Possible-Translations Semantics

(Extended Abstract)

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Abstract

This text aims at providing a bird's eye view of possible-translations semantics ([10, 24]), defined, developed and illustrated as a very comprehensive formalism for obtaining or for representing semantics for all sorts of logics. With that tool, a wide class of complex logics will very naturally turn out to be (de)composable by way of some suitable combination of simpler logics. Several examples will be mentioned, and some related special cases of possible-translations semantics, among which are society semantics and non-deterministic semantics, will also be surveyed.

1 Logics, translations, possible-translations

Let a *logic* $\mathcal{L}$ be a structure of the form $\langle \mathcal{S}, \Vdash \rangle$, where $\mathcal{S}$ denotes its *language* (its set of *formulas*) and $\Vdash \subseteq \mathsf{Pow}(\mathcal{S}) \times \mathsf{Pow}(\mathcal{S})$ represents its associated *consequence relation* (cr), somehow defined so as to embed some formal model of reasoning. Call any subset of $\mathcal{S}$ a *theory*. As usual, capital Greek letters will denote theories, and lowercase Greek will denote formulas; a sequence such as $\Gamma, \alpha, \Gamma' \Vdash \Delta', \beta, \Delta$ should be read as asserting that $\Gamma \cup \{\alpha\} \cup \Gamma' \Vdash \Delta' \cup \{\beta\} \cup \Delta$.

Morphisms between any two of the above structures will be called *translations*. So, given any two logics, $\mathcal{L}_1 = \langle \mathcal{S}_1, \Vdash_1 \rangle$ and $\mathcal{L}_2 = \langle \mathcal{S}_2, \Vdash_2 \rangle$, a mapping $t : \mathcal{S}_1 \to \mathcal{S}_2$ will constitute a translation from $\mathcal{L}_1$ into $\mathcal{L}_2$ just in case the following holds:

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(T1) $\Gamma \Vdash_1 \Delta \Rightarrow t(\Gamma) \Vdash_2 t(\Delta)$

A translation is said to be *conservative* in case the converse of (T1) holds, i.e.:

(T2) $\Gamma \Vdash_1 \Delta \Leftarrow t(\Gamma) \Vdash_2 t(\Delta)$

Given a logic $\mathcal{L}{=}\langle \mathcal{S}, \Vdash \rangle$, a *possible-translations representation* (ptr) over it is a structure of the form $\langle \mathsf{Log}, \mathsf{Tr}, \mathsf{Reg} \rangle$, where $\mathsf{Log} = \{\langle \mathcal{S}_j, \Vdash_j \rangle\}_{j\in J}$ is an indexed set of logics (also called *factors* or *ingredients* of this ptr), $\mathsf{Tr} = \{t_j : \mathcal{S} \to \mathcal{S}_j\}_{j\in J}$ is an indexed set of translations, and $\mathsf{Reg} \subseteq \mathsf{Pow}(\mathsf{Tr})$. To any such ptr one can immediately associate three levels of consequence relations: A *local* pt-cr, $\Vdash^j_{\mathsf{pt}}$, for each $t_j \in \mathsf{Tr}$, a *regional* pt-cr, $\Vdash^R_{\mathsf{pt}}$, for each $R \in \mathsf{Reg}$, and a *global* pt-cr, $\Vdash_{\mathsf{pt}}$. These relations will be defined by setting:

(L-pt) $\Gamma \Vdash^j_{\mathsf{pt}} \Delta$ iff $t_j(\Gamma) \Vdash_j t_j(\Delta)$

(R-pt) $\Gamma \Vdash^R_{\mathsf{pt}} \Delta$ iff $(\eth t_j \in R)[\Gamma \Vdash^j_{\mathsf{pt}} \Delta]$,
where $\eth$ is some (generalized) quantifier

(G-pt) $\Gamma \Vdash_{\mathsf{pt}} \Delta$ iff $(\forall R \in \mathsf{Reg})[\Gamma \Vdash^R_{\mathsf{pt}} \Delta]$

Obviously, (L-pt) is just a particular case of (R-pt). Taking $\mathsf{Reg} = \{\{t_j\} : t_j \in \mathsf{Tr}\}$ makes the regional pt-cr perfectly dispensable —we will call any ptr with that characteristic a *simple* ptr and write it more simply as $\langle \mathsf{Log}, \mathsf{Tr} \rangle$. There are usually many ways of obtaining the same global pt-cr. Suppose for instance that '$\eth = \forall$' in (R-pt). Then, $\Vdash_{\mathsf{pt}}$ will be exactly the same, for every Reg such that $\bigcup \mathsf{Reg} \supseteq \mathsf{Tr}$.

Given two logics $\mathcal{L}_1{=}\langle \mathcal{S}_1, \Vdash_1 \rangle$ and $\mathcal{L}_2{=}\langle \mathcal{S}_2, \Vdash_2 \rangle$, we will say that $\mathcal{L}_1$ is *sound* with respect to $\mathcal{L}_2$ in case $\Vdash_1 \subseteq \Vdash_2$. Similarly, we will say that $\mathcal{L}_1$ is *complete* with respect to $\mathcal{L}_2$ in case $\Vdash_1 \supseteq \Vdash_2$. Notice that translations can be endomorphisms. In particular, any logic is sound and complete with respect to itself, the identity endomorphism always constituting thus a trifling example of a ptr. A ptr over a logic $\mathcal{L}{=}\langle \mathcal{S}, \Vdash \rangle$ is said to be *adequate* in case $\mathcal{L}$ is sound and complete with respect to $\langle \mathcal{S}, \Vdash_{\mathsf{pt}} \rangle$. Thus, an adequate ptr can be seen as a way of combining a set of translations so as to obtain a very particular conservative translation. Finally, a *possible-translations semantics* (pts) is simply a possible-translations representation in which all factors are defined by 'semantic means' (in contrast to, say, 'abstract deductive' or 'proof-theoretical' means). This characterization certainly looks very vague, but I will show in more detail in the following subsections how the canonical semantic notions work and how they can be seen as special cases of simple pts, according to the above definitions.

