m) \quad (\Gamma, \sim^{m+1}\varphi \not\models \sim^{m}\varphi, \Delta) \quad m\text{-verificatio}$$

$$m\text{-falsificatio} \quad (\Gamma, \sim^{m}\varphi \not\models \sim^{m+1}\varphi, \Delta) \quad (N2.m)$$

Then:

A **decent negation** should respect (N1.m) and (N2.m), for every m.

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- (*m*-nonbot)  $(\Gamma, \sim^{m+1} \varphi \not\models \Delta)$
- (*m*-nontop)  $(\Gamma \not \vdash \sim^{m+1} \varphi, \Delta)$

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- (*m*-nonbot)  $(\Gamma, \sim^{m+1} \varphi \not\models \Delta)$
- (*m*-nontop)  $(\Gamma \not\models \sim^{m+1} \varphi, \Delta)$
- (*m*-paradoxical inequivalence)  $(\Gamma, \sim^m \varphi \dashv \not \vdash \sim^{m+1} \varphi, \Delta)$

On what negation is not:

[Marcos 2005]

### $(N1.m) | (\Gamma, \sim^{m+1} \varphi \not\vdash \sim^{m} \varphi, \Delta) \quad m\text{-verificatio}$

*m*-falsificatio  $(\Gamma, \sim^m \varphi \not\models \sim^{m+1} \varphi, \Delta)$ 



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A decent negation should respect (N1.m) and (N2.m), for every m.

Let  $\neg$  be classical negation, and let  $\Box$  and  $\Diamond$  come from Normal Modal Logics.

On what negation is not:

[Marcos 2005]

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Let ¬ be classical negation, and let □ and ◊ come from Normal Modal Logics. Here are a few more consequences, involving *paranormality:* 

On what negation is not:

[Marcos 2005]

### $(N1.m) \quad (\Gamma, \sim^{m+1} \varphi \not\vdash \sim^{m} \varphi, \Delta) \quad m\text{-verificatio}$

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Let  $\neg$  be classical negation, and let  $\Box$  and  $\Diamond$  come from Normal Modal Logics. Here are a few more consequences, involving *paranormality:* The operator  $\neg \varphi \stackrel{\text{def}}{=} \Box \neg \varphi$  defines a decent (paracomplete) negation. The operator  $\neg \varphi \stackrel{\text{def}}{=} \Diamond \neg \varphi$  defines a decent (paraconsistent) negation.

On what negation is not:

[Marcos 2005]

(N2.m)

### $(N1.m) | (\Gamma, \sim^{m+1} \varphi \not\vdash \sim^{m} \varphi, \Delta) \quad m\text{-verificatio}$

*m*-falsificatio  $(\Gamma, \sim^m \varphi \not\vdash \sim^{m+1} \varphi, \Delta)$ 

Then:

A **decent negation** should respect (N1.m) and (N2.m), for every m.

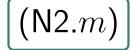
But the above negative rules might not be enough...

On what negation is not:

[Marcos 2005]

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A decent negation should respect (N1.m) and (N2.m), for every m.

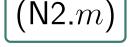
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These should also be avoided!!

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> Given any consistent tarskian logic  $\mathcal{L}$ , one can always find an inconsistent logic  $\mathcal{IL}$  such that:  $\Gamma \models_{\mathcal{IL}}^{\mathsf{m}} \beta, \Delta \quad \text{iff} \quad \Gamma \models_{\mathcal{L}}^{\mathsf{m}} \beta, \Delta$ yet:  $\mathcal{S} \not\models_{\mathcal{IL}}^{\mathsf{m}}$ .

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Now, what about ~-inconsistency??

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Then, using the same trick as before:

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