Errata to the paper 'A Taxonomy of **C**-systems', and more

last updated Feb 2005

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This note contains a collection of important corrections, problems, comments and clarifications to the paper:

> Walter A. Carnielli and João Marcos. A taxonomy of **C**-systems. In W. A. Carnielli, M. E. Coniglio, and I. M. L. D'Ottaviano, editors, *Paraconsistency: The Logical Way to the Inconsistent,* Proceedings of the II World Congress on Paraconsistency, held in Juquehy, BR, May 8–12, 2000, volume 228 of *Lecture Notes in Pure and Applied Mathematics*, pages 1–94. Marcel Dekker, 2002. Preprint available at:
> `http://www.cle.unicamp.br/e-prints/abstract_5.htm`.

It also contains a few solutions to problems that had been left open. Many of the suggested changes are implemented in the above paper's successor:

> Walter A. Carnielli, Marcelo E. Coniglio, and João Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, 2nd edition, volume 14. Kluwer Academic Publishers, 2005. Preprint available at:
> `http://www.cle.unicamp.br/e-prints/vol_5,n_1,2005.html`.

All contributions are welcome and will be credited to their authors!

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If you have not found *any* of the following slips in the paper,
maybe you have not read it carefully enough?

General: Deduction Metatheorem (DM) (M. E. Coniglio)

The Theorem 3.1 (p.48) is indeed correct for C_{min} as stated, for the mentioned reasons. Moreover, this obviously continues provable if new *axioms* are added to the logic. Nevertheless, if one extends this logic by adding new *rules*, then the (DM) often fails! Unfortunately, for lack of care in the presentation of our logics, we introduced them by adding new rules instead of the corresponding axioms... The problem is that we *do* want the (DM) to be valid in all our logics.

Consider a particular example. The logic **bC** is defined in Section 3.2 from C_{min} by adding the rule (bc1) $\circ A, A, \neg A \vdash B$ to the axioms and rules of the latter (p.50). Take now the 8-valued matrices from Theorem 3.53, and notice that they validate all axioms of C_{min} and its only rule, *modus ponens* (MP). Moreover, if one now defines a matrix for the consistency connective such that $v(\circ A) = \frac{4}{7}$ for every value of A, then also the rule (bc1) is validated by the matrices (its premises can never be simultaneously distinguished). But now notice $\circ A \to (A \to (\neg A \to B))$ is not validated by those matrices: Just take A and B as atomic sentences p and q such that $v(p) = \frac{1}{7}$ and $v(q) = \frac{6}{7}$. Then $v(\circ A \to (A \to (\neg A \to B))) = \frac{6}{7}$, while 1 is the only distinguished value of the matrices. So, in this formulation, **bC** would *not* respect the (DM). This was not what we intended, but it neatly illustrates what might happen when one thinks in terms of sequents but writes down Hilbertian axioms instead (in terms of sequents, the (DM) becomes just a rule for implication introduction). *Nostra culpa...*

To fix that flaw, all logics that we introduced by adding new inference rules should instead have been defined by adding the corresponding implicational axioms. Then, given that the (DM) will hold good, the initial rule will be readily derivable, by (MP). Therefore:

Page 50, Fact 3.8

Status unknown: We are not sure as yet if this is true. To be sure, one would have to check in detail whether the (DM) is still derivable from the axiom (Min1) and the rules (MP) and $(A \to B), (B \to C) \vdash (A \to C)$. To play safe, it is better to stick to the original axiom (Min2) all along, and use the last rule as derived.

At any rate, at least the following alternative formulation of the above mentioned Fact can be proven: '(Min2) can be substituted, in C_{min}, by the axiom $\vdash (A \to B) \to ((B \to C) \to (A \to C))$.'