One last methodological discrimination is sometimes useful. In case one starts with a logic $\mathcal{L}$ and then finds a set of factors for it in an adequate ptr, one will call the process *splitting logics*; in case one starts with the factors and then build a logic for which the corresponding ptr is adequate, the process will be called *splicing logics*. The immense majority of examples from the literature on *combining logics* is of a more synthetic character: More and more logics are spliced as time goes by. Here, on the contrary, it will be often natural to use ptr's in order to analyze some given logics, splitting them into simpler components in order to understand them. *Frango ut patefaciam.*

Digression 1.1 (*Categorial*) If one considers the category where logics are the objects and translations are the arrows, the diagrams we get for the ptr's all look like there were sunbeams irradiating from a common core. The logic that originates from the combination can be seen as the colimit of this diagram. In [11] the authors show how to generalize this construction for arbitrary diagrams. This should be compared to what is done in [29] in understanding *fibring* (a more general form of combination, check [23, 4]) as a categorial construction. A first advance in that direction, generalizing the basic construction of fibring, can be found in [16]. A different semantically-driven generalization of fibring, *cryptofibring*, is categorially investigated in [7]. □

Digression 1.2 (*Historical*) Possible-translations semantics were first introduced in [9], restricted to the use of finite-valued truth-functional factors. The embryo was then frozen for a period, and in between 1997 and 1998 it was publicized under the denomination 'non-deterministic semantics', in [12], and in several talks by Carnielli and a few by myself. Noticing that the non-deterministic element was but a particular accessory of the more general picture, from 1999 on the semantics retook its earlier denomination ([10, 24, 14, 15, 26]). □

1.1 What is a logic?

To be sure, this is a question that will *not* be answered in this section. Any number of answers to it can be found in the literature, if you dig hard enough. I will here instead recall how some among the most popular answers can be recast in the present framework.

Given a logic $\mathcal{L} = \langle \mathcal{S}, \Vdash \rangle$ as above, we will call it an MCT logic in case its cr is subject to the following restrictions:

(C1)	$(\Gamma, \varphi \Vdash \varphi, \Delta)$	(overlap)
(C2)	$(\Gamma \Vdash \varphi, \Delta)$ and $(\Gamma', \varphi \Vdash \Delta') \Rightarrow (\Gamma', \Gamma \Vdash \Delta, \Delta')$	(cut)
(C3)	$(\Gamma \Vdash \Delta) \Rightarrow (\Gamma', \Gamma \Vdash \Delta, \Delta')$	(dilution)

Call any clause of the form $\Gamma \Vdash \Delta$ an *inference*. Theories that appear at the left-hand side of the $\Vdash$ are also dubbed *countertheories*, or *premises* assumed by the inference; theories that appear at the right-hand side of the $\Vdash$ are also called *alternatives* sanctioned by the inference. An SCT consequence relation (cf. [31]) is a particular case of an MCT consequence relation, in which each inference has a single formula as alternative (no real 'alternative' in that case, is it?). Such alternative is often called *conclusion* of the inference. SCT logics are also called *single-conclusion*, in contrast to the more symmetrical (*multiple-premise*) *multiple-conclusion* MCT logics. It would be just as natural, of course, to consider here a SPMCT logic to be defined by the same restrictions above, but on a single-premise-multiple-conclusion environment. Very uncommon in practice, the SPMCT case works pretty much like the SCT case in most circumstances. Below I will only mention SPMCT logics explicitly, thus, when relevant.

Here are some degenerate examples of logics. Let a logic $\langle \mathcal{S}, \Vdash \rangle$ be called *overcomplete* in case its cr is characterized by one of the following universal properties:

(C0.0.0)	$(\Gamma \Vdash \Delta)$	(triviality)
(C0.0.1)	$(\Gamma, \alpha \Vdash \Delta)$	(nihilism)
(C0.1.0)	$(\Gamma \Vdash \beta, \Delta)$	(dadaism)
(C0.1.1)	$(\Gamma, \alpha \Vdash \beta, \Delta)$	(semitriviality)

Note, by the way, that THE trivial logic is characterized by the nonproper cr over the language $\mathcal{S}$. Clearly, SCT logics must identify trivial and dadaistic logics, and identify nihilistic and semitrivial logics. When we talk about THE dadaistic logic in a given language we will be referring to the logic having a non-trivial dadaistic cr. Similarly, THE nihilistic logic will refer to the logic having a non-trivial nihilistic cr, and THE semitrivial logic will denote the logic having a non-dadaistic non-nihilistic cr.

A formula β of a logic $\mathcal{L}$ is said to be a *thesis* of this logic in case $(\Gamma \Vdash \beta, \Delta)$, for any choice of Γ and Δ; an *antithesis* of this logic is any formula α such that $(\Gamma, \alpha \Vdash \Delta)$, for any choice of Γ and Δ. An arbitrary thesis is sometimes denoted by $\top$, and an arbitrary antithesis is sometimes denoted by $\bot$.

Theorem 1.1.1 (i) Every multiple-conclusion overcomplete logic is MCT. Every single-conclusion overcomplete logic is SCT.
(ii) The empty language defines a unique MCT / SCT logic.
(iii) Any arbitrary intersection of MCT / tarkian logics defined over some fixed language defines a MCT / SCT logic.

Proof:
(i): Just check that properties (C1)–(C3) of a MCT / SCT logic hold for each of the above four kinds of overcomplete logics.
(ii): Indeed, in the MCT case, $\mathsf{Pow}(\varnothing) \times \mathsf{Pow}(\varnothing) = \{\langle \varnothing, \varnothing \rangle\}$ and $\langle \varnothing, \langle \varnothing, \varnothing \rangle \rangle$ is obviously trivial. Similarly for the SCT case.
(iii): Given some language $\mathcal{S}$ and any indexed set of MCT / SCT logics $\{\langle \mathcal{S}, \Vdash_i \rangle\}_{i \in I}$, it is easy to see that $\langle \mathcal{S}, \bigcap_{i \in I}(\Vdash_i) \rangle$ is also a MCT / SCT logic. In particular, note that, in the MCT case, $\bigcap_{i \in I}(\Vdash_i) = \{\langle \varnothing, \varnothing \rangle\}$ iff $(\mathcal{S} = \varnothing)$, and then you're in case (ii); besides, note that the condition $I = \varnothing$ puts you directly in case (i). Similarly for the SCT case. □

Theorem 1.1.2 Fix some MCT / SCT logic $\mathcal{L}$ over some non-empty language $\mathcal{S}$. Then:
(i) $\mathcal{L}$ is the trivial logic iff there is at least one formula in its language which is both a thesis and an antithesis of $\mathcal{L}$.
(ii) $\mathcal{L}$ is the nihilistic logic iff all of its formulas are antitheses of it.
(iii) $\mathcal{L}$ is the dadaistic logic iff all of its formulas are theses of it.
(iv) $\mathcal{L}$ is the semitrivial logic iff any formula implies any other (or the same) formula, but no antitheses nor theses are present in the language of this logic.

Proof: Immediate. □

Several other restrictions and extensions of the above notion of logic are studied in [25], from an abstract viewpoint. As in that paper, a logic here will be called *minimally decent* in case it is not overcomplete.

1.2 What is the canonical notion of entailment?

Let $\mathcal{V}$ denote an arbitrary set of *truth-values*, where $\mathcal{D}^{\mathcal{V}} \subseteq \mathcal{V}$ denotes its subset of *designated* values (the 'true truth-values'), and $\mathcal{U}^{\mathcal{V}} = \mathcal{V} \setminus \mathcal{D}^{\mathcal{V}}$ denotes its subset of *undesignated* values (the 'false truth-values'). Given a language $\mathcal{S}$, let a *valuation* over it be any mapping $\S^{\mathcal{V}} : \mathcal{S} \to \mathcal{V}$. Call any collection of valuations over $\mathcal{S}$ a (MCT) *semantics* sem over $\mathcal{S}$. This semantics will be called *κ-valued* if κ is the greatest cardinality of truth-values of the valuations in sem, that is, $\kappa = \sup_{\S^{\mathcal{V}} \in \mathsf{sem}}(|\mathcal{V}|)$. To any valuation $\S^{\mathcal{V}} \in \mathsf{sem}$ and any semantics sem one can associate *canonical* notions of *local entailment*, $\vDash^{\S}_{\mathsf{sem}}$ and *global entailment*, $\vDash_{\mathsf{sem}}$, by setting:

$$\text{(L-ce)} \quad \Gamma \vDash^{\S^{\mathcal{V}}}_{\mathsf{sem}} \Delta \ \text{ iff } \ (\S^{\mathcal{V}}(\Gamma) \cap \mathcal{U}^{\mathcal{V}} \neq \varnothing \ \text{ or } \ \S^{\mathcal{V}}(\Delta) \cap \mathcal{D}^{\mathcal{V}} \neq \varnothing)$$

$$\text{(G-ce)} \quad \Gamma \vDash_{\mathsf{sem}} \Delta \ \text{ iff } \ (\forall \S^{\mathcal{V}} \in \mathsf{sem})[\Gamma \vDash^{\S^{\mathcal{V}}}_{\mathsf{sem}} \Delta]$$

An *ordinary* MCT semantics is one in which a fixed cardinal of designated / undesignated values is set throughout all the valuations of the semantics. Obviously, any semantics can be made ordinary by just adding to each valuation a convenient number of truth-values that will not be used. Similarly to above, a SCT (*ordinary*) *κ-valued semantics* will be defined just like an MCT (ordinary) κ-valued semantics, only that all inferences will have exactly one formula at their right-hand sides.

Theorem 1.2.1 (i) Any MCT / SCT κ-valued semantics induces at least one MCT / SCT logic by way of one of its associated canonical entailment relations.
(ii) Consider any covering of the valuations of a given MCT / SCT semantics. Each layer of the covering can now be said to determine a new (universal) 'regional semantics', and the intersection of all the entailments associated to the latter gives you back the global entailment.

Proof: (i): It is easy to check that any $\vDash$ defined as in (L-ce) or in (G-ce) respects the properties (CR1)–(CR3). Note that this holds good irrespective of κ or of the number of valuations in sem.
(ii): Just recall Theorem 1.1.1(iii). □

Given the above results, one sees that any semantic structure of the form $\langle \mathcal{S}, \vDash \rangle$ defines an MCT and a SCT logic, and the logics corresponding to the global entailment relation can be obtained through the intersection of all local (or regional) entailment relations. As before, given a logic $\mathcal{L} = \langle \mathcal{S}, \Vdash \rangle$ and a semantics sem over $\mathcal{S}$, one can now very naturally talk about $\mathcal{L}$ being *locally sound* with respect to some $\S \in \mathsf{sem}$ in case $\Vdash \subseteq \vDash^{\S}_{\mathsf{sem}}$, and being *globally sound* with respect to sem in case $\Vdash \subseteq \vDash_{\mathsf{sem}}$. Similarly for local and global completeness and adequacy. The statement of the following result parallels that of Theorem 1.1.2.

Theorem 1.2.2 Here is how you can obtain adequate ordinary semantics for each variety of overcomplete logic:
(i) For the trivial logic, consider the empty semantics (empty set of truth-values).
(ii) For the nihilistic logic, consider some semantics whose valuations make everything false.
(iii) For the dadaistic logic, consider some semantics whose valuations make everything true.
(iv) For the semitrivial logic, consider some semantics whose valuations either make everything true or make everything false.

Proof: Let $\mathcal{S}$ be an arbitrary fixed language, let $\mathcal{D}_n$ and $\mathcal{U}_n$ be pairwise disjoint arbitrary sets of truth-values, for each $1 \leq n \leq 4$, such that $\mathcal{U}_2 \neq \varnothing$, $\mathcal{D}_3 \neq \varnothing$, $\mathcal{D}_4 \neq \varnothing$ and $\mathcal{U}_4 \neq \varnothing$. For each n, let $\mathsf{val}(\mathcal{D}_n) = \{\S : \S(\mathcal{S}) \subseteq \mathcal{D}_n\}$ denote the sets of all valuations over $\mathcal{S}$ whose counterdomains range only over designated values, and let $\mathsf{val}(\mathcal{U}_n) = \{\S : \S(\mathcal{S}) \subseteq \mathcal{U}_n\}$ do a similar thing for undesignated values. Consider now semantics such that $\mathsf{sem}_1 \subseteq \mathsf{val}(\mathcal{D}_1) \cap \mathsf{val}(\mathcal{U}_1) = \varnothing$, $\mathsf{sem}_2 \subseteq \mathsf{val}(\mathcal{U}_2)$, $\mathsf{sem}_3 \subseteq \mathsf{val}(\mathcal{D}_3)$ and $\mathsf{sem}_4 \subseteq \mathsf{val}(\mathcal{D}_4) \cup \mathsf{val}(\mathcal{U}_4)$. It is easy, then, to check that: (i) sem_1 is adequate for the trivial logic; (ii) sem_2 is adequate for the nihilistic logic; (iii) sem_3 is adequate for the dadaistic logic; (iv) sem_4 is adequate for the semitrivial logic. □

1.3 What can be done with translations between logics?

The general definitions of translation and of conservative translation that you found at the beginning of the present section were studied in detail in [12, 19], and interesting specializations of these notions were proposed in [20]. Typical examples of everyday translations are given by the endomorphisms that define uniform substitutions in a logic whose language is formed by a free algebra (of formulas). One can here also easily check that:

Theorem 1.3.1 (i) A logic can always be conservatively translated into itself.
(ii) To check soundness or completeness of a given logic with respect to some MCT / SCT semantics amounts to checking the identity mapping from the language into itself to be a translation.