Page 50, (bc1), line −19

$\circ A, A, \neg A \vdash B \rightsquigarrow \vdash \circ A \to (A \to (\neg A \to B))$

Page 51, (RA0), line 20

$\circ B, (A \to B), (A \to \neg B) \vdash \neg A \rightsquigarrow \vdash \circ B \to ((A \to B) \to ((A \to \neg B) \to \neg A))$

Page 51, (bc0), line −17

$\circ A, A, \neg A \vdash \neg B \rightsquigarrow \vdash \circ A \to (A \to (\neg A \to \neg B))$

Page 51, (RA1), line −15

$\circ B, (\neg A \to B), (\neg A \to \neg B) \vdash A \rightsquigarrow \vdash \circ B \to ((\neg A \to B) \to ((\neg A \to \neg B) \to A))$

Page 56, (bc2), line −14

$\neg\bullet A \vdash \circ A \rightsquigarrow \vdash \neg\bullet A \to \circ A$

Page 56, (bc3), line −11

$\neg\circ A \vdash \bullet A \rightsquigarrow \vdash \neg\circ A \to \bullet A$

Page 57, (bc4), line 4

$\bullet A \vdash \neg\circ A \rightsquigarrow \vdash \bullet A \to \neg\circ A$

Page 57, (bc5), line 5

$\circ A \vdash \neg\bullet A \rightsquigarrow \vdash \circ A \to \neg\bullet A$

Page 58, last paragraph before Subsection 3.5

Change all rules for the corresponding implicational axioms.

Page 58, (ci1), line -8
$\bullet A \vdash A \rightsquigarrow \vdash \bullet A \to A$
Page 58, (ci2), line -7
$\bullet A \vdash \neg A \rightsquigarrow \vdash \bullet A \to \neg A$
Page 58, (ci), line -4
$\bullet A \vdash (A \wedge \neg A) \rightsquigarrow \vdash \bullet A \to (A \wedge \neg A)$
Page 64, line -4
$\div A \vdash \neg A \rightsquigarrow \vdash \div A \to \neg A$
Page 69, (cl), line 16
$\neg(A \wedge \neg A) \vdash \circ A \rightsquigarrow \vdash \neg(A \wedge \neg A) \to \circ A$
Page 72, (bun), line 22
$(A \to (\circ B \wedge (B \wedge \neg B))) \vdash \neg A \rightsquigarrow \vdash (A \to (\circ B \wedge (B \wedge \neg B))) \to \neg A$
Page 73, (cd), line -8
$\neg(\neg A \wedge A) \vdash \circ A \rightsquigarrow \vdash \neg(\neg A \wedge A) \to \circ A$
Page 73, (cb), line -7
$(\neg(A \wedge \neg A) \vee \neg(\neg A \wedge A)) \vdash \circ A \rightsquigarrow \vdash (\neg(A \wedge \neg A) \vee \neg(\neg A \wedge A)) \to \circ A$
Page 73, lines -3 and -2
$\neg(A \wedge \neg A), (A \to B), (A \to \neg B) \vdash \neg A \rightsquigarrow \vdash \neg(A \wedge \neg A) \to ((A \to B) \to ((A \to \neg B) \to \neg A)))$
Page 75, (ce), line 15
$A \vdash \neg\neg A \rightsquigarrow \vdash A \to \neg\neg A$
Page 77, (ca1)–(ca3), lines 22–24
$(\circ A \wedge \circ B) \vdash \circ(A \wedge B) \rightsquigarrow \vdash (\circ A \wedge \circ B) \to \circ(A \wedge B)$
$(\circ A \wedge \circ B) \vdash \circ(A \vee B) \rightsquigarrow \vdash (\circ A \wedge \circ B) \to \circ(A \vee B)$
$(\circ A \wedge \circ B) \vdash \circ(A \to B) \rightsquigarrow \vdash (\circ A \wedge \circ B) \to \circ(A \to B)$
Page 78, line 7
$\bullet(A \to B) \vdash (\bullet A \vee \bullet B) \rightsquigarrow \vdash \bullet(A \to B) \to (\bullet A \vee \bullet B)$
Page 78, line -3
$A^{(n)}, A, \neg A \vdash B \rightsquigarrow \vdash A^{(n)} \to (A \to (\neg A \to B))$
Page 80, (co1)–(co3), lines -13 to -11
$(\circ A \vee \circ B) \vdash \circ(A \wedge B) \rightsquigarrow \vdash (\circ A \vee \circ B) \to \circ(A \wedge B)$
$(\circ A \vee \circ B) \vdash \circ(A \vee B) \rightsquigarrow \vdash (\circ A \vee \circ B) \to \circ(A \vee B)$
$(\circ A \vee \circ B) \vdash \circ(A \to B) \rightsquigarrow \vdash (\circ A \vee \circ B) \to \circ(A \to B)$
Page 81, (co1)–(co3), lines -16 to -14
$\circ(A \wedge B) \vdash (\circ A \vee \circ B) \rightsquigarrow \vdash \circ(A \wedge B) \to (\circ A \vee \circ B)$
$\circ(A \vee B) \vdash (\circ A \vee \circ B) \rightsquigarrow \vdash \circ(A \vee B) \to (\circ A \vee \circ B)$
$\circ(A \to B) \vdash (\circ A \vee \circ B) \rightsquigarrow \vdash \circ(A \to B) \to (\circ A \vee \circ B)$
Page 82, (cj1)–(cj3), lines -11 to -9
$\bullet(A \wedge B) \dashv\vdash (\bullet A \wedge B) \vee (\bullet B \wedge A) \rightsquigarrow \vdash \bullet(A \wedge B) \leftrightarrow (\bullet A \wedge B) \vee (\bullet B \wedge A)$
$\bullet(A \vee B) \dashv\vdash (\bullet A \wedge \neg B) \vee (\bullet B \wedge \neg A) \rightsquigarrow \vdash \bullet(A \wedge B) \leftrightarrow (\bullet A \wedge \neg B) \vee (\bullet B \wedge \neg A)$
$\bullet(A \to B) \dashv\vdash (A \wedge \bullet B) \rightsquigarrow \vdash \bullet(A \to B) \leftrightarrow (A \wedge \bullet B)$
Page 92, (M1n)and (M2n), lines -24 to -22
Change for the corresponding implicational forms.