Proof: (i): Just consider the identity mapping $t : \varphi \mapsto \varphi$, for every $\varphi \in \mathcal{S}$.
(ii): Considering a logic $\mathcal{L}_a = \langle \mathcal{S}, \Vdash \rangle$ and a SCT semantic structure $\mathcal{L}_b = \langle \mathcal{S}, \vDash \rangle$, to show that $\mathcal{L}_a$ is sound with respect to $\mathcal{L}_b$ you have to show that the identity mapping, as in part (i), is a translation from $\mathcal{L}_a$ into $\mathcal{L}_b$. Similarly, to show that $\mathcal{L}_a$ is complete with respect to $\mathcal{L}_b$ your task is showing that the identity mapping is a translation from $\mathcal{L}_b$ into $\mathcal{L}_a$. □

Here are some degenerate examples of translations:

Theorem 1.3.2 For arbitrary logics (not necessarily MCT nor SCT) over some fixed language $\mathcal{S}$:
(i) Any logic is translatable into the trivial logic.
(ii) Any single-conclusion logic is translatable into any logic having a thesis. Any single-premise logic is translatable into any logic having an antithesis.
(iii) The dadaistic logic is conservatively translatable into any logic having a thesis. The nihilistic logic is conservatively translatable into any logic having an antithesis. The semitrivial logic is conservatively translatable into any logic respecting (C1) and having no theses and no antitheses.
(iv) Given a logic with no (anti)theses at all, NO logic having a(n anti)thesis whatever is translatable into the former.
(v) Any logic having no theses nor antitheses is translatable into the semitrivial logic.

Proof: (i): Choose any $\alpha \in \mathcal{S}$, and set $t : \varphi \mapsto \alpha$, for every $\varphi \in \mathcal{S}$.
(ii): For the first part, set $t : \varphi \mapsto \top$, for every $\varphi \in \mathcal{S}$. For the second part, $t : \varphi \mapsto \bot$ will do the job.
(iii): Similar to (ii).
(iv): Let $\langle \mathcal{S}_1, \Vdash_1 \rangle$ be a logic with a thesis $\top$, and let $\langle \mathcal{S}_2, \Vdash_2 \rangle$ be a logic with no thesis. If there would be some translation $t : \mathcal{S}_1 \to \mathcal{S}_2$, then, in particular, $\Vdash_2 t(\top)$ would need to hold, but there is no formula in $\mathcal{L}_2$ with that property. Similarly for an antithesis.
(v): Exercise. □

Problem 1.3.3 For more esoteric non-MCT logics, such as non-monotonic logics and other context-dependent applications it might seem more natural to work with a definition of translation that directly involves the inferences, instead of the formulas. In that case, a translation from $\langle \mathcal{S}_1, \Vdash_1 \rangle$ into $\langle \mathcal{S}_2, \Vdash_2 \rangle$ had better be defined, say, as a mapping $t : \mathsf{Pow}(\mathcal{S}_1) \to \mathsf{Pow}(\mathcal{S}_2)$ instead of $t : \mathcal{S}_1 \to \mathcal{S}_2$, as before. It might be better as well to think of a logic directly as a set of theories, instead of a set of formulas, endowed with a consequence relation. The properties of this sort of definitions are yet to be investigated in more detail. An advance in that direction was already made in [17], where the authors conceive SCT logics as two-sorted first-order structures (the sort of 'formulas' and the sort of 'theories'), and talk about 'transfers' as morphisms among those structures (of which translations between SCT logics, in the above sense, are but particular cases).

1.4 What are possible-translations semantics?

We have defined above the notion of a possible-translations representation (ptr) based on the combination of a collection of factors through local ($\Vdash^j_{\mathsf{pt}}$), regional ($\Vdash^R_{\mathsf{pt}}$) and global ($\Vdash_{\mathsf{pt}}$) consequence relations (cr). A possible-translations semantics (pts) was then characterized as a ptr based on factors defined by 'semantic means'. Moreover, the above sections have shown a conventional rendering of the received notion of 'semantics', slightly generalized in accordance with the principles of the theory of valuations (cf. [18]) and of abstract multiple-conclusion deductive systems (cf. [32, 30]).

There are several ways of combining logics. In a very pleasant paper, [3], Blackburn and de Rijke survey the reasons one might have for splicing logics, and propose a catalogue of the forms of combination based on the increasing level of involvement of the ingredient logics: They come up with nice pictures for 'refining structures', then 'classification structures', then 'totally fibred structures'. Another taxonomy is delineated at [8, 4, 28], where 'synchronization' and 'parameterization' appear as distinguished special cases of 'fibring'. How would the general picture for the combination through a possible-translations representation look like?

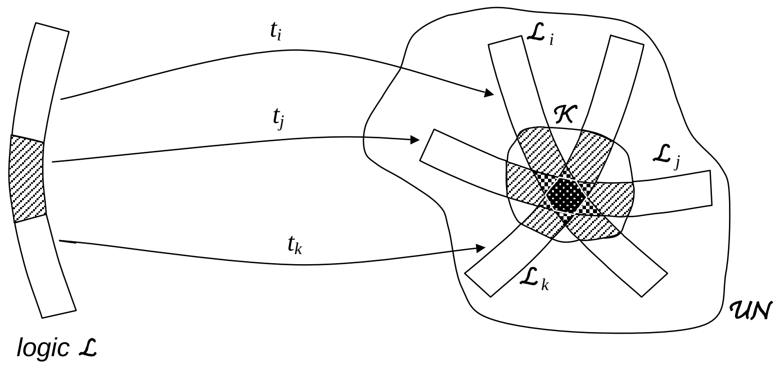

Figure 1: *The logical Rosetta Stone.*

An insightful analogy may be provided by concentrating on the situation in which a logic is split into its simpler components and comparing it to the deciphering of the 'Rosetta Stone' (cf. [15]). Carved in 196B.C. and found by Napoleon troops in July 1799 near the homonymous village (Rashid) located in the western delta of the Nile, the Rosetta Stone is a basalt slab containing three different inscriptions of a text written by a group of priests to honor the Egyptian pharaoh. Why is it important? Because it finally allowed scholars to decipher the Hierogliphic writing, a problem that had been open for several hundred years! After the work of Thomas Young, a British physicist, and Jean-François Champollion, a French Egyptologist, the code was finally broken, and a phonetic value was attached to hieroglyphs that had previously been thought to have a purely symbolic value. How was it done? The three scripts in the stone were the Hieroglyphic (used for important or religious documents), the Demotic (everyday Egyptian script) and the Greek (language of the rulers of Egypt at that time). With the aid of both Greek and Coptic (language of the Christian descendants of the ancient Egyptians), Champollion was able to decipher the Demotic writing, and from that he was able to trace back the meaning of the Hierogliphic signs. But how did they know that the three scripts represented the same text, to start with? Because the stone *said so*, at the very end of its Greek inscription! Another beautiful example of self-reference, therefore.

Based on the above story, Figure 1 gives a schematic illustration of what is going on when a ptr is designed. The Rosetta Stone is the 'logical universe' $\mathcal{UN}$ where all ingredient logics can be found, resembling perhaps an egg with the sunny side up. The long curved format of the logic represents the form of reasoning sanctioned by it. You can see that the morphisms (possible-translations) are intended to preserve that format. At a distinguished hachured region of each logic you may find its circumstantial theses and antitheses. Each translation should in particular take theses into theses, and antitheses into antitheses. The region where they can be found in $\mathcal{UN}$ is at its yolk $\mathcal{K}$. The appetizing part is the one in which the ingredients are cooked together so as to give us the corresponding possible-translations structure.

The next result shows some simple examples of ptr and pts:

Theorem 1.4.1 (i) Any logic has an adequate possible-translations representation.
(ii) Any (MCT / SCT) semantics can be seen as a possible-translations semantics with any positive number of factors.

Proof: (i): Just consider the identical mapping from the language into itself.
(ii): For any given semantic structure, you can define the natural 1-factor pts by way of the identical mapping, which does exactly the same job as the former semantics —though it does not really tell you more than you already knew. Now, assume you have a MCT / SCT semantics $\mathsf{sem} = \{\S_k\}_{k\in K}$. In case there is more than one valuation in sem, a second natural pts is obtainable in any case if you pick the single-valuation SCT semantics $\mathsf{sem}_k = \S_k$, for each $k \in K$, and consider as translations $|K|$ applications of the

identical mapping. Any pts that extends one of the above natural possible-translations semantics by the addition of redundant factors and translations leaves the resulting global pt-entailment untouched. □

One can count now on a more sophisticated interplay between local and global notions at hand: If an MCT / SCT semantics can be seen as a general way of gluing arbitrary collections of valuations, a possible-translations semantics can be seen as a more general way of gluing collections of any arbitrary kind of previously given semantics.

Call a semantics *unitary* in case it is defined by way of a single valuation, or a single factor; call it *large* in case the cardinality of the set of valuations or the set of factors is at least as big as the cardinality of the underlying language. Obviously, any unitary semantics is ordinary from its very inception; unitary semantics can be made large, and large semantics can always be made ordinary at request, by the addition of redundant valuations or truth-values. We already knew from Theorem 1.2.1(ii) than any MCT / SCT semantics can be reduced to the intersection of unitary MCT / SCT semantics; the last result above suggests now that any semantics can ultimately and quite naturally be converted into a large possible-translations semantics whose factors are all unitary semantics themselves.

Moreover:

Theorem 1.4.2 If you are talking about logics characterized by MCT / SCT entailments, or by simple possible-translations representations:
(i) Global soundness implies local soundness.
(ii) Local completeness implies global completeness.
In overcomplete logics:
(iii) Local soundness automatically transfers to global soundness.
(iv) Global completeness automatically transfers to local completeness.

Proof: Parts (i) and (ii): Just recall the definitions of (L-ce) and (G-ce) (subsection 1.2), (L-pt) and (G-pt) (section 1).
Parts (iii) and (iv): You need no more than 1 valuation to define an overcomplete logic, as we saw in Theorem 1.2.2. □

Note that, in non-overcomplete logics, there is no reason in general for global soundness to be expected to transfer to local soundness, or for local completeness to be expected to transfer to global completeness.

1.5 Which logics have adequate semantics?

Right now we have two things called MCT: The abstract consequence relations characterized by way of clauses (C1)–(C3) in subsection 1.1 and the semantics to which canonical entailment relations were associated in subsection 1.2. A similar thing can be said about abstract SCT consequence relations and SCT semantics. The attentive reader will certainly have noticed, though, that we have not as yet established a relation between the homonymous creatures! This subsection will correct this slip for the benefit of the interested.

Consider first the SCT case. Given a single-conclusion logic $\langle \mathcal{S}, \Vdash \rangle$ and a countertheory $\Pi \subseteq \mathcal{S}$, the *right-closure* of Π, denoted by $\Pi^{\mathbf{c}}$, is the set of all of its derived consequences, that is, the set $\{\pi : \Pi \Vdash \pi\}$.

Theorem 1.5.1 (i) In any SCT logic, $\Pi^{\mathbf{cc}} = \Pi^{\mathbf{c}}$, that is, $\Pi^{\mathbf{c}} \Vdash \pi \Leftrightarrow \Pi \Vdash \pi$. (ii) In any SCT logic $\langle \mathcal{S}, \Vdash \rangle$, given arbitrary $\Sigma \cup \Delta \cup \{\varphi\} \subseteq \mathcal{S}$, to check whether $\Sigma, \Delta \Vdash \varphi$ holds is equivalent to checking whether $(\forall \delta \in \Delta)\Sigma \Vdash \delta$ implies $\Sigma \Vdash \varphi$.

Proof: Immediate. □

Theorem 1.5.2 (*Lindenbaum-like*) Each SCT logic has at least as many (but no less than one) sound SCT unitary semantics as the number of its right-closed theories.