The necessary changes at other places that might have not been listed above are all straightforward. Notice that in a few places, to prove or disprove (IpE), axioms are in general more than one needs, as rules might well do the job. At any rate, the weaker logics from above that contain only the rules instead of the corresponding implicational axioms might also be interesting or more appropriate in a few situations, and they deserve further study.

Pages 31–32, definitions of linguistic and deductive extensions
Notice that in general, in the literature, it is not required that such extensions be 'proper'. The adaptations in that case are straightforward.
As remarked in the paper, in the above comment, and also at some other remarks below, in most, if not all, cases that we talk about 'extensions' we are in fact assuming to be talking about logics that extend other logics by the addition of new axioms or of rules that do not invalidate the Deduction Metatheorem.

Page 35, definitions (Eq1) and (Eq2), and Fact 2.8 (M. E. Coniglio)
That (Eq1) defines an equivalence relation for formulas is an easy consequence of (Con1) and (Con3) (p.31). But (Eq2) does *not* in general define an equivalence relation for sets of formulas under exactly the same conditions. For that effect one needs to restrict the notion of a consequence relation, either by adding the property:

(Con4) $[(\forall B \in \Delta)\ \Gamma \Vdash B$ and $\Delta \Vdash A]$ implies $\Gamma \Vdash A$ (transitivity for sets)

or else by adding the property:

(Con5) $\Gamma \Vdash A$ implies $\Gamma^{\mathsf{fin}} \Vdash A$, for some finite $\Gamma^{\mathsf{fin}} \subseteq \Gamma$ (compacity)

First, it is obvious that adding either (Con4) or (Con5) to (Con1) and (Con3) will do the job. Second, it is equally easy to check that (Con4) derives (Con3) in the presence of (Con1) and (Con2). (Hint: Instantiate, in (Con4), Γ as $\Delta \cup \Gamma$, Δ as $\Gamma \cup \{\alpha\}$, and α as β.)
Finally, to check that (Con4) is *not* derivable from (Con1)–(Con3), consider, for instance, the logic $\mathbf{L}_{\mathbb{R}}$ having the set $\mathbb{R}$ of real numbers as its set of formulas and a consequence relation $\Vdash$ defined as follows:

$\Gamma \Vdash A$ iff $A \in \Gamma$, or $A = \frac{1}{n}$ for some $n \in \mathbb{N}$, $n \geq 1$, or
there is a sequence $(A_n)_{n\in\mathbb{N}}$ contained in Γ such that
$(A_n)_{n\in\mathbb{N}}$ converges to A.

It is easy to see that $\mathbf{L}_{\mathbb{R}}$ satisfies (Con1), (Con2) and (Con3). On the one hand, (Con4) is not valid in $\mathbf{L}_{\mathbb{R}}$. Indeed, take $\Gamma = \varnothing$, $\Delta = \{\frac{1}{n} : n \in \mathbb{N},\ n \geq 1\}$ and $\alpha = 0$. Then the antecedent of (Con4) is true: Every element of Δ is a thesis, and Δ contains the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ that converges to 0. On the other hand, the consequent of (Con6) is false: 0 is not a thesis in $\mathbf{L}_{\mathbb{R}}$.
The above example of $\mathbf{L}_{\mathbb{R}}$ was proposed in:

J.-Y. Béziau. *Research on Universal Logic: Excessivity, negation, sequents* (in French). PhD thesis, Université Denis Diderot (Paris 7), France, 1995.

If those properties are presupposed from the start there is no need to adjust the statement of Fact 2.8.

Page 38, Fact 2.11(i)
Every theory which is contradictory with respect to $\sim$ is explosive $\rightsquigarrow$ Every theory which is contradictory with respect to $\sim$ is trivial

Fact 2.11(ii)
A logic with a supplementing negation $\sim$ cannot be $\sim$-contradictory nor trivial, given part (a) of (D10), in the previous page.

Page 38, Fact 2.13(ii)
If **L** is finitely trivializable $\leadsto$ If **L** is non-trivial yet finitely trivializable

Page 39, second paragraph, **Page 43**, lines 17–19,
Page 45, line 8, **Page 46**, third paragraph,
Page 52, line 23, and a few other places,
Jaśkowski's logic **D2**
There is an awful lot of misunderstanding and confusion in the literature about the logic **D2**, one of the earliest samples of the paraconsistent vintage, introduced in:

> Stanisław Jaśkowski. A propositional calculus for inconsistent deductive systems (in Polish). *Studia Societatis Scientiarum Torunensis*, Sectio A, 5:57–77, 1948. Translated into English in *Studia Logica*, 24:143–157, 1967, and in *Logic and Logical Philosophy*, 7:35–56, 1999.
>
> Stanisław Jaśkowski. On the discussive conjunction in the propositional calculus for inconsistent deductive systems (in Polish). *Studia Societatis Scientiarum Torunensis*, Sectio A, 8:171–172, 1949. Translated into English in *Logic and Logical Philosophy*, 7:57–59, 1999.

Misled by decades of biased presentations of this logic, our paper commits basically the same mistakes in its presentation. However, it should be clear to anyone that reads the above papers, once and for all, that the logic presented in our paper is *not* Jaśkowski's **D2**. Let's call **J** the logic defined, as in the paper, by setting $\Gamma \Vdash_{\mathbf{J}} \alpha$ iff $\Diamond\Gamma \vDash_{S5} \Diamond\alpha$. This 'pre-discussive' logic **J** is indeed implicitly considered by Jaśkowski in his papers, but it does not represent the 'discussive' logic **D2**.
To define **D2**, Jaśkowski in fact uses the above '$\Diamond$-translation', but only after he preprocesses the classical connectives, in the following way. Let For denote the set of formulas of classical propositional logic, in a language containing the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$, and let For^{M} denote the set of formulas of a language containing also the unary modal connectives $\Diamond$ and $\Box$. Consider a mapping $j : For \rightarrow For^{\mathsf{M}}$ such that:

(i) $p^* = p$ for every atomic sentence p
(ii) $(\neg A)^* = \neg A^*$
(iii) $(A \wedge B)^* = A^* \wedge \Diamond B^*$
(iv) $(A \vee B)^* = A^* \vee B^*$
(v) $(A \rightarrow B)^* = \Diamond A^* \rightarrow B^*$
(vi) $(A \leftrightarrow B)^* = (\Diamond A^* \rightarrow B^*) \wedge (\Diamond B^* \rightarrow \Diamond A^*)$

Then, **D2** is the logic defined by setting $\Gamma \Vdash_{\mathbf{D2}} \alpha$ iff $\Diamond(\Gamma^*) \vDash_{S5} \Diamond(\alpha^*)$, where $\Gamma^* = \{\gamma^* : \gamma \in \Gamma\}$. It should be noticed that clause (iii) comes from the 1949 2-pages paper, that was only officially translated into English very recently; all the other clauses are indigenous to the 1948 paper. Without (iii), the resulting conjunction is left-adjunctive but not left-disadjunctive, as in the case of the logic **J** mentioned in our paper. Too much fuss has been made in the literature about the alleged 'non-adjunctive' character of **D2**. With the above definition, however, **D2** is perfectly adjunctive. Moreover, it *validates all axioms and rules of positive classical logic*, and yet $\neg$ *is non-explosive.*