Proof: You have to take the truth-values from somewhere, and all that you have at this point is a logic $\langle \mathcal{S}, \Vdash \rangle$ with its underlying language $\mathcal{S}$ and its cr $\Vdash$. So, given any theory $\Delta \in \mathcal{S}$, take $\mathcal{V} = \mathcal{S}$ and $\mathcal{D} = \Delta^{\mathbf{c}}$ to be, respectively, the sets of truth-values and of designated values. Now, take the unitary semantics sem_Δ given by the identical mapping which takes each formula into itself. This defines a local / global entailment $\vDash_\Delta$ such that $\Gamma \vDash_\Delta \varphi$ iff ($\Gamma \not\subseteq \Delta^{\mathbf{c}}$ or $\varphi \in \Delta^{\mathbf{c}}$). Now, suppose you have (*a*) some $\Gamma \Vdash \varphi$ such that (*b*) $\Gamma \subseteq \Delta^{\mathbf{c}}$; all you need now is to show that (*c*) $\varphi \in \Delta^{\mathbf{c}}$. From (*b*) and (CR1), it follows that (*d*) $\Delta^{\mathbf{c}} \Vdash \gamma$, for every $\gamma \in \Gamma$. From (*a*) and (CR3) you have that (*e*) $\Delta^{\mathbf{c}}, \Gamma \Vdash \varphi$. From (*d*) and (*e*), by repeated applications of (CR2), you conclude that $\Delta^{\mathbf{c}} \Vdash \varphi$. But this finally implies (*c*), by definition of right-closure and Theorem 1.5.1(i). One defines, thus, a sound semantics corresponding to each right-closed theory of the underlying language. The collection of all such semantics is sometimes referred to as the LINDENBAUM BUNDLE.
Now, even if there are no non-empty theories, as in the case of the empty logic from Theorem 1.1.1(ii), you can count on a sound (and complete) unitary semantics, as in Theorem 1.2.2(i). □

Theorem 1.5.3 (*Wójcicki-like*) Any SCT logic has an adequate semantics.

Proof: Given a SCT logic $\langle \mathcal{S}, \Vdash \rangle$, define $\vDash_\Delta$, for each $\Delta \subseteq \mathcal{S}$, as in Theorem 1.5.2. Next, take the intersection of the Lindenbaum bundle, i.e., of all the unitary semantics thereby induced. Accordingly, define $\vDash = \bigcap_{\Delta \subseteq \mathcal{S}} (\vDash_\Delta)$. Now, such $\vDash$ is obviously sound for $\Vdash$. To check the converse, completeness, assume that $\Gamma \vDash \varphi$. Thus, $\Gamma \vDash_\Delta \varphi$, for every $\Delta \in \mathcal{S}$, and then it follows, by definition of $\vDash_\Delta$, that $(\forall \gamma \in \Gamma)\Delta^{\mathbf{c}} \Vdash \gamma$ implies $\Delta^{\mathbf{c}} \Vdash \varphi$. By part (i) of

Theorem 1.5.1, this amounts to the same as saying that $(\forall\gamma \in \Gamma)\Delta \Vdash \gamma$ implies $\Delta \Vdash \varphi$. But, by part (ii) of the same theorem, this is equivalent to writing $\Delta, \Gamma \Vdash \varphi$. In the particular case where $\Delta = \varnothing$ you will finally find what you want. $\square$

Corollary 1.5.4 Every SCT logic $\langle \mathcal{S}, \Vdash \rangle$ has an adequate ordinary κ-valued semantics, with $\kappa \leq |\mathcal{S}|$. $\square$

The previous result is very general, but a κ-valued semantics is more interesting in case its truth-values are well-behaved with respect to the underlying language, for instance, in case one can count on truth-functionality. The contrast between designated and undesignated values casts though a shadow of *bivalence*. Indeed:

Theorem 1.5.5 (*Suszko-like*) Every SCT logic has an adequate κ-valued SCT semantics, for $\kappa \leq 2$.

Proof: To make things easier, given a κ-valued SCT semantics, first you should make it ordinary. Next, for any κ-valuation § of the ordinary semantics $\mathsf{sem}(\kappa)$, and every consequence relation based on $\mathcal{V}_\kappa$ and $\mathcal{D}_\kappa$, define $\mathcal{V}_2 = \{T, F\}$ and $\mathcal{D}_2 = \{T\}$ and set the characteristic total function $b_\S : \mathcal{S} \to \mathcal{V}_2$ to be such that $b_\S(\varphi) = T$ iff $\S(\varphi) \in \mathcal{D}$. Now, collect all such bivaluations $b_\S$'s into a new semantics $\mathsf{sem}(2)$, and notice that $\Gamma \vDash_{\mathsf{sem}(2)} \varphi$ iff $\Gamma \vDash_{\mathsf{sem}(\kappa)} \varphi$. $\square$

Everything can be easily dualized to the SPMCT case. Only that now, given a single-premise logic $\langle \mathcal{S}, \Vdash \rangle$ and a theory $\Pi \subseteq \mathcal{S}$, you had better work with the *left-closure* of Π, denoted by ${}^{\mathbf{c}}\Pi$, as the set of all of its deriving premises, that is, the set $\{\pi : \pi \Vdash \Pi\}$. The rest is straightforward to adapt.

I will now briefly show how the above constructions can be modified for the MCT case (cf. [30]). As usual, call $\langle \Sigma, \Pi \rangle$ a *partition* of the set $\Theta \subseteq \mathcal{S}$ in case $\Sigma \cup \Pi = \Theta$ and $\Sigma \cap \Pi = \varnothing$.

Theorem 1.5.6 (*Cut for sets*) Given a MCT logic $\langle \mathcal{S}, \Vdash \rangle$:
If $\Gamma, \Sigma \Vdash \Pi, \Delta$, for every partition $\langle \Sigma, \Pi \rangle$ of Θ then $\Gamma \Vdash \Delta$.

Proof: Exercise. Use (C2) (cut), and induction on the cardinality of the Θ. $\square$

Theorem 1.5.7 (*L-theorem*) Each MCT logic has some sound MCT unitary semantics.

Proof: The overcomplete case is done. Otherwise, given a minimally decent logic $\langle \mathcal{S}, \Vdash \rangle$, call any partition $\langle \Sigma, \Pi \rangle$ of its language $\mathcal{S}$ *closed* in case $\Sigma \nVdash \Pi$. For every closed partition $\langle \Sigma, \Pi \rangle$ of $\mathcal{S}$, define the unitary semantics in which $\mathcal{V} = \mathcal{S}$, $\mathcal{D} = \Sigma$ and $\mathcal{U} = \Pi$. The local / global canonical entailment ${}_\Sigma{\vDash}_\Pi \subseteq \mathsf{Pow}(\mathcal{S}) \times \mathsf{Pow}(\mathcal{S})$ induced by that definition will be such that $\Gamma\,{}_\Sigma{\vDash}_\Pi \Delta$ iff

$\Gamma \cap \Pi \neq \varnothing$ or $\Delta \cap \Sigma \neq \varnothing$. Now, given an arbitrary inference $\Gamma \Vdash \Delta$, one can in particular conclude, by (C3) (dilution), that $\Gamma, \Sigma \Vdash \Pi, \Delta$. Supposing by absurd that both $\Gamma \cap \Pi = \varnothing$ and $\Delta \cap \Sigma = \varnothing$, one would be forced to conclude that $\Gamma \subseteq \Sigma$ and $\Delta \subseteq \Pi$, given that $\langle \Sigma, \Pi \rangle$ is a partition. From the above it follows that $\Sigma \Vdash \Pi$. This is impossible, for the partition $\langle \Sigma, \Pi \rangle$ is supposed to be closed. □

Theorem 1.5.8 (*W-theorem*) Any MCT logic has an adequate semantics.

Proof: Given an MCT logic $\langle \mathcal{S}, \Vdash \rangle$, define ${}_{\Sigma}{\vDash}_{\Pi}$, for each closed partition $\langle \Sigma, \Pi \rangle$ of $\mathcal{S}$, as in Theorem 1.5.7. Call cp the set of all such closed partitions. Take again the intersection of all the unitary semantics thereby induced, thus defining $\vDash \; = \bigcap_{\langle \Sigma, \Pi \rangle \in \mathsf{cp}} ({}_{\Sigma}{\vDash}_{\Pi})$. Soundness is easy to check. To check completeness, assume $\Gamma \nVdash \Delta$. Given the cut for sets (Theorem 1.5.6) we know that there will be some partition $\langle \Sigma, \Pi \rangle$ of $\mathcal{S}$ such that $\Gamma, \Sigma \nVdash \Pi, \Delta$. From (C3) (dilution), we know that such partition must be closed. Moreover, given (C1) (overlap), one must conclude that $\Gamma \subseteq \Sigma$ and $\Delta \subseteq \Pi$, and so $\Gamma {}_{\Sigma}{\nvDash}_{\Pi} \Delta$, thus $\Gamma \nvDash \Delta$. □

Corollary 1.5.9 Every MCT logic $\langle \mathcal{S}, \Vdash \rangle$ has an adequate ordinary κ-valued semantics, with $\kappa \leq |\mathcal{S}|$. □

Theorem 1.5.10 (*S-theorem*) Every MCT logic has an adequate κ-valued MCT semantics, for $\kappa \leq 2$. □

One can conclude from the above results that:

Theorem 1.5.11 (i) Every SCT / MCT logic has an adequate possible-translations semantics, in fact even a possible-translations semantics based on 2-valued factors (copies of classical logic).
(ii) The local and the global consequence relations associated to any simple possible-translations representation or possible-translations semantics based on SCT / MCT factors is SCT / MCT.

Proof: From Theorems 1.5.3 and 1.4.1. □

It is noteworthy that the above results for canonical semantics have pretty much the same flavor of a pts: Each unitary semantics can be seen as determining a translation, and the intersection of all of the appropriate unitary semantics in each case gives you the desired conservative translation.

2 Further illustrations

We have seen, in the previous section, that every MCT / SCT logic has an adequate MCT / SCT (2-valued) semantics. Moreover, any logic (MCT, SCT, or not) has an adequate possible-translations representation (ptr), and if it has an adequate semantics (MCT, SCT, or not) then it can be given an adequate possible-translations semantics (pts).

What about other less trivial examples of possible-translations semantics, not obtained by plain use of brute force, as above? Indeed, notice that the previous adequacy results were often either uninformative (when a logic was used to represent itself) or non-constructive (when a κ-valued semantics was posited but no recursive method was presented so as to define it). The situation can be improved in a some cases. In the case of sufficiently expressive finite-valued truth-functional logics, for instance, a constructive method can be designed for the specification of a recursive set of clauses that describe the 2-valued semantics announced by Theorem 1.5.5 (cf. [6, 5]).

Moreover, to get even more concrete, one can use a ptr to provide, say, a pts based on a couple of well-behaved and well-known finite-valued truth-functional factors for logics having NO adequate finite-valued truth-functional semantics, as done in [10, 24, 14, 26] for several paraconsistent and paracomplete logics. Also, deductive limits for infinite hierarchies of logics can very naturally be spliced, and decidability transferred from the factors to the product, as in [24, 14]. Moreover, truth-functional finite-valued logics can themselves be split in terms of 2-valued logics, that is, fragments of classical logic ([24, 27]), copies of classical logic can be combined into fragments of modal logics, and so on and so forth.

The final version of the paper will display a few representative such examples in detail.

3 Some other related semantic structures

The advantage of possible-translations semantics lies in its generality. It is no overstatement to assert that pretty much anything that one might want to call a semantics can be recast in the present framework. This leads us immediately to the main disadvantage of possible-translations semantics: its generality! Anything that is universally true can easily turn out to be also universally irrelevant. It is very important thus to characterize some interesting subclasses of possible-translations semantics, defined by stricter terms. Clauses restricting the set of translations or the factors involved are often helpful, often inevitable. With that in mind, *society semantics* ([13, 24, 21, 22]), *dyadic semantics* ([6, 5]), and (dynamic and static) *non-deterministic semantics* ([2, 1]) can all be precisely characterized as specialized forms of possible-translations semantics.

This will be done in detail in the final version of the paper.

References

[1] Arnon Avron. Non-deterministic semantics for families of paraconsistent logics. Presented at the III World Congress on Paraconsistency, held in Toulouse, FR, July 2003. To appear in Proceedings, 2005.
`http://antares.math.tau.ac.il/~aa/articles/int-c.ps.gz`.

[2] Arnon Avron and Iddo Lev. Non-deterministic multiple-valued structures. *Journal of Logic and Computation*, 2005. In print.
`http://antares.math.tau.ac.il/~aa/articles/nmatrices.ps.gz`.

[3] Patrick Blackburn and Maarten de Rijke. Why combine logics? *Studia Logica*, 59(1):5–27, 1997.

[4] Carlos Caleiro. *Combining Logics*. PhD thesis, IST, Universidade Técnica de Lisboa, PT, 2000.
`http://wslc.math.ist.utl.pt/ftp/pub/CaleiroC/00-C-PhDthesis.ps`.

[5] Carlos Caleiro, Walter A. Carnielli, Marcelo E. Coniglio, and João Marcos. Dyadic semantics for many-valued logics. Research report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2003. Presented at III World Congress on Paraconsistency, Toulouse, FR, July 28–31, 2003.
`http://wslc.math.ist.utl.pt/ftp/pub/CaleiroC/03-CCCM-dyadic2.pdf`.

[6] Carlos Caleiro, Walter A. Carnielli, Marcelo E. Coniglio, and João Marcos. Suszko's Thesis and dyadic semantics. Research report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2003. Presented at III World Congress on Paraconsistency, Toulouse, FR, July 28–31, 2003.
`http://wslc.math.ist.utl.pt/ftp/pub/CaleiroC/03-CCCM-dyadic1.pdf`.