Actually, in our paper we wrote several times that **D2** is an **LFI**, and we were indeed not wrong about that, even in the present updated formulation of the logic. Let's prove it. Consider the following set of abbreviations on For:

$\top \stackrel{\text{def}}{=} (A \vee \neg A)$, for any formula A (a top particle)

$\bot \stackrel{\text{def}}{=} \neg\top$ (a bottom particle)

$\blacksquare A \stackrel{\text{def}}{=} (\neg A \to \bot)$

$\blacklozenge A \stackrel{\text{def}}{=} \neg\blacksquare\neg A$

$\circ A \stackrel{\text{def}}{=} (\blacklozenge A \to \blacksquare A)$ (a consistency connective)

It is easy to check that $\circ$ has indeed the expected behavior of a consistency connective, namely: (a) $\circ p, p \nVdash_{\mathbf{D2}} q$; (b) $\circ p, \neg p \nVdash_{\mathbf{D2}} q$; (c) $\circ A, A, \neg A \Vdash_{\mathbf{D2}} B$. To check (a), just take an $S5$-model containing a sole world w in which p is true but q is false. To check (b), take again an $S5$-model containing a sole world w, but now let both p and q be false in it. To check (c), notice that it corresponds in the end to checking, in $S5$, the validity of the inference $(\Diamond A \to \Box A), \Diamond A, \Diamond\neg A \vDash_{S5} \Diamond B$. You might use your preferred $S5$-decision procedure to check that. As a consequence, one may now safely conclude that **D2** is a **dC**-*system based on classical logic*.

One final observation about **D2**. The fact that it is defined by way of a (double) translation into the modal logic $S5$ has led some people to believe and assert that **D2** is a 'modal paraconsistent logic'. It should be remarked, however, that **D2** fails one of the main characterizing properties of a normal modal logic: the *replacement property* (called, in our paper, (IpE), for 'intersubstitutivity of provable equivalents'). Indeed, while it is true, for instance, that $\neg(A \wedge \neg A) \dashv\Vdash_{\mathbf{D2}} (B \vee \neg B)$, the following inference fails: $\neg\neg(A \wedge \neg A) \Vdash_{\mathbf{D2}} \neg(B \vee \neg B)$. Indeed, for a counter-model to the latter inference, just take for A an atomic sentence p and consider a modal model with two worlds w and v such that w sees v, p is true in w but false in v. On that matter, check also the paper: (**Chapter 3.2** of the present thesis)

> João Marcos. Modality and paraconsistency. In M. Bilkova and L. Behounek, editors. *The Logica Yearbook 2004*, Proceedings of the XVIII International Symposium promoted by the Institute of Philosophy of the Academy of Sciences of the Czech Republic. Filosofia, Prague, 2005. Preprint available at:
> `http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-ModPar.pdf`.

Page 42, Fact 2.14

One needs to assume here that $\neg A$ does not always denote a bottom particle.

Page 44, Fact 2.15(ii)

Any explosive logic is partially explosive $\leadsto$ Any non-trivial explosive logic is partially explosive

Page 45, Definition (D27) of positive-preservation, and G. Priest &
Page 46, Definition (D28) of a **C**-system A. Avron &
M. E. Coniglio & J. Marcos & W. Carnielli

As it is, the definition of a **C**-system allows for some degenerate examples, such as that of a logic **L** that is a **C**-system based on whatever constitutes the 'negationless fragment' of **L** itself. This is not very informative. A better and more careful way of implementing the same intuition is as follows. Consider a set of connectives $\Sigma 1$ and let **L**1 be a consistent logic (that is, neither paraconsistent nor trivial) whose

formulas are written with the help of $\Sigma 1$. Let $\neg$ be some symbol for negation and let **L**2 be a logic whose formulas are written with the help of a set of connectives $\Sigma 2$ such that $\neg \in \Sigma 2 - \Sigma 1$. We say that **L**2 is a **C**-*system based on* **L**1 *with respect to* $\neg$ if:
(a) **L**2 is a conservative extension of **L**1, and $\neg$ is not definable in **L**1,
(b) **L**2 is an **LFI** (with respect to $\neg$), where $\Delta(A) = \{\circ A\}$ (recall (D15)).

Page 45, proof of Fact 2.19
Notice that one needs to really guarantee somehow that (1) ‘$\circ A$ is a not a top particle’, (2) ‘$\{\circ A, A\}$ is not always trivial’ and (3) ‘$\{\circ A, \neg A\}$ is not always trivial’, for the proposed definition of $\circ A$ as $(A \to \bot) \vee (\neg A \to \bot)$. To wit, some properties of the symbol $\neg$ in the given paraconsistent logic must be known in advance. Indeed, if $A \leftrightarrow \neg A$ is provable, for instance, then both (2) and (3) fail.
Part (i) is in general unproblematic. Indeed, in positive classical logic, $\vdash ((A \wedge B) \to C) \leftrightarrow ((A \to C) \vee (B \to C))$, so, if $\circ A$, as above defined, is a top particle in a logic such as the one mentioned in the statement of the Fact, then $(A \wedge \neg A) \vdash \bot$, and the logic would not be paraconsistent.
For parts (ii) and (iii) it is enough to consider that the negation symbol $\neg$ has the two following ‘negative properties’:

(*verificatio*) $(\exists A)\ \neg A \nVdash A$ (*falsificatio*) $(\exists A)\ A \nVdash \neg A$

This justifies the ‘in general’ used in the proof of the Fact. Notice that these rules are the weakest forms of some basic characterizing negative rules for negation proposed in the paper: (**Chapter 4.1** of the present thesis)

João Marcos. On negation: Pure local rules. *Journal of Applied Logic*, 2005.
Preprint available at:
`http://www.cle.unicamp.br/e-prints/vol_4,n_4,2004.html`.

Page 45, definition (D27)
Extend the definition in the natural way for the case of logics containing more than one negation symbol.

Page 46, lines -18 and -17, bold paraconsistency
We announced that all of our **C**-systems would be boldly paraconsistent, but we did not prove that in the paper. Given the significance of this claim, it is only fair that we sketch here its proof.
Consider any of the 8K maximal paraconsistent 3-valued logics from Subsection 3.11, of which each of the other **C**-systems of the paper is a deductive fragment. Assume $\Gamma \nvDash \sigma(p_0, \ldots, p_n)$ for some appropriate choice of formulas. In particular, by (Con2), it follows that $\nvDash \sigma(p_0, \ldots, p_n)$. Now, consider a variable p not in $p_0, \ldots, p_n$. Let p be assigned the value $\frac{1}{2}$, and extend this assignment to the variables $p_0, \ldots, p_n$ so as to give the value 0 to $\sigma(p_0, \ldots, p_n)$. It is obvious that, in this situation, $p, \neg p \nvDash \sigma(p_0, \ldots, p_n)$.

Page 51, Theorem 3.12 M. E. Coniglio & J. Marcos
The ‘real proof’ is in fact not that naive. Given a derivation of a formula of the language of C_{min} in which (bc1) is used, there can always happen, in theory, that there is *another* derivation of the same formula that does not use (bc1) but that still makes use of the new connective $\circ$ of the extended language.

This result is not really worth the painful induction over the Hilbertian derivations. An alternative, and simpler, way of verifying the Theorem is by looking directly at the recursive semantics associated to both logics, and checking that the corresponding decision procedures for formulas *not* containing the consistency connective in the case of **bC** validates exactly the same formulas as the decision procedure of C_{min} does. For such procedures, check, further on, the paper mentioned in the comment to '**Page 66**, Fact 3.50'.

Page 57, Theorem 3.25
Wrong choice of matrices for the independence proofs, as it is (for instance, the axiom (bc1) is not validated by them). The easiest way of fixing this is by changing the matrix of negation, both in part (i) and in part (ii), for one such that $v(\neg A) = 0$ if $v(A) \in \{1, \frac{2}{3}\}$, and $v(\neg A) = 1 - v(A)$ otherwise.
It is also possible to prove the same theorem using 3-valued matrices, instead of 4-valued ones. Consider again the same matrices for $\wedge$, $\vee$, $\to$ and $\neg$ as in Theorem 3.23. For part (i), take $v(\circ A) = 1$ and $v(\bullet A) = \frac{1}{2}$ if $v(\circ A) \in \{1, 0\}$ and $v(\circ A) = 0$ and $v(\bullet A) = 1$ otherwise. For part (ii), take $v(\circ A) = \frac{1}{2}$ if $v(\circ A) \in \{1, 0\}$ and $v(\circ A) = 0$ otherwise, and take $v(\bullet A) = 1$ for every value of A.