[7] Carlos Caleiro and Jaime Ramos. Cryptomorphisms at work. Abstract, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2004. Presented at XVII International Workshop on Algebraic Development Techniques, March 27–30, 2004, Barcelona, ES. Long version to be submitted.
`http://wslc.math.ist.utl.pt/ftp/pub/CaleiroC/04-CR-fiblog17.pdf`.

[8] Carlos Caleiro, Cristina Sernadas, and Amílcar Sernadas. Mechanisms for combining logics. Research report, Section of Computer Science, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 1999.
`http://wslc.math.ist.utl.pt/ftp/pub/SernadasA/99-CSS-comblog.ps`.

[9] Walter Carnielli. Many-valued logics and plausible reasoning. In *Proceedings of the* XX *International Congress on Many-Valued Logics,* held at the University of Charlotte / NC, US, 1990, pages 328–335. IEEE Computer Society, 1990.

[10] Walter A. Carnielli. Possible-translations semantics for paraconsistent logics. In D. Batens, C. Mortensen, G. Priest, and J. P. Van Bendegem, editors, *Frontiers of Paraconsistent Logic,* Proceedings of the I World Congress on Paraconsistency, held in Ghent, BE, July 29–August 3, 1997, pages 149–163. Research Studies Press, Baldock, UK, 2000.

[11] Walter A. Carnielli and Marcelo E. Coniglio. A categorial approach to the combination of logics. *Manuscrito—Revista Internacional de Filosofia*, XXII(2):69–94, 1999.

[12] Walter A. Carnielli and Itala M. L. D'Ottaviano. Translations between logical systems: A manifesto. *Logique et Analyse (N.S.)*, 40(157):67–81, 1997.

[13] Walter A. Carnielli and Mamede Lima-Marques. Society semantics and multiple-valued logics. In W. Carnielli and I. M. L. D'Ottaviano, editors, *Advances in Contemporary Logic and Computer Science: Proceedings of the* XI *Brazilian Logic Conference on Mathematical Logic*, Salvador, BR, May 6–10, 1996, volume 235 of *Contemporary Mathematics*, pages 33–52. American Mathematical Society, 1999.

[14] Walter A. Carnielli and João Marcos. Limits for paraconsistent calculi. *Notre Dame Journal of Formal Logic*, 40(3):375–390, 1999.
http://projecteuclid.org/Dienst/UI/1.0/Display/euclid.ndjfl/1022615617.

[15] Walter A. Carnielli and João Marcos. *Ex contradictione non sequitur quodlibet.* In R. L. Epstein, editor, *Proceedings of the* II *Annual Conference on Reasoning and Logic*, held in Bucharest, RO, July 2000, volume 1, pages 89–109. Advanced Reasoning Forum, 2001.
http://www.advancedreasoningforum.org/Journal-BARK/V1TOC/v1toc.html.

[16] Marcelo E. Coniglio. Categorial combination of logics: Completeness preservation. Preprint, Section of Computer Science, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2001. Submitted for publication.
http://wslc.math.ist.utl.pt/ftp/pub/ConiglioM/catcomb.ps.

[17] Marcelo E. Coniglio and Walter A. Carnielli. Transfers between logics and their applications. *Studia Logica*, 72(3):367–400, 2002.

[18] Newton C. A. da Costa and Jean-Yves Béziau. Théorie de la valuation. *Logique et Analyse (N.S.)*, 37(146):95–117, 1994.

[19] Jairo J. da Silva, Itala M. L. D'Ottaviano, and Antônio M. Sette. Translations between logics. In C. H. Montenegro X. Caicedo, editor, *Models, Algebras and Proofs: Proceedings of the* X *Latin American Symposium on Mathematical Logic,* held in Bogotá, CO, July 1995, pages 435–448. Marcel Dekker, New York, 1999.

[20] Richard L. Epstein. *Propositional Logics: The semantic foundations of logic.* Wadsworth-Thomson Learning, 2000.

[21] Victor L. Fernández. Society Semantics for n-valued Logics (in Portuguese). Master's thesis, State University of Campinas, BR, 2001.
http://www.cle.unicamp.br/prof/coniglio/Victesis.ps.

[22] Victor L. Fernández and Marcelo E. Coniglio. Combining valuations with society semantics. *Journal of Applied Non-Classical Logics*, 13(1):21–46, 2003.
http://www.cle.unicamp.br/e-prints/abstract_11.html.

[23] Dov M. Gabbay. *Fibring Logics.* Oxford Logic Guides 38. Clarendon Press, 1999.

[24] João Marcos. Possible-Translations Semantics (in Portuguese). Master's thesis, State University of Campinas, BR, 1999.
http://www.cle.unicamp.br/students/J.Marcos/index.htm.

[25] João Marcos. On negation: Pure local rules. *Journal of Applied Logic*, 2005. In print. Preprint available at:
http://www.cle.unicamp.br/e-prints/vol_4,n_4,2004.html.

[26] João Marcos. Possible-translations semantics for some weak classically-based paraconsistent logics. Research report, CLC, Department of Mathematics, Instituto Superior Técnico, 1049-001 Lisbon, PT, 2004.
http://wslc.math.ist.utl.pt/ftp/pub/MarcosJ/04-M-PTS4swcbPL.pdf.

[27] João Marcos and Jean-Yves Béziau. Many values, many representations. Forthcoming.

[28] Amílcar Sernadas and Cristina Sernadas. Combining logic systems: Why, how, what for? *CIM Bulletin*, 15:9–14, December 2003.
http://wslc.math.ist.utl.pt/ftp/pub/SernadasA/03-SS-fiblog22.pdf.

[29] Amílcar Sernadas, Cristina Sernadas, and Carlos Caleiro. Fibring of logics as a categorial construction. *Journal of Logic and Computation*, 9(2):149–179, 1999.
http://wslc.math.ist.utl.pt/ftp/pub/SernadasA/98-SSC-fiblog.ps.

[30] D. J. Shoesmith and Timothy J. Smiley. *Multiple-Conclusion Logic*. Cambridge University Press, Cambridge–New York, 1978.

[31] Alfred Tarski. Über den Begriff der logischen Folgerung. *Actes du Congrès International de Philosophie Scientifique*, 7:1–11, 1936.

[32] Jan Zygmunt. *An Essay in Matrix Semantics for Consequence Relations*. Wydawnictwo Uniwersytetu Wrocławskiego, Wrocław, 1984.