Page 58–59, and **Page 62**, Theorem 3.41,
on the axiomatization of the logic **Ci**
The more we have tried to clarify and motivate the whole thing, the axiomatization(s) of **Ci** still remained somewhat hard to swallow. This is an important point, of course, as every other subsequent logic in the paper will extend this fundamental logic. A simple set of axioms for **Ci** is obtained if one just adds to **bC** the following new axioms:

(ci) $\vdash \neg\circ A \to (A \wedge \neg A)$
(cis) $\vdash \circ\circ A$
(inc) $\vdash \bullet A \leftrightarrow \neg\circ A$

Remember to check also the paper mentioned in the comment to '**Page 66**, Fact 3.50', below.

Page 60, Fact 3.32 (M. E. Coniglio)
(or in any extension of this logic) $\leadsto$ (or in any axiomatic extension of this logic)

Page 61, Fact 3.36 (M. E. Coniglio)
The addition of (RC): [...] to **Ci** causes its collapse into classical logic $\leadsto$ The least extension of **Ci** that satisfies (RC): [...] and the Deduction Metatheorem collapses into classical logic

Page 61, Fact 3.37(i) (M. E. Coniglio & J. Marcos)
This part of the Fact is false. Indeed, to see that (bc2) is independent from the other axioms of **Ci**, consider again the 3-valued matrices for $\wedge$, $\vee$, $\to$ and $\neg$ as in Theorem 3.23, and define $v(\circ A) = 1$ and $v(\bullet A) = 0$ if $v(\circ A) \in \{1, 0\}$ and $v(\circ A) = 0$ and $v(\bullet A) = \frac{1}{2}$ otherwise. To see that (bc3) is independent from the other axioms of **Ci**, do again as above, but now define $v(\circ A) = \frac{1}{2}$ and $v(\bullet A) = 0$ if $v(\circ A) \in \{1, 0\}$ and $v(\circ A) = 0$ and $v(\bullet A) = 1$ otherwise.

Page 64, (Alt11), line 9
This has led to some confusion, and it must be clarified once and for all. If one is talking about a non-trivial extension of positive classical logic, (Alt10) and (Alt12) alone define all properties of classical negation. (Alt11) will be derivable from the other axioms and rules, and it is thus *not* necessary to talk about this last axiom at any point in the text, once it can be checked that one can count on the previous two axioms for negation. Classical logic is indeed axiomatizable by (MP) and axioms (Min1)–(Min9) of positive classical logic, plus (Alt10) and (Alt12). To be sure, this fact was already remarked in this paper, at p.50, lines 18–19.

Page 66, Fact 3.50 (M. E. Coniglio & J. Marcos)
The ‘only if’ part holds good, but the ‘if’ part is too restrictive as it is, and the alleged proof is wrong. In fact, as we know from Facts 3.32 and 3.33, p.60, all formulas preceded by a $\circ$ or a $\bullet$ ‘behave classically’ in **Ci**. And so does any complex boolean combination of such formulas. Accordingly, given a classical theorem $A(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are its atomic subformulas, then the following can be proven: $\vdash_{\mathbf{Ci}} \circ A(\circ p_1, \ldots, \circ p_n)$. Moreover, given any formula $A(C_1, \ldots, C_n)$, where $C_1, \ldots, C_n$ are all tops or bottoms of **Ci**, then $\circ A$ is also a top. There are, thus, many ‘provably consistent’ formulas of **Ci** left off by the statement of the above Fact.
All this is of course much easier to verify semantically. For that you might check, for instance, the paper: (**Chapter 2.2** of the present thesis)

João Marcos. Possible-translations semantics for some weak classically-based paraconsistent logics. Research report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2004.
`http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-PTS4swcbPL.pdf`

Page 67, Theorem 3.51(i)
This is true in fact for any **LFI**, and not only for extensions of **Ci**.

Page 68, line 5 (M. E. Coniglio)
(RC) cannot be added to **Ci** $\rightsquigarrow$ (RC) cannot be added to **Ci**, together with the Deduction Metatheorem

Page 79, and **Page 95**, line -10, and **Page 101**, lines 8–12,
on the logic C_{Lim} and non-finitely gently explosive paraconsistent logics
The logic C_{Lim} is not compact, thus it is also not finitely gently explosive. Indeed, let p be an atomic sentence, and let $\Gamma^{\kappa} = \{p^n : 0 \leq n < \kappa \leq \omega\}$, where the formulas p^n are defined as at the end of p.74. Then, $\Gamma^{\omega}, \neg p \vdash B$, for every B, in every logic C_n, $1 \leq n < \omega$, thus $\Gamma^{\omega}, \neg p \vdash B$ is a sound inference in C_{Lim}. Suppose now that there is a finite subset $\Gamma_{\mathsf{fin}} \subseteq \Gamma$ such that $\Gamma_{\mathsf{fin}}, \neg p \vdash B$ holds good in C_{Lim}. Then, if A^m is the largest formula in Γ_{fin}, the derivation $\Gamma^m, \neg p \vdash B$ will also hold good, by monotonicity. But that same derivation does not hold good in C_{m+1}, and this logic extends C_{Lim}. Absurd.

Page 81, line 7
Cito $\rightsquigarrow$ **Cigo** (the ‘g’ is from axiom (cg), on p.74)
Citoe $\rightsquigarrow$ **Cigoe**

Page 88, Fact 3.75
The definitions of the congruence matrices have caused some confusion. To be sure, only one of them is always available for sure in the 8K logics: the one that makes $v(A \equiv B) = 1$ when $v(A) = \frac{1}{2} = v(B)$. The other matrix is only definable in *some* of the 8K logics.

Page 91, Fact 3.79
Every deductive extension $\rightsquigarrow$ Every non-linguistic deductive extension, that is, every deductive extension over a fixed language,

Page 93, last 4 lines (M. E. Coniglio & J. Marcos)
being thus algebraizable in the sense of Blok-Pigozzi (though [...]) $\rightsquigarrow$ being thus *equivalential* in the sense of Blok-Pigozzi (though [...]). That being known, to be BP-algebraizable they will only need to be shown, in addition, to be weakly algebraizable. For the argument, check the Theorem 3.16 of:

> Josep Maria Font, Ramon Jansana, and Don Pigozzi. A survey of abstract algebraic logic. *Studia Logica*, 74(1/2):13–97, 2003. Abstract algebraic logic, Part II (Barcelona, 1997).

The same theorem shows that *all* our present **LFI**s are at least protoalgebraizable, as all of them extend positive classical logic and contain thus an appropriate implication such that $\vdash A \to A$ and $A, A \to B \vdash B$.
(Notice that this partly settles a taxonomical question that appears on **Page 95**.)

Page 95, line 11, proof of Theorem 3.83 (M. E. Coniglio)
$\{0, a, 1, u\}$ and $\{0, b, 1, u\}$ are two filters $\rightsquigarrow$ $\{a, 1, u\}$ and $\{b, 1, u\}$ are two filters

Page 99, second paragraph
The logics **mbC** and **mCi** were mentioned in passing, but their axiomatizations were not clarified beyond any doubt. To obtain the logic **mbC**, indeed, all one needs to do is to 'delete axiom (Min11): $\neg\neg A \to A$' from the set of axioms of **bC**. Now, the axiomatization of **mCi** is trickier, because of the intended relation of classical opposition between the $\circ$ and the $\bullet$. The most obvious way of obtaining **mCi**, in a sense, seems to be through an infinite set of axioms, namely, by adding to **mbC** the following new axioms:

(ci)	$\vdash \neg\circ A \to (A \wedge \neg A)$
$(\text{cc})_n$	$\vdash \circ\neg^n\circ A$, where $\neg^0 A = A$, and $\neg^{n+1} A = \neg\neg^n A$, for every $n \in \mathbb{N}$
(inc)	$\vdash \bullet A \leftrightarrow \neg\circ A$

Compare this to the set of axioms for **Ci** proposed above, in the comments to '**Page 58–59** etc'. Notice that, in **mCi**, the so-called 'Guillaume's Theses' from Fact 3.38, p.61, are no longer true, thus the need of (cis_n).
Check also, again, the paper mentioned in the comment to '**Page 66**, Fact 3.50'.

Page 108, item [111]
Theory of logical calculi $\rightsquigarrow$ *Theory of Logical Calculi